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Scheduling the Two-Way Traffic on a Single-Track Railway with a Siding

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Abstract—The paper is concerned with scheduling the two-way traffic between two stations connected by a single-track railway with a siding. It is shown that if, for each station, the order in which trains leave this station is known or can be found, then for various objective functions an optimal schedule can be constructed in polynomial time using the method of dynamic programming. Based on this result, the paper also presents a polynomial-time algorithm minimising the weighted number of late trains.

Keywords: dynamic programming, polynomial algorithm, railway planning, scheduling theory

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1. INTRODUCTION

This paper presents a generalisation of the problem of scheduling the movement of trains on a single-track railway, previously considered in [1]. For the objective functions, considered in [1], the proofs below are a new justification of the corresponding algorithms.

The single-track railways are part of many railway networks and often are used for transportation within factories. The considered problem also arises in situations when one of the tracks of a two-track railway becomes inaccessible due to maintenance or accidents.

Detailed surveys on models and methods for railway planning are presented in [2–4]. This paper is a sequel of [1] where the reader can find the related literature review. In particular, [1] analyses the publications [5–11] which contain interesting results on planning the movement of trains on a single-track railway.

The considered problem can be stated as follows. There are two sets of trains: N_1 and N_2. Trains in the set N_1 are at station 1 and must go to station 2, whereas the trains in the set N_2 are at station 2 and must go to station 1. The station number that is opposite to the station with number $s \in \{1, 2\}$ will be denoted by $\bar{s}$. There is a siding between stations, permitting oncoming trains to pass each other, which can accommodate one train. In the siding, there is the main line for non-stop movement of trains and the additional line for a train to wait. The train on the main line goes through the siding without stopping. The speed of trains is the same for all trains and is constant. The time required for a train to cover the distance between station 1 and the siding, and between station 2 and the siding, is p_1 and p_2 respectively. Without loss of generality, it will be assumed that $p_1 \geqslant p_2$. The number of trains in N_1 is n_1, and the number of trains in N_2 is n_2. The trains can depart their stations starting from the point in time $t = 0$.

There are given minimal possible times between departures of two trains from the same station, arrivals of two trains at the siding, and between the arrival of a train at a station and the departure of a train from the same station. It is assumed that all these times are equal to β. It is assumed that $\beta < p_2 \leqslant p_1$. The time interval of length $\beta > 0$ will be called the safety interval. The assumption that the all listed above safety intervals have the same length simplifies the presentation of the results below, although these results can be generalised to the case when this assumption does not hold.

It is required to construct a schedule of train movements σ, satisfying the conditions listed above, i.e., to specify:

- the departure time $S_s^i(\sigma)$ for each train $i \in N_s, s \in \{1, 2\}$;
- the time $\tau_s^i(\sigma)$, which each train $i \in N_s, s \in \{1, 2\}$ spends in the siding.

Therefore, the arrival time of train $i \in N_s$ at its destination is

$$C_s^i(\sigma) = S_s^i(\sigma) + p_1 + p_2 + \tau_s^i(\sigma). \tag{1.1}$$

Consider the problem of minimisation of the objective function

$$\gamma(\sigma) = \bigodot_{i=1}^{n_1} \varphi_1^i(C_1^i(\sigma)) \odot \bigodot_{i=1}^{n_2} \varphi_2^i(C_2^i(\sigma)), \tag{1.2}$$

where $\varphi_s^i(\cdot)$ is a non-decreasing function, and $\odot$ is some associative and commutative operation such that for any numbers a_1, a_2, b_1, b_2, satisfying the inequalities $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$,

$$a_1 \odot b_1 \leqslant a_2 \odot b_2. \tag{1.3}$$

The operation $\odot$ can be, for example, addition or the operation of taking the maximum. Many objective functions, used in practice, have the form (1.2), for example, the maximum lateness, the weighted sum of train arrival times, and the number of late trains.

Following [1], the considered problem can be denoted as $S2S1|siding = 1, t_j = t| \odot \varphi_j$, where $S2$ designates two stations, $S1$ specifies a single siding, $siding = 1$ defines the siding capacity, $t_j = t$ indicates that all trains have the same speed.

In what follows, if it is clear what schedule σ is considered, the notations C_s^i, S_s^i, τ_s^i will be used instead of the notations $C_s^i(\sigma)$, $S_s^i(\sigma)$, $\tau_s^i(\sigma)$.

2. SCHEDULE PROPERTIES

The algorithm below constructs a schedule considering only the departure times of the trains that do not stop in the siding. In order to show that the information available at these points in time is sufficient for the construction of an optimum schedule, consider properties of feasible schedules.

Definition 1. Train $i \in N_s$, $s \in \{1, 2\}$, will be called an express if it goes through the siding without stopping, i.e., if $\tau_s^i = 0$. Trains which stop in the siding we be referred to as non-expresses.

Definition 2. Train i from station s is active at time t, if $S_s^i \leqslant t \leqslant C_s^i$.

Lemma 1. *If two trains from different stations are active simultaneously at some point in time, then one of them is a non-express and the other is an express that passes this non-express.*

Proof. Consider a schedule where two trains, train i from station s and train j from station $\bar{s}$, are active simultaneously at some point in time. Since the trains depart from different stations, the existence of the safety interval implies that $S_s^i \neq C_{\bar{s}}^j$ and $S_{\bar{s}}^j \neq C_s^i$. Therefore, if i and j are both expresses, then at the point in time $\max\{S^i_s, S^j_{\bar{s}}\}$, these trains should move towards each other which contradicts the feasibility of the schedule.

These trains cannot be two non-expresses either. Indeed, let i and j be non-expresses. Without loss of generality, assume that train i departs first. Then, taking into account that $C^i_s \neq S^j_{\bar{s}}$, at some point in time, these trains must move toward each other. Hence, one of them must pass the other and therefore cannot stop in the siding.

Thus, one of these trains must be an express and the other must be a non-express. Without loss of generality, assume that i is an express and j in a non-express. Since the arrival time of one of these trains cannot coincide with the departure time of the other, for some period since the point in time $\max\{S^i_s, S^j_{\bar{s}}\}$, they move towards each other. Therefore, non-express j must be in the siding when i goes through the siding. □

Lemma 2. *There exists an optimal schedule such that, for any two trains i and j from the same station s, the inequality $S^i_s < S^j_s$ implies $C^i_s < C^j_s$.*

Proof. Assume the contrary. Let, for two trains i and j from station s, the inequalities $S^i_s < S^j_s$ and $C^i_s \geqslant C^j_s$ hold. Because of the existence of the safety interval, the latter inequality implies $C^i_s > C^j_s$. Thus, train j overtakes train i, and therefore train j arrives at the siding at least β after i. On the other hand, by Lemma 1, since i is a non-express, any train g from station $\bar{s}$ which is active in the time interval $[S^j_s, C^j_s]$ must be an express that passes i when i is in the siding. According to the same lemma, train j also must allow g to pass it, which is impossible since the siding can accommodate only one train. Consequently, in the time interval $[S^j_s, C^j_s]$ there are no active trains from station $\bar{s}$, and therefore i can leave the siding β before the arrival of train j at the siding. In this case the new arrival time of train i is $C^j_s - \beta$. Since the objective function is non-decreasing, such change of the schedule does not increase the objective function. □

Thus, it will be assumed that no train overtakes a non-express from the same station. According to the following lemma, it is possible to assume that for each non-express there is at least one train from the opposite station which passes it.

Lemma 3. *There exists an optimal schedule such that each non-express leaves the siding simultaneously with some express.*

Proof. Consider a non-express j from station s. If no express passes j, let $t = S^j_s + p_s$, i.e., in this case t is the arrival time of j at the siding. If there are expresses that pass j, let $t = S^i_{\bar{s}} + p_{\bar{s}}$, where i is the last of these expresses, i.e., in this case t is the moment when the last express goes through the siding.

From the point in time t until the point in time C^j_s between the siding and station $\bar{s}$ there are no active trains from station $\bar{s}$, because otherwise, according to Lemma 1, such train must pass j which contradicts the choice of the point in time t. Consequently, j can leave the siding at the point in time t which does not lead to the increase of the objective function. If $t = S^j_s + p_s$, then in the new schedule j is an express. □

In what follows, only schedules which satisfy Lemma 2 and Lemma 3 will be considered. Each schedule defines a sequence of the departure times of expresses. Two expresses cannot leave simultaneously the opposite stations, because in this case they are moving towards each other which contradicts the feasibility of the schedule. On the other hand, taking into account the existence of the safety interval, two trains cannot leave simultaneously the same station. Therefore, in the defined sequence all expresses have different departure times. The following lemma shows what trains can be active between the consecutive departure times of two expresses, and therefore plays a key role in the proof that a schedule can be constructed by making decisions only at the departure times of expresses.

Lemma 4. *Let e and e' be two consecutive expresses, departing from stations x and x' respectively. Then, for any train j from station s which is active in the time interval $[S^e_x, C^{e'}_{x'}]$ and neither e nor e' passes it, either $s = x$ and $S^j_s < S^e_x$, or $s = x'$ and $S^j_s > S^{e'}_{x'}$.*

Proof. Let $S^j_s < S^e_x$ and $s \neq x$. Since j is active at some point in the time interval $[S^e_x, C^{e'}_{x'}]$, train j is active at the point in time S^e_x. According to Lemma 1, e must pass train j which contradicts the choice of j. Suppose that $S^j_s > S^{e'}_{x'}$ and $s \neq x'$. Train j is active at some point in the time interval $[S^e_x, C^{e'}_{x'}]$, and hence $S^j_s \leqslant C^{e'}_{x'}$, therefore at the point in time S^j_s train j is active together with the express e', and according to Lemma 1, e' must pass j, which contradicts the selection of j.

Suppose now that $S^j_s \in [S^e_x, S^{e'}_{x'}]$. Since all departure times of expresses are different and since e and e' are two consecutive expresses, train j cannot be an express. According to the choice of j, the expresses e and e' do not pass j, and therefore by Lemma 3, j is stationary in the siding when some express g goes through the siding. Since e and e' are two consecutive expresses, express g departs from its station either before S^e_x or after $S^{e'}_{x'}$. Assume that g departs before S^e_x. If g departs from station x, then it arrives at the siding before e. Consequently, j also arrives at the siding before e. Then, by $S^j_s \in [S^e_x, S^{e'}_{x'}]$, trains e and j are active at the point in time S^j_s. Since g passes j, the train j departs from station $\bar{x}$ and, according to Lemma 1, e must pass j which contradicts the choice of j. Suppose that g departs from station $\bar{x}$ and the inequality $S^g_{\bar{x}} < S^e_x$ holds. Since two expresses from different stations cannot be active simultaneously, $C^g_{\bar{x}} < S^e_x \leqslant S^j_s$ which contradicts that train g passes j.

Suppose that g departs after $S^{e'}_{x'}$. If g arrives at the siding before e', then these two trains are from different stations and are active simultaneously at the departure time of g. This, by Lemma 1, contradicts that g is an express. Suppose that g arrives at the siding after e'. If j also arrives at the siding after e', then by $S^j_s \in [S^e_x, S^{e'}_{x'}]$ it departs from the station $\bar{x}'$ and at the point in time $S^{e'}_{x'}$ is active simultaneously with e'. Then, by Lemma 1, e' passes train j which contradicts the selection of j. If j arrives at the siding before e', then g passes it only if e' passes j which contradicts the choice of j. $\square$

3. EXPRESS TYPES AND DEPARTURE TIMES OF TRAINS

All expresses can be partitioned into six types depending on the station from which the express departs, the existence of a non-express which this express passes, and if such a non-express exists, whether or not this train remains in the siding after the express goes through the siding.

Definition 3. The set of all expresses that pass the same non-express will be called a batch.

Definition 4. A pair (s, b), where s specifies the station from which the express departs and b assumes

- 0, if the express goes through an empty siding;
- 1, if the express is part of a batch and is not last in this batch;
- 2, if the express is the last in a batch;

will be called the type of this express.

As will be shown below, the process of schedule construction can be viewed as a sequential process, where each step determines the type of the next express. It is shown below that, without loss of generality, the set of considered schedules can be limited to the schedules where each express departs as early as possible, and the formulae will be given for the calculation of these moments.

Lemma 5. *There exists an optimal schedule in which for any express i of type (s, b), where $b \neq 1$, and express i' of type (s', b'), where $b' \neq 0$, which immediately follows express i and passes some* *non-express g', train g' has the following departure time:*

$$S_{s'}^{g'} = \begin{cases} C_s^i + \beta, & \textit{if } \ s = s' \\ S_s^i + \beta, & \textit{if } \ s \neq s', \ b = 0 \\ S_s^i + 2p_s + \beta, & \textit{if } \ s \neq s', \ b = 2. \end{cases} \tag{3.1}$$

Proof. Suppose that the lemma does not hold for two consecutive expresses i and i' and a non-express g' such that i' passes g'. If $s = s'$, then g' departs from station $\bar{s}$. Since $b \neq 1$, express i does not pass g', and therefore, according to Lemma 1, g' cannot be active simultaneously with i. In other words,

$$[S_{\bar{s}}^{g'}, C_{\bar{s}}^{g'}] \cap [S_s^i, C_s^i] = \emptyset. \tag{3.2}$$

Because i' passes g' and i departs earlier than i', the inequalities $C_{\bar{s}}^{g'} > S_{s'}^{i'} > S_s^i$ hold. Therefore, according to (3.2), $S_{\bar{s}}^{g'} > C_s^i$, and taking into account the safety interval, $S_{\bar{s}}^{g'} \geqslant C_s^i + \beta$.

If there exists a train that causes train g' to depart after the point in time $C_s^i + \beta$, then this train must be active at some point in the time interval $[C_s^i, S_{s'}^{i'} + p_{s'}]$, where $S_{s'}^{i'} + p_{s'}$ is the time of arrival of train i' at the siding. According to Lemma 4 and Lemma 2, in this time interval, there can be active only train g such that i passes g and some train j from station s such that $S_s^j > S_{s'}^{i'}$. Taking into account that i passes g,

$$S_{\bar{s}}^{g'} \geqslant C_s^i + \beta > S_{\bar{s}}^g + \beta.$$

Because g and g' depart from the same station, by Lemma 2, train g cannot cause g' to depart after the point in time $C_s^i + \beta$. Train j cannot do it either, because g' and i' meet in the siding before the arrival of j at the siding.

Assume that $s \neq s'$. Then, g' departs from station s and $S_{s'}^{i'} > C_s^i$ because, by Lemma 1, the expresses i and i' cannot be active simultaneously. If $S_s^{g'} < S_s^i$, then taking into account Lemma 2, $C_s^{g'} < C_s^i < S_{s'}^{i'}$ which contradicts that i' passes g'. Therefore, $S_s^{g'} > S_s^i$ and, taking into account the safety interval, $S_s^{g'} \geqslant S_s^i + \beta$.

It can be shown that if $b = 0$, then $S_s^i + \beta$ is the earliest possible departure time of g'. Indeed, otherwise there exists a train, active at some point in the time interval $[S_s^i, S_{s'}^{i'} + p_{s'}]$, which causes g' to depart later. According to Lemma 4, only train j from station s, satisfying $S_s^j < S_s^i$, or train j' from station s', satisfying $S_{s'}^{j'} > S_{s'}^{i'}$ can be active on this interval. It is easy to see that by Lemma 2, train j cannot cause g' to depart later than $S_s^i + \beta$. Train j' cannot do this either, because g' and i' meet in the siding before j' arrives at the siding.

If $b = 2$, then in contrast to the case when $b = 0$, in the time interval $[S_s^i, S_{s'}^{i'} + p_{s'}]$ there is an active train g from station $\bar{s}$ such that i passes g and g determines the earliest possible departure time of g'. Indeed, since g and g' are non-expresses from different stations, by Lemma 1 they cannot be active simultaneously. Therefore, $C_{\bar{s}}^g < S_{s'}^{g'}$ which taking into account the safety interval and the equality $C_{\bar{s}}^g = S_s^i + 2p_s$, which follows from Lemma 3, implies that the smallest possible departure time of g' is $S_s^i + 2p_s + \beta$.

The above implies that train g' can have a departure time specified by Eq. (3.1) without changing the schedule of the other trains. So, any optimal schedule can be transformed into a schedule that satisfies this lemma. $\square$

Consider express i of type (s, b), and express i' of type (s', b') which immediately follows i. Lemma 4 implies that $S_{s'}^{i'}$ is determined only by express i, non-express g which i passes if such a non-express exists, and non-express g' which i' passes if such a non-express exists. Taking into account Lemma 5, it will be assumed that the departure time of g' is determined by (3.1), and therefore depends only on S_s^i. Furthermore, in the case when $b = 2$, by Lemma 3, express i also determines the time when non-express g leaves the siding. In the case when $b = 1$, non-express g does not affect the departure time of i'. Since this observation holds for every express and the objective function is non-decreasing, it is reasonable that i' departs as early as possible.

As far as the most early departure time of i' is concerned, it is easy to see that if $s \neq s'$, then by virtue of Lemma 1, the inequality $S_{s'}^{i'} \geqslant C_s^i + \beta$ must hold, whereas if $s = s'$, then the inequality $S_{s'}^{i'} \geqslant S_s^i + \beta$ must hold. As far as the restrictions on $S_{s'}^{i'}$ imposed by non-express g are concerned, consider the case $s \neq s'$. Then, g and i' depart from the same station and, because of the existence of the safety interval, $S_{\bar{s}}^g + \beta \leqslant C_s^i$ and $C_s^i + \beta \leqslant S_{s'}^{i'}$. So, in this case, g does not impose any restrictions on the departure time of i'. Assume that $s = s'$. It is easy to see that in this case, if $b = 1$ then g does not impose any restrictions of the departure time of i', and if $b = 2$, then by Lemma 1 and Lemma 3 and the existence of the safety interval

$$S_{s'}^{i'} \geqslant C_{\bar{s}}^g + \beta = S_s^i + 2p_s + \beta.$$

Assume that g' exists. If $g = g'$, then, as it has been mentioned before, this train does not impose any restrictions on $S_{s'}^{i'}$. If $g \neq g'$, assume that g' has the earliest possible departure time, defined by Lemma 5. Since g' should arrive at the siding at least β prior to i', the inequality $S_{s'}^{i'} \geqslant S_{\bar{s}'}^{g'} + p_{\bar{s}'} + \beta - p_{s'}$ holds.

The following lemma summaries the discussion above and show how to calculate the earliest possible departure time for i'.

Lemma 6. *There exists an optimal schedule such that, for any expresses i of type (s, b) and immediately following it i' of type (s', b'), train i' has departure time*

$$S_{s'}^{i'} = \begin{cases} S_s^i + \beta, & \text{if } s = s' \text{ and } b = 1 \\ S_s^i + \beta, & \text{if } s = s' \text{ and } b = b' = 0 \\ \max\{S_s^i + 2p_s + \beta, S_{\bar{s}'}^{g'} + p_{\bar{s}'} + \beta - p_{s'}\}, & \text{if } s = s',\ b = 2,\ b' \neq 0 \\ S_s^i + 2p_s + \beta, & \text{if } s = s',\ b = 2,\ b' = 0 \\ S_{\bar{s}'}^{g'} + p_{\bar{s}'} + \beta - p_{s'}, & \text{if } s = s',\ b = 0,\ b' \neq 0 \\ C_s^i + \beta, & \text{if } s \neq s',\ b' = 0 \\ \max\{C_s^i + \beta, S_{\bar{s}'}^{g'} + p_{\bar{s}'} + \beta - p_{s'}\}, & \text{if } s \neq s',\ b' \neq 0, \end{cases} \tag{3.3}$$

where g' is the non-express that i' passes if such non-express exists, and $S_{\bar{s}'}^{g'}$ is calculated according to Lemma 5.

Let express i of type (s, b) and express i' of type (s', b') be two consecutive expresses. In what follows, there will be considered only schedules where the difference $S_{s'}^{i'} - S_s^i$ is defined by (3.3). Since the difference between their departure times depends only on the types, this difference will be denoted by $h(s, b, s', b')$. Taking into account (3.1) and (3.3),

$$h(s, b, s', b') = \begin{cases} \beta, & \text{if } s = s' \text{ and } b = 1 \\ \beta, & \text{if } s = s' \text{ and } b = b' = 0 \\ \max\{2p_s + \beta, 2(p_{\bar{s}} + \beta)\}, & \text{if } s = s',\ b = 2,\ b' \neq 0 \\ 2p_s + \beta, & \text{if } s = s',\ b = 2,\ b' = 0 \\ 2(p_{\bar{s}} + \beta), & \text{if } s = s',\ b = 0,\ b' \neq 0 \\ p_s + p_{\bar{s}} + \beta, & \text{if } s \neq s',\ b' = 0 \\ p_s + p_{\bar{s}} + \beta, & \text{if } s \neq s',\ b = 0,\ b' \neq 0 \\ \max\{p_s + p_{\bar{s}} + \beta, 3p_s + 2\beta - p_{\bar{s}}\}, & \text{if } s \neq s',\ b = 2,\ b' \neq 0. \end{cases} \tag{3.4}$$

The following four lemmas and the corollary together with Lemmas 3–6 show that the express types determine the schedule.

Lemma 7. *If the last express departs from station s, then it is the last train departing from station s.*

Proof. Let train i from station s be the express with the latest departure time. Suppose that there exists train j from station s such that $S_s^j > S_s^i$. Since j cannot be an express, some express i' from station $\bar{s}$ should pass it. Trains i and i' are both expresses, and according to Lemma 1, they cannot be active simultaneously. This implies $S_{\bar{s}}^{i'} > C_s^i$ which contradicts the choice of i. □

Lemma 8. *Let train i from station s be the express with the latest departure time among all expresses and let train j from station $\bar{s}$ satisfy $C_{\bar{s}}^j > S_s^i$. Then j is the last train from station $\bar{s}$ and i passes j.*

Proof. If $S_{\bar{s}}^j \leqslant S_s^i$, then by Lemma 2, j is a non-express and i passes j. If $S_{\bar{s}}^j > S_s^i$, then j is a non-express by the choice of i. Taking into account that $S_{\bar{s}}^j > S_s^i$ and that i is the last express, i should pass j. Because an express cannot pass two non-expresses, j is the last train from station $\bar{s}$. □

Lemma 9. *The first express always is the first train that departs from the corresponding station.*

Proof. Let train i from station s be the express with the earliest departure time among all expresses. Assume that there exists train j from station s such that $S_s^j < S_s^i$. Because i is the earliest express, train j is a non-express and some train i' from station $\bar{s}$ passes it. By Lemma 1, i and i' cannot be active simultaneously, and by Lemma 2, $C_{\bar{s}}^{i'} < S_s^i$ which contradicts the choice of i. Thus, i is the first train from station s. □

Lemma 10. *Let express i from station s have the earliest departure time among all expresses. If train i' from station s' satisfies the equality $S_{s'}^{i'} \leqslant S_s^i$, then i' is the first train from station $\bar{s}$ and i passes this train.*

Proof. By Lemma 9, $s' = \bar{s}$, and by virtue of the choice of i, train i' can not be an express. Therefore, according to Lemma 2, i should pass this train. Since an express can not pass several non-expresses, i' is the first train from station $\bar{s}$. □

Observe that if the first express passes a train, then this train must arrive at the siding β prior to the arrival there of this express. This observation together with two latter lemmas implies the following corollary.

Corollary. *There exists an optimal schedule in which the first express has the departure time*

$$t = \begin{cases} 0, & \text{if } b = 0 \\ \max\{0, p_{\bar{s}} + \beta - p_s\}, & \text{if } b \neq 0, \end{cases} \tag{3.5}$$

where (s, b) is the type of this express.

4. STATES AT THE DEPARTURE TIMES OF THE EXPRESSES

Let i be an express of type (s, b). The number of trains at station s at the point in time S_s^i, including i, will be denoted by k_s. As far as the station $\bar{s}$ is concerned, consider the set of trains, comprised of the non-express that i passes (if it exists) plus all other train that are at station $\bar{s}$ at the point in time $S_{\bar{s}}^i$. The number of trains in this set will be denoted by $k_{\bar{s}}$. It is easy to see that the 4-tuple (k_1, k_2, s, b) satisfies the following constraints:

(1) $k_s \geqslant 1$ and $k_{\bar{s}} \geqslant 0$;

(2) if $b \neq 0$, then $k_{\bar{s}} \geqslant 1$;

(3) if $b = 1$, then $k_s \geqslant 2$.

Any 4-tuple, satisfying the conditions mentioned above, will be called a state. Thus, the departure time of each express is associated with some state.

Lemma 11. *Let* (k_1, k_2, s, b) *and* (k'_1, k'_2, s', b') *be states, corresponding to two consecutive expresses. Then*

$$k'_s = k_s - 1, \qquad k'_{\bar{s}} = \begin{cases} k_{\bar{s}} - 1, & \text{if } b = 2 \\ k_{\bar{s}}, & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. Assume that the state (k_1, k_2, s, b) corresponds to the departure time of express i and the state (k'_1, k'_2, s', b') corresponds to the departure time of express i'. According to Lemma 4, i' must pass every train which departs from station s between the points in time S^i_s and $S^{i'}_{s'}$. Since an express cannot pass two trains, in the time interval $[S^i_s, S^{i'}_{s'}]$, besides i, only one train can depart from station s. If train that i' passes exists, then it is counted in the calculation of k'_s. So, $k'_s = k_s - 1$.

If $b = 0$, then, at the point in time S^i_s, there are $k_{\bar{s}}$ trains at station $\bar{s}$. By Lemma 4, i' passes each train that departs from station $\bar{s}$ between the points in time S^i_s and $S^{i'}_{s'}$, and therefore, such train is counted in the calculation of $k'_{\bar{s}}$. Hence, in this case $k'_{\bar{s}} = k_{\bar{s}}$. If $b = 1$, then no train can depart from station $\bar{s}$ in the time interval $[S^i_s, S^{i'}_{s'}]$ except the train that is passed by expresses i and i'. Indeed, otherwise, by Lemma 4, either i or i' must pass this train, whereas an express cannot pass two trains. Hence, $k'_{\bar{s}} = k_{\bar{s}}$ when $b = 1$. For $b = 2$, by Lemma 3, the train that i passes leaves the siding at the same time as i and therefore is not counted in the calculation of $k'_{\bar{s}}$. By Lemma 4, i' passes any other train that departs from station $\bar{s}$ between the points in time S^i_s and $S^{i'}_{s'}$, and therefore this train is counted in the calculation of $k'_{\bar{s}}$. Hence, $k'_{\bar{s}} = k_{\bar{s}} - 1$ when $b = 2$. □

5. ORDERED COST FUNCTIONS

Assume that, for each station, there is given a linear order on the set of trains departing from this station such that, for any trains i and j from station s, where j precedes i in this order, and for any points in time $t_1 < t_2$,

$$\varphi^j_s(t_1) \odot \varphi^i_s(t_2) \leqslant \varphi^j_s(t_2) \odot \varphi^i_s(t_1). \tag{5.1}$$

Lemma 12. *There exists an optimal schedule in which the trains depart from each station according to the linear order for this station.*

Proof. Suppose that there is no schedule, satisfying the lemma. Then, for every optimal schedule σ, there exists a pair of trains i and j from the same station s, for which

$$C^i_s(\sigma) < C^j_s(\sigma)$$

and train j precedes i in the linear order for station s. For each schedule σ among all these pairs choose a pair, where $C^i_s(\sigma)$ is minimal, and denote this pairs $i(\sigma)$ and $j(\sigma)$ and the station, from which these trains depart, denote by $s(\sigma)$. Among all optimal schedules select a schedule, for which $C^{i(\sigma)}_{s(\sigma)}(\sigma)$ is maximal. Denote this schedule by η.

The value of the objective function for the schedule η is specified by the expression (1.2), which since $\odot$ is associative and commutative, can be written as

$$\gamma(\eta) = \left(\varphi^{i(\eta)}_{s(\eta)}\left(C^{i(\eta)}_{s(\eta)}(\eta)\right) \odot \varphi^{j(\eta)}_{s(\eta)}\left(C^{j(\eta)}_{s(\eta)}(\eta)\right)\right) \odot \left(\bigodot_{g \notin \{i(\eta), j(\eta)\}} \varphi^g_{s(\eta)}\left(C^g_{s(\eta)}(\eta)\right) \odot \bigodot_{g=1}^{n_{\bar{s}(\eta)}} \varphi^g_{\bar{s}(\eta)}\left(C^g_{\bar{s}(\eta)}(\eta)\right)\right).$$

Consider schedule π, according to which all trains, except $i(\eta)$ and $j(\eta)$, are scheduled as in η. Train $i(\eta)$ proceeds in schedule π as train $j(\eta)$ in schedule η, whereas train $j(\eta)$ proceeds in schedule π as $i(\eta)$ in schedule η. Thus,

$$C^g_{s(\eta)}(\pi) = C^g_{s(\eta)}(\eta), \quad g \in \{1, 2, \ldots, n_{s(\eta)}\} \backslash \{i(\eta), j(\eta)\},$$
$$C^g_{\bar{s}(\eta)}(\pi) = C^g_{\bar{s}(\eta)}(\eta), \quad g \in \{1, 2, \ldots, n_{\bar{s}(\eta)}\},$$
$$C^{i(\eta)}_{s(\eta)}(\pi) = C^{j(\eta)}_{s(\eta)}(\eta), \quad C^{j(\eta)}_{s(\eta)}(\pi) = C^{i(\eta)}_{s(\eta)}(\eta).$$

The value of the objective function, corresponding to the schedule π, is

$$\gamma(\pi) = \left(\varphi^{i(\eta)}_{s(\eta)}(C^{j(\eta)}_{s(\eta)}(\eta)) \odot \varphi^{j(\eta)}_{s(\eta)}(C^{i(\eta)}_{s(\eta)}(\eta))\right) \odot \left(\bigodot_{g \notin \{i(\eta), j(\eta)\}} \varphi^g_{s(\eta)}(C^g_{s(\eta)}(\eta)) \odot \bigodot_{g=1}^{n_{\bar{s}(\eta)}} \varphi^g_{\bar{s}(\eta)}(C^g_{\bar{s}(\eta)}(\eta))\right).$$

By virtue of (5.1) and $C^{i(\eta)}_{s(\eta)}(\eta) < C^{j(\eta)}_{s(\eta)}(\eta)$, the inequality

$$\varphi^{i(\eta)}_{s(\eta)}\left(C^{j(\eta)}_{s(\eta)}(\eta)\right) \odot \varphi^{j(\eta)}_{s(\eta)}\left(C^{i(\eta)}_{s(\eta)}(\eta)\right) \leqslant \varphi^{i(\eta)}_{s(\eta)}\left(C^{i(\eta)}_{s(\eta)}(\eta)\right) \odot \varphi^{j(\eta)}_{s(\eta)}\left(C^{j(\eta)}_{s(\eta)}(\eta)\right)$$

holds which, according to (1.3), implies $\gamma(\pi) \leqslant \gamma(\eta)$.

Consequently, schedule π is optimal. If in this schedule the trains from each station depart in the given linear order, then the lemma is proven. If there exists a pair $i(\pi)$ and $j(\pi)$ and $C^{i(\pi)}_{s(\pi)}(\pi) > C^{i(\eta)}_{s(\eta)}(\eta)$, then this contradicts the choice of η.

It remains to consider the case $C^{i(\pi)}_{s(\pi)}(\pi) = C^{i(\eta)}_{s(\eta)}(\eta)$. In this case, π can be transformed into the schedule α where all trains, except $i(\pi)$ and $j(\pi)$, are scheduled as in π; train $i(\pi)$ proceeds in schedule α as train $j(\pi)$ in schedule π, and train $j(\pi)$ proceeds in schedule α as $i(\pi)$ in schedule π. As above, $\gamma(\alpha) \leqslant \gamma(\pi)$, i.e., schedule α is optimal. The lemma follows from the observation that such transformation is possible only finite number of times because, in all pairs that trigger this transformation, the trains with the smaller departure time are different. □

In what follows, only schedules, in which trains from each station depart according to the given linear order, will be considered. The departure time of each express can be obtained by the sequential application of (3.4), starting from the departure time of the first express given by (3.5). Taking into account (3.4) and (3.5), it can be seen that the departure time of any express can be written as

$$t = m_1 p_1 + m_2 p_2 + m_3 \beta,$$

where m_1, m_2 and m_3 are some integers. According to (3.4) and (3.5), from the departure of one express to the departure of the next one, the coefficients m_1 and m_2 increase at most by 3 and decrease at most by 1, whereas the coefficient m_3 increases at most by 2. Thus, the coefficients m_1 and m_2 do not exceed $3(n_1+n_2)$ and are not less than $-(n_1+n_2)$, whereas the coefficient m_3 does not exceed $2(n_1+n_2)$. Consequently, all possible departure times of the expresses belong to the set

$$\begin{aligned} T = \Big\{ t | \ & t \geqslant 0, t = m_1 p_1 + m_2 p_2 + m_3 \beta, \\ & m_1 \in \{-(n_1+n_2), \ldots, 0, 1, \ldots, 3(n_1+n_2)\}, \\ & m_2 \in \{-(n_1+n_2), \ldots, 0, 1, \ldots, 3(n_1+n_2)\}, m_3 \in \{0, 1, \ldots, 2(n_1+n_2)\} \Big\}. \end{aligned} \tag{5.2}$$

The cardinality of T is $O((n_1+n_2)^3)$.

Number trains at each station in the reverse order of the order of their departure times, i.e., for any trains i and j from the same station s,

$$i < j \quad \text{implies} \quad S^i_s > S^j_s.$$

For any state (k_1, k_2, s, b), let $\Omega(k_1, k_2, s, b)$ be the set of all states (k'_1, k'_2, s', b'), satisfying Lemma 11 and the condition: if $b = 1$, then $s' = s$ and $b' \neq 0$. If $\Omega(k_1, k_2, s, b) = \emptyset$, then state (k_1, k_2, s, b) is called final. According to Lemmas 7, 8 and 11, an express is the last in a schedule if and only if its departure time corresponds to the final state.

Let i be an express that departs at time $t \in T$ associated with state (k_1, k_2, s, b). Then, $i = k_s$. The corresponding schedule induces the sequence of expresses, comprised of the express i and all expresses that depart after express i. This sequence, in turn, induces the sequence of states that correspond to the departure times of these expresses. Thus, the first state in this sequence is (k_1, k_2, s, b), the last is a final state, and for any two consecutive states (k'_1, k'_2, s', b') and (k''_1, k''_2, s'', b''),

$$(k''_1, k''_2, s'', b'') \in \Omega(k'_1, k'_2, s', b'). \tag{5.3}$$

Call feasible any sequence of states, in which any two consecutive states (k'_1, k'_2, s', b') and (k''_1, k''_2, s'', b'') satisfy (5.3) and the last in this sequence is a final state.

It is easy to see that, for any feasible sequence of states $(k^1_1, k^1_2, s^1, b^1), \ldots, (k^a_1, k^a_2, s^a, b^a)$, there exists a schedule such that the train $k^1_{s^1}$ is an express and all subsequent expresses are the trains $k^2_{s^2}, \ldots, k^a_{s^a}$. Observe that there are infinitely many such schedules, including schedules, in which the trains depart as early as possible.

Consider all schedules, in which express i from station s departs at the point in time t associated with state (k_1, k_2, s, b). Let $G(t, k_1, k_2, s, b)$ be the minimal value of

$$\bigodot_{j\in\{1,\ldots,k_1\}} \varphi^j_1\left(C^j_1(\sigma)\right) \odot \bigodot_{g\in\{1,\ldots,k_2\}} \varphi^g_2\left(C^g_2(\sigma)\right) \tag{5.4}$$

among all these schedules σ. As has been mentioned above, since all cost functions are nondecreasing, it suffices to consider only schedules in which trains depart as early as possible. Then, $G(t, k_1, k_2, s, b)$ can be viewed as the minimal value attained on all feasible sequences that start with state (k_1, k_2, s, b).

Assume that a feasible sequence, on which the value $G(t, k_1, k_2, s, b)$ is attained, consists of only one state, which therefore is final. It is easy to see that only states $(1, 0, 1, 0)$, $(0, 1, 2, 0)$, $(1, 1, 1, 2)$, $(1, 1, 2, 2)$ are final states. Thus,

$$G(t, 1, 0, 1, 0) = \varphi^1_1(t + p_1 + p_2), \tag{5.5}$$

$$G(t, 0, 1, 2, 0) = \varphi^1_2(t + p_1 + p_2), \tag{5.6}$$

$$G(t, 1, 1, 1, 2) = \varphi^1_1(t + p_1 + p_2) \odot \varphi^1_2(t + 2p_1), \tag{5.7}$$

$$G(t, 1, 1, 2, 2) = \varphi^1_1(t + 2p_2) \odot \varphi^1_2(t + p_1 + p_2). \tag{5.8}$$

If (k_1, k_2, s, b) is not a final state, then

$$G(t, k_1, k_2, s, b) = \Phi(t, k_1, k_2, s, b) \odot \min_{(k'_1,k'_2,s',b')\in\Omega(k_1,k_2,s,b)} G(t + h(s, b, s', b'), k'_1, k'_2, s', b'), \tag{5.9}$$

where

$$\Phi(t, k_1, k_2, s, b) = \begin{cases} \varphi^{k_s}_s(t + p_1 + p_2) \odot \varphi^{k_{\bar{s}}}_{\bar{s}}(t + 2p_s), & \text{if } b = 2 \\ \varphi^{k_s}_s(t + p_1 + p_2), & \text{otherwise.} \end{cases} \tag{5.10}$$

Denote by $X(n_1, n_2)$ the set of all feasible pairs (s, b) for n_1 and n_2. Then, taking into account Lemma 9 and Corollary, the optimal value of the objective function is

$$\gamma^* = \min_{(s,b)\in X(n_1,n_2)} G(S^{n_s}_s, n_1, n_2, s, b),$$

where $S_s^{n_s}$ is calculated using (3.5). This value can be obtained by dynamic programming, using (5.5)–(5.8) and the Bellman equation (5.9).

Observe that if $n_1 = 0$ or $n_2 = 0$, then the problem is trivial. Indeed, in this case all trains are expresses that depart in the optimal schedule every β time units. In the case when $n_1 \neq 0$ and $n_2 \neq 0$, the computational complexity of the proposed algorithm is $O(n_1 n_2 (n_1 + n_2)^3)$, where $n_1 n_2$ reflects the cardinality of the set of all states, whereas $(n_1 + n_2)^3$ reflects the cardinality of T.

6. A MODIFIED ALGORITHM FOR TWO IMPORTANT OBJECTIVE FUNCTIONS

It will be shown that in the case of some frequently used in practice objective functions the computational complexity of the algorithm can be substantially reduced. Consider the set of all schedules in which express i from station s departs at time t associated with state (k_1, k_2, s, b). Each such schedule determines a feasible sequence of states that starts with (k_1, k_2, s, b). According to Lemmas 3, 5 and 6, the point in time t and this feasible sequence give the arrival times of each train from the set $\{1, 2, \ldots, k_1\}$ from station 1 and from the set $\{1, 2, \ldots, k_2\}$ from station 2. Therefore, for any train j that departs from station s' and belongs to one of these sets,

$$C_{s'}^{j} = t + K(j, s', l), \tag{6.1}$$

where $K(j, s', l)$ is determined by the corresponding feasible sequence of states l. Denote by $\mathcal{L}(k_1, k_2, s, b)$ the set of all feasible sequences in which (k_1, k_2, s, b) is the first state.

6.1. Maximum Lateness

Consider the maximum lateness problem

$$L_{\max}(\sigma) = \max_{i \in N_s,\ s \in \{1,\,2\}} \{C_s^i(\sigma) - d_s^i\}, \tag{6.2}$$

where d_s^i is the time by which it is desired for train i from station s to arrive at its destination. In the scheduling literature this time is commonly referred to as the due date. It is easy to see that, in this case, the order of departures from each station is specified by the due dates, i.e., for any trains i and j from the same station s, the inequality $d_s^i < d_s^j$ implies the inequality $S_s^i < S_s^j$.

In this case, the Bellman equation (5.9) is

$$G(t, k_1, k_2, s, b) = \max \Bigg\{ \Phi(t, k_1, k_2, s, b),$$

$$\min_{(k_1', k_2', s', b') \in \Omega(k_1, k_2, s, b)} \{G(t + h(s, b, s', b'), k_1', k_2', s', b')\} \Bigg\}, \tag{6.3}$$

where

$$\Phi(t, k_1, k_2, s, b) = \begin{cases} \max\left\{t + p_1 + p_2 - d_s^{k_s}, t + 2p_s - d_{\bar{s}}^{k_{\bar{s}}}\right\}, & \text{if } b = 2 \\ t + p_1 + p_2 - d_s^{k_s}, & \text{otherwise.} \end{cases} \tag{6.4}$$

By virtue of (6.1), the value $G(t, k_1, k_2, s, b) - t$ depends only on the feasible sequence of states. Denote this difference by $F(k_1, k_2, s, b)$ and introduce the notation $L(k_1, k_2, s, b) = \Phi(t, k_1, k_2, s, b) - t$, i.e.,

$$L(k_1, k_2, s, b) = \begin{cases} \max\left\{p_1 + p_2 - d_s^{k_s}, 2p_s - d_{\bar{s}}^{k_{\bar{s}}}\right\}, & \text{if } b = 2 \\ p_1 + p_2 - d_s^{k_s}, & \text{otherwise.} \end{cases}$$

According to (5.5)–(5.8), the values of F for the final states are

$$F(1,0,1,0) = p_1 + p_2 - d_1^1, \tag{6.5}$$

$$F(0,1,2,0) = p_1 + p_2 - d_2^1, \tag{6.6}$$

$$F(1,1,1,2) = \max\left\{p_1 + p_2 - d_1^1, 2p_1 - d_2^1\right\}, \tag{6.7}$$

$$F(1,1,2,2) = \max\left\{2p_2 - d_1^1, p_1 + p_2 - d_2^1\right\}. \tag{6.8}$$

Subtracting t from both sides of the Eq. (6.3),

$$F(k_1,k_2,s,b) = G(t,k_1,k_2,s,b) - t$$

$$= \max\left\{\Phi(t,k_1,k_2,s,b) - t, \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} G(t + h(s,b,s',b'), k_1', k_2', s', b') - t\right\}$$

$$= \max\Bigg\{L(k_1,k_2,s,b), \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} \min_{l\in\mathcal{L}(k_1',k_2',s',b')} \max\Bigg\{\max_{j\in\{1,\dots,k_1'\}}\left\{t + h(s,b,s',b') + K(j,1,l) - d_1^j - t\right\}, \max_{g\in\{1,\dots,k_2'\}}\left\{t + h(s,b,s',b') + K(g,2,l) - d_2^g - t\right\}\Bigg\}\Bigg\}$$

$$= \max\Bigg\{L(k_1,k_2,s,b), \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} \min_{l\in\mathcal{L}(k_1',k_2',s',b')} \max\Bigg\{\max_{j\in\{1,\dots,k_1'\}}\left\{K(j,1,l) - d_1^j\right\}, \max_{g\in\{1,\dots,k_2'\}}\left\{K(g,2,l) - d_2^g\right\}\Bigg\} + h(s,b,s',b')\Bigg\}$$

$$= \max\left\{L(k_1,k_2,s,b), \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)}\left\{G(0,k_1',k_2',s',b') + h(s,b,s',b')\right\}\right\}$$

$$= \max\left\{L(k_1,k_2,s,b), \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)}\left\{F(k_1',k_2',s',b') + h(s,b,s',b')\right\}\right\}.$$

As in above, Lemma 9 and Corollary lead to the following optimal value of the objective function

$$\min_{(s,b)\in X(n_1,n_2)}\left\{F(n_1,n_2,s,b) + S_s^{n_s}\right\}, \tag{6.9}$$

where $S_s^{n_s}$ is calculated using (3.5). This value can be obtained by dynamic programming, using (6.5)–(6.8) and the obtained above Bellman equation

$$F(k_1,k_2,s,b) = \max\left\{L(k_1,k_2,s,b), \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)}\left\{F(k_1',k_2',s',b') + h(s,b,s',b')\right\}\right\}.$$

As before, if $n_1 = 0$ or $n_2 = 0$, then the problem becomes trivial and in the optimal schedule all trains are expresses that depart from the same station every β time units. In the case when $n_1 \neq 0$

and $n_2 \neq 0$, since the set $\Omega(k_1, k_2, s, b)$ contains not more than six elements, the computational complexity is determined by the number of states which implies $O(n_1 n_2)$.

6.2. Weighted Sum of Arrival Times

The algorithm below allows to minimise the weighted sum of arrival times

$$\sum_{i\in N_s,\ s\in\{1,\,2\}} w_s^i C_s^i(\sigma), \tag{6.10}$$

where w_s^i is the weight (priority) of train i from station s. If, in some schedule, for two trains i and j from the same station s, $w_s^i < w_s^j$ and $S_s^i < S_s^j$, then train j can depart instead of train i and train i can depart instead of train j without changing the rest of the schedule. As a result, the value of (6.10) will decrease. Therefore, in what follows, will be considered only schedules where, for any two trains i and j from the same station s, the inequality $w_s^i < w_s^j$ implies $S_s^i > S_s^j$.

As before, consider express i from station s that departs at time t associated with state (k_1, k_2, s, b). Instead of function $F(k_1, k_2, s, b)$, introduced in the previous section, consider

$$H(k_1, k_2, s, b) = G(t, k_1, k_2, s, b) - t\left(\sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2} w_2^g\right).$$

Then, according to (5.5)–(5.8),

$$H(1,0,1,0) = (p_1 + p_2)w_1^1, \tag{6.11}$$
$$H(0,1,2,0) = (p_1 + p_2)w_2^1, \tag{6.12}$$
$$H(1,1,2,2) = 2p_2 w_1^1 + (p_2 + p_1)w_2^1, \tag{6.13}$$
$$H(1,1,1,2) = 2p_1 w_2^1 + (p_2 + p_1)w_1^1. \tag{6.14}$$

Taking into account (5.9), (6.1) and that $\odot$ is addition,

$$\begin{aligned}
H(k_1, k_2, s, b) &= G(t, k_1, k_2, s, b) - t\left(\sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2} w_2^g\right)\\
&= \Phi(t, k_1, k_2, s, b) + \min_{(k_1', k_2', s', b')\in\Omega(k_1,k_2,s,b)} G(t + h(s,b,s',b'), k_1', k_2', s', b') - t\left(\sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2} w_2^g\right)\\
&= \Phi(t, k_1, k_2, s, b) + \min_{(k_1', k_2', s', b')\in\Omega(k_1,k_2,s,b)}\ \min_{l\in\mathcal{L}(k_1', k_2', s', b')} \left\{\sum_{j=1}^{k_1'} w_1^j(t + h(s,b,s',b') + K(j,1,l))\right.\\
&\qquad \left. + \sum_{g=1}^{k_2'} w_2^g(t + h(s,b,s',b') + K(g,2,l))\right\} - t\left(\sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2} w_2^g\right)\\
&= \Phi(t, k_1, k_2, s, b) + \min_{(k_1', k_2', s', b')\in\Omega(k_1,k_2,s,b)}\ \min_{l\in\mathcal{L}(k_1', k_2', s', b')} \left\{\sum_{j=1}^{k_1'} w_1^j K(j,1,l)\right.\\
&\quad \left. + \sum_{g=1}^{k_2'} w_2^g K(g,2,l) + \left(\sum_{j=1}^{k_1'} w_1^j + \sum_{g=1}^{k_2'} w_2^g\right) h(s,b,s',b') + t\left(\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g\right)\right\}.
\end{aligned}$$

By virtue of Lemma 11,

$$\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g = \begin{cases} -w_s^{k_s} - w_{\bar{s}}^{k_{\bar{s}}}, & \text{if } b = 2 \\ -w_s^{k_s}, & \text{otherwise.} \end{cases} \tag{6.15}$$

Consequently,

$$t\left(\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g\right)$$

does not depend on the state (k_1', k_2', s', b'). Therefore

$$\begin{aligned} H(k_1, k_2, s, b) &= \Phi(t, k_1, k_2, s, b) + t\left(\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g\right) \\ &+ \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} \left\{ \min_{l\in\mathcal{L}(k_1',k_2',s',b')} \left\{ \sum_{j=1}^{k_1'} w_1^j K(j,1,l) + \sum_{g=1}^{k_2'} w_2^g K(g,2,l) \right\} \right. \\ &\left. + \left(\sum_{j=1}^{k_1'} w_1^j + \sum_{g=1}^{k_2'} w_2^g\right) h(s,b,s',b') \right\} \\ &= \Phi(t, k_1, k_2, s, b) + t\left(\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g\right) \\ &+ \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} \left\{ G(0, k_1', k_2', s', b') + \left(\sum_{j=1}^{k_1'} w_1^j + \sum_{g=1}^{k_2'} w_2^g\right) h(s,b,s',b') \right\}. \end{aligned}$$

Then, according to (6.15) and because, by (5.10),

$$\Phi(t, k_1, k_2, s, b) = \begin{cases} w_s^{k_s}(t + p_1 + p_2) + (t + 2p_s) w_{\bar{s}}^{k_{\bar{s}}}, & \text{if } b = 2 \\ w_s^{k_s}(t + p_1 + p_2), & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \Phi(t, k_1, k_2, s, b) &+ t\left(\sum_{j=1}^{k_1'} w_1^j - \sum_{j=1}^{k_1} w_1^j + \sum_{g=1}^{k_2'} w_2^g - \sum_{g=1}^{k_2} w_2^g\right) \\ &= \begin{cases} w_s^{k_s}(p_1 + p_2) + 2p_s w_{\bar{s}}^{k_{\bar{s}}}, & \text{if } b = 2 \\ w_s^{k_s}(p_1 + p_2), & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} H(k_1, k_2, s, b) &= \Psi(k_1, k_2, s, b) \\ &+ \min_{(k_1',k_2',s',b')\in\Omega(k_1,k_2,s,b)} \left\{ H(k_1', k_2', s', b') + \left(\sum_{j=1}^{k_1'} w_1^j + \sum_{g=1}^{k_2'} w_2^g\right) h(s,b,s',b') \right\}, \end{aligned} \tag{6.16}$$

where

$$\Psi(k_1, k_2, s, b) = \begin{cases} w_s^{k_s}(p_1 + p_2) + 2p_s w_{\bar{s}}^{k_{\bar{s}}}, & \text{if } b = 2 \\ w_s^{k_s}(p_1 + p_2), & \text{otherwise.} \end{cases}$$

By Lemma 9 and Corollary, the optimal value of the objective function is

$$\min_{(s,b)\in X(n_1,n_2)} \left\{ H(n_1,n_2,s,b) + \left(\sum_{j=1}^{n_1} w_1^j + \sum_{g=1}^{n_2} w_2^g \right) S_s^{n_s} \right\},$$

where $S_s^{n_s}$ is calculated using (3.5). This value can be found by dynamic programming using (6.11)–(6.14) and the Bellman equation (6.16). As in the case of $L_{\max}$, the computational complexity of the proposed algorithm is $O(n_1 n_2)$.

7. WEIGHTED NUMBER OF LATE TRAINS

Consider minimisation of the weighted number of late trains

$$\sum_{i\in N_s,\, s\in\{1,\,2\}} w_s^i U_s^i(\sigma), \tag{7.1}$$

where

$$U_s^i(\sigma) = \begin{cases} 0, & \text{if } C_s^i(\sigma) \leqslant d_s^i \\ 1, & \text{if } C_s^i(\sigma) > d_s^i, \end{cases} \tag{7.2}$$

and the due date d_s^i, as before, is the time by which it is desired for train i from station s to arrive at its destination. Contrary to the previous sections, in this case, the cost functions are not ordered. Moreover, each train can be an express, or a non-express, which some express passes at the siding, or a train that is late. By virtue of the objective function, all late trains can depart in any order after the arrival of all trains that are on time. Therefore, in what follows, only the choice of trains that are on time and their scheduling are considered.

Each train i from station s that is on time arrives by its due date d_s^i. Taking into account Subsection 6.1, it will be assumed that all trains from the same station that are on time depart in a nondecreasing order of their due dates. As in Subsection 6.1, it will be assumed that all trains from each station are numbered in a nonincreasing order of their due dates.

Let express i depart from station s at the point in time t. As before, the departure time of this express will be associated with a state (k_1, k_2, s, b), where k_s is the number of trains at station s at the departure time of express i, and $k_{\bar{s}}$ is the number of trains g from station $\bar{s}$ that satisfy at least one of the following two conditions:

- the express i passes train g;
- at the point in time t, train g is situated at station $\bar{s}$.

Express i and the train that i passes (if such train exists) meet their due dates, therefore taking into account that $i = k_s$,

$$t + p_1 + p_2 \leqslant d_s^{k_s}. \tag{7.3}$$

In the case $b = 2$, the inequality

$$t + 2p_s \leqslant d_{\bar{s}}^{k_{\bar{s}}} \tag{7.4}$$

holds. Observe that some trains from station s that belong to the set $\{1, \ldots, k_s - 1\}$ can be late. Therefore, when $b = 1$, the next express will be $j \in \{1, \ldots, k_s - 1\}$, which is not necessarily the train $k_s - 1$, and

$$t + \beta + p_1 + p_2 \leqslant d_s^j, \tag{7.5}$$

$$t + \beta + 2p_s \leqslant d_{\bar{s}}^{k_{\bar{s}}}. \tag{7.6}$$

Assume that after i the next express is i' and its departure time is associated with state (k_1', k_2', s', b'). By Lemma 11 and because some trains that depart after the point in time t can be late,

$$k_s' \leqslant k_s - 1; \tag{7.7}$$

$$k_{\bar{s}}' \leqslant k_{\bar{s}} - 1, \text{ if } b = 2; \quad k_{\bar{s}}' = k_{\bar{s}}, \text{ if } b = 1; \quad \text{and} \quad k_{\bar{s}}' \leqslant k_{\bar{s}}, \text{ if } b = 0. \tag{7.8}$$

Analogously to (7.3),

$$t + h(s, b, s', b') + p_1 + p_2 \leqslant d_{s'}^{k_{s'}}, \tag{7.9}$$

and in the case $b' = 2$,

$$t + h(s, b, s', b') + 2p_{s'} \leqslant d_{\bar{s}'}^{k_{\bar{s}'}}, \tag{7.10}$$

furthermore, in the case when $b' = 1$ and the next express is $j' \in \{1, \dots, i' - 1\}$,

$$t + h(s, b, s', b') + \beta + p_1 + p_2 \leqslant d_{s'}^{j'}, \tag{7.11}$$

$$t + h(s, b, s', b') + \beta + 2p_{s'} \leqslant d_{\bar{s}'}^{k_{\bar{s}'}}. \tag{7.12}$$

The set of all states (k_1', k_2', s', b'), satisfying (7.7)–(7.12), will be denoted by $\Psi(t, k_1, k_2, s, b)$. In other words, if there exists an express that is next after i, then its departure time is associated with a state in $\Psi(t, k_1, k_2, s, b)$, and every state in this set specifies the next possible express after i.

Consider all schedules, according to which train k_s from station s is an express that has the departure time t associated with the state (k_1, k_2, s, b). Let $G(t, k_1, k_2, s, b)$ be the minimal possible value of

$$\sum_{j \in \{1, \dots, k_1\}} w_1^j U_1^j(\sigma) + \sum_{g \in \{1, \dots, k_2\}} w_2^g U_2^g(\sigma) \tag{7.13}$$

on the set of all these schedules σ. Observe that if $\Psi(t, k_1, k_2, s, b) = \emptyset$, then $b \neq 1$. If $\Psi(t, k_1, k_2, s, b) = \emptyset$, then by Lemmas 7 and 8, k_s is the last on time train from station s, and if there exists an on time train from station $\bar{s}$ with the departure time greater than t, then it is $k_{\bar{s}}$ and k_s passes non-express $k_{\bar{s}}$. Therefore, for every $t \in T$, where T is defined in (5.2), and every state (k_1, k_2, s, b) such that $\Psi(t, k_1, k_2, s, b) = \emptyset$,

$$G(t, k_1, k_2, s, b) = \begin{cases} \sum\limits_{i < k_s} w_s^i + \sum\limits_{i \leqslant k_{\bar{s}}} w_{\bar{s}}^i, & \text{if } b = 0 \\ \sum\limits_{i < k_s} w_s^i + \sum\limits_{i < k_{\bar{s}}} w_{\bar{s}}^i, & \text{if } b = 2. \end{cases} \tag{7.14}$$

If, for some (k_1, k_2, s, b) and $t \in T$ such that (7.3)–(7.6) hold, $\Psi(t, k_1, k_2, s, b) \neq \emptyset$, then

$$G(t, k_1, k_2, s, b) = \min_{(k_1', k_2', s', b') \in \Psi(t, k_1, k_2, s, b)} \Big\{ G(t + h(s, b, s', b'), k_1', k_2', s', b') + W(k_1, k_2, s, b, k_1', k_2') \Big\}, \tag{7.15}$$

where

$$W(k_1, k_2, s, b, k_1', k_2') = \begin{cases} \sum\limits_{k_s' < i < k_s} w_s^i + \sum\limits_{k_{\bar{s}}' < i \leqslant k_{\bar{s}}} w_{\bar{s}}^i, & \text{if } b \neq 2 \\ \sum\limits_{k_s' < i < k_s} w_s^i + \sum\limits_{k_{\bar{s}}' < i < k_{\bar{s}}} w_{\bar{s}}^i, & \text{if } b = 2. \end{cases} \tag{7.16}$$

Let D be the set of all states (k_1, k_2, s, b) that satisfy (7.3)–(7.6) for $t = S_s^{k_s}$ calculated using (3.5). Then the optimal value of the objective function

$$\min_{(k_1,k_2,s,b)\in D} \left\{ G(S_s^{k_s}, k_1, k_2, s, b) + \sum_{i>k_s} w_s^i + \sum_{i>k_{\bar{s}}} w_{\bar{s}}^i \right\}$$

can be calculated by dynamic programming, using (7.14)–(7.15).

In the case when $n_1 \neq 0$ and $n_2 \neq 0$, taking into account the cardinality of T and the cardinality of the set of all states, the computational complexity of this algorithm is $O(n_1^2 n_2^2 (n_1 + n_2)^3)$. If $n_1 = 0$ or $n_2 = 0$, an optimal schedule can be constructed using a more efficient algorithm. In order to describe the idea of this algorithm, without loss of generality, assume that $n_1 \neq 0$ and number all trains in a nondecreasing order of the due dates. According to the algorithm, the schedule is constructed where the trains depart in the increasing order of their numbers, i.e., in a nondecreasing order of the due dates. Thus, the first train has the departure time $t = 0$, the second departs at $t = \beta$, etc. Let i be the train with the smallest number among the trains that do not meet their due dates. Among all trains j such that $j \leqslant i$ choose train j' with the minimal w_1^j. This train will be late and is excluded from the set of considered trains. Then, the process repeats for the new set of trains.

8. CONCLUSION

The algorithms, presented above, were developed for the case of two stations and a single siding that can accommodate only one train. The directions of further research may include cases with several sidings, sidings with different capacity, and a more complex structure of the railway network. Another direction of research is problems with restrictions on the departure times.

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