EMBEDDINGS FOR THE SPACE LD^2 ON SETS OF FINITE PERIMETER

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Abstract. In the article we consider the space $LD^p_\gamma(\Omega)$, being the space of L^p-bounded deformations in an open set Ω of finite perimeter and having the L^p-integrability of the boundary values. We demonstrate the embedding result $L^{pN/(N-1)}(\Omega) \subset LD^p_\gamma(\Omega)$. The necessity of these type of embeddings appears in the theory, describing the motion of rigid bodies in a viscous fluid.

1. Presentation of the problem

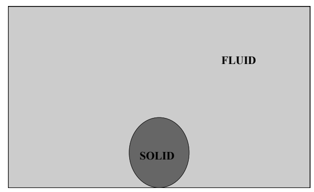
The problem of motion of rigid bodies in a viscous fluid, filling a bounded domain, was studied by many authors Hoffmann, Starovoitov [21], San Martín, Starovoitov, Tucsnak [30], Feireisl, Hillairet, Nečasová [13], Bost, Cottet, Maitre [8], Gunzburger, Lee, Seregin [16], Takahashi [32], Judakov [39] and etc.. The authors of these works considered no-slip conditions on the boundaries of the rigid bodies and the domain. Hesla [19], Hillairet [20], Starovoitov [31] have shown that such modelling gives a paradoxical mathematical result of no collisions of the rigid bodies and no collisions of them with the boundary of the domain.

One of possibilities to include collisions is to consider the slippage on the boundaries. The slippage is prescribed by Navier boundary conditions. Firstly the slippage have been considered by Neustupa, Penel [27], [28]. They have investigated a prescribed collision of a ball with a wall, when the slippage is allowed on the boundaries of the ball and of the wall. The case of the motion of a single body, moved in the whole space $\mathbb{R}^3$, has been studied in [29]. Recently Gérard-Varet, Hillairet [14] have proved a local-in-time existence result: up to collisions. In [15] it has been shown that a rigid ball touches the boundary of the wall in a finite period of time in the case of Navier boundary conditions on the boundaries of the ball and the wall. In the article [9] the Navier condition on the boundary of the body and the non-slip condition on the boundary of the domain has been considered and shown the global-in-time solvability result of the weak solution.

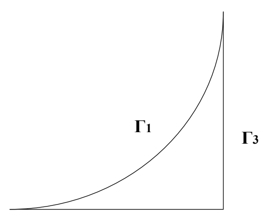
One of the main obstacles to study, in a general situation, the motion of many rigid bodies, which collide, is the absence of embedding results for the

Date: February 14, 2018.

space of L^2 integrable bounded deformations in domains with bad regularity of boundary.



(a) Body touching the boundary



(b) Cuspidal subregion of interest generated after touching

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let us consider a vector-function $\mathbf{v} : \mathbf{x} \in \Omega \to \mathbb{R}^N$, and define the tensor of deformation $\mathbb{D}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ with the components

$$d_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad i, j = 1, ...N.$$

Definition 1.1. *We define the space of functions of $L^p(\Omega)$-bounded deformation with $p \geq 1$ as*

$$LD^p(\Omega) = \{\mathbf{v} \in L^p(\Omega) : \mathbb{D}\mathbf{v} \in L^p(\Omega)\},$$

endowed by the norm $||\mathbf{v}||_{LD^p(\Omega)} = ||\mathbf{v}||_{L^p(\Omega)} + ||\mathbb{D}\mathbf{v}||_{L^p(\Omega)}$.

Since at the moment of collision of the rigid body with the boundary of the domain and/or of collision of two bodies, the fluid will occupy a domain with cusps. In the following considerations we will be interested in embedding results involving the space $LD^2(\Omega)$ for Ω with the cusps. The reason is that under the study of the the motion of colliding rigid bodies it appears the important question of the validation of the convective term

$$\int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbb{D}\boldsymbol{\psi}\, d\mathbf{x} \qquad \text{for the test function} \qquad \boldsymbol{\psi} \in LD^2(\Omega) \tag{1}$$

in cuspidal domains Ω (see the definition 2.1 in [9]). This question could be solved if we will demonstrate that the solution $\mathbf{u} \in LD^2(\Omega)$ is integrable at least in $L^4(\Omega)$.

Let us do a short description of existing embedding results for cuspidal domains. There are well known embedding results [1], [18], [26], involving the Sobolev space $W_2^1(\Omega)$, for cuspidal domains Ω. The methods, applied in these works, can not be applied for $LD^2(\Omega)$, since the technique, used in these works, destroy the norm of our space $LD^2(\Omega)$. The optimal embedding theorem $W_2^1(V(x^\alpha)) \hookrightarrow L^r(V(x^\alpha))$ for $r \in [1, \frac{2(\alpha+1)}{\alpha-1}]$ in the cuspidal domain

$$V(x^\alpha) = \{\mathbf{x} = (x, y) : \ 0 < x < 1, \quad 0 < y < x^\alpha\} \subset \mathbb{R}^2$$

was obtained in the article [25]. The embedding result $W_2^1(V(x^\alpha)) \hookrightarrow L^q(\partial V(x^\alpha))$, with $1 \leqslant q \leqslant 2$ for optimal values of α : $\alpha < 1 + \frac{2}{q}$, was shown in [2]. For a more complete description of optimal embedding results in cuspidal domains, we refer to [7], [22], [24], [26] and [37].

Now let us study the following example.

Example 1.2. *Let us consider the cuspidal domain $V(x^2)$. This type of cuspidal domains appears at the moment of touching of a ball moving in a 2D-fluid with a plane wall. We take the vector function*

$$\mathbf{w} = ((s-1)yx^{-s}, x^{1-s}) \tag{2}$$

with a real parameter s that will be chosen later on. Following the calculations of [3]*, p. 219-221, we have*

$$\mathbb{D}\mathbf{w} = \begin{bmatrix} -s(s-1)yx^{-s-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} ||\mathbf{w}||^2_{L^q(V(x^2))} &\leq C \int_0^1 \left(\int_0^{x^2} (y^p x^{-ps} + x^{-p(s-1)}) dy \right) dx \leq C \int_0^1 x^{-p(s-1)+2}) dx, \\ ||\mathbb{D}\mathbf{w}||^2_{L^2(V(x^2))} &\leq C \int_0^1 x^{6-2(s+1)} dx. \end{aligned}$$

Considering $q = 2$, we obtain that

$$\mathbf{w} \in LD^2(V(x^2)) \qquad \textit{for any} \quad s < 1 + \frac{3}{2}.$$

Moreover if we consider $q = 2 + \varepsilon$ we conclude

$$\mathbf{w} \notin L^{2+\varepsilon}(V(x^2)) \qquad \textit{for} \quad s = 1 + \frac{3}{2+\varepsilon}.$$

Hence having LD^2-integrability of functions in the cuspidal domains we can not expect the L^q-integrability for $q > 2$ and can not solve positively the above posed question of the validation of the convective term (1).

In what follows we will discuss the behaviour of boundary values of LD^2-functions. To do it let us present some notations, definitions and results given in [4], [23], [36] and [35] which we will use. Let us denote the Lebesgue measure as $\mathcal{L}^d$ in $\mathbb{R}^d$ and the d-dimensional Hausdorff measure as $\mathcal{H}^d$ in $\mathbb{R}^N$ for $d \leq N$.

Definition 1.3. *Let $E \subset \mathbb{R}^N$ be a bounded $\mathcal{L}^N-$measurable set. We denote the characteristic function of the set E by χ_E. If $\chi_E \in BV(\mathbb{R}^N)$, then E is called set with finite perimeter. It means that the generalized gradient $\nabla\chi_E = (\mu_1, ..., \mu_N) = \mu$ is the vector of bounded Radon measures μ_i, $i = 1, ...N$, satisfying*

$$\int_E div\, \phi\, d\mathbf{x} = -\int_\Omega (\phi, d\mu), \qquad \textit{for any } \phi = (\phi_1, ..., \phi_N) \in C_c^1(\mathbb{R}^N)$$

and the number, called the perimeter of E,

$$P(E) = |\nabla\chi_E|(\mathbb{R}^N) = \sup_\phi \left\{ \int_E div\, \phi\, d\mathbf{x} : \quad |\phi| \leq 1, \quad \phi \in C_c^1(\mathbb{R}^N) \right\}$$

is finite.

The following results are presented on the pages 154-156 of [35] and on the page 159, Proposition 3.62 of [4].

Proposition 1.4. *1) The set of all sets with finite perimeters forms algebra, that is if E, F have finite perimeters then the sets $\mathbb{R}^N \backslash E$, $E \cup F$, $E \cap F$ also have finite perimeters;*

2) If the set E is an open set, that has Lipschitz boundary, then E is a set with finite perimeter and $P(E) = \mathcal{H}^{N-1}(\partial E)$.

Remark 1.5. *Let us consider the motion of few rigid bodies inside of a bounded domain, occupied by a fluid. If we assume that the rigid bodies and the domain have Lipschitz boundaries, then by Proposition 1.4 we conclude that the fluid in the domain occupies an open set with finite perimeter (See Figure 1(c)). By this reason we will be interested to derive an embedding result $LD^2(\Omega) \hookrightarrow L^q(\Omega)$ for arbitrary set Ω with finite perimeter (with some additional information).*

Let us introduce the concept of essential boundary (see the pages 256, 258 of [35] and on the page 158 of [5]). Let $\omega_N(\rho) = \mathcal{L}^N(B_\rho(\mathbf{x}))$ be the volume of the ball $B_\rho(\mathbf{x})$ with the radius $\rho > 0$ and the center $\mathbf{x} \in \mathbb{R}^N$. We define the unit sphere

$$\mathbb{S}^{N-1} = \{\mathbf{a} \in \mathbb{R}^N : \quad |\mathbf{a}| = 1\}$$

(c) Domain with finite perimeter

in $\mathbb{R}^N$ and the hyperplane

$$P_{\mathbf{a}} = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} \cdot \mathbf{a} = 0\}$$

orthogonal to $\mathbf{a}$, crossing the zero point of $\mathbb{R}^N$.

Definition 1.6. *Let E be a given subset of $\mathbb{R}^N$. A point $\mathbf{x} \in$ is point of density (rarefaction) of the set E if*

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(\mathbf{x}))}{\omega_N(\rho)} = 1 \ (0).$$

We denote by E^* the set of all points of density of E and E_* the complement of the set of points of rarefaction of E. The set $\partial^* E = E^* \backslash E_*$ is called the essential boundary of the set E.

Let us recall some facts about sets with finite perimeter and its essential boundary. For more details we refer the reader to the works [12], [5], [11], [17], [38] and [36].

Proposition 1.7. *Let E be a set with finite perimeter and let $\partial^* E$ be its essential boundary. Then:*

1) (the page 205 of [11]*) The boundary $\partial^* E$ is countably $\mathcal{H}^{N-1}$-rectifiable, that is*

$$\partial^* E = \cup_{n=1}^{\infty} K_n \cup S, \qquad \textit{where} \quad \mathcal{H}^{N-1}(S) = 0$$

and K_n is a compact subset of a C^1 hypersurface in $\mathbb{R}^N$, that is

$$K_n = \Phi_n(A_n), \qquad \textit{where} \quad \Phi_n \in C^1, \quad A_n \subset \mathbb{R}^{N-1} \textit{ is compact}$$

and the sets K_n are disjoint pairs.

2) (the page 205 of [11]*, the pages 227-228 of* [36] *and the pages 154, 158 of* [5]*) the unit normal $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x})$ exists for $\mathcal{H}^{N-1}-$a.a. points $\mathbf{x} \in \partial^* E$.*

3) (the page 233 of [36]*) For a given $\mathbf{a} \in \mathbb{S}^{N-1}$ let $l_{\mathbf{a}}(\mathbf{x})$ be the line parallel to the vector $\mathbf{a}$ and crossing through $\mathbf{x} \in P_{\mathbf{a}}$. Then for $\mathcal{L}^{N-1}$-a.a. $\mathbf{x} \in P_{\mathbf{a}}$ the set $l_{\mathbf{a}}(\mathbf{x}) \cap E_*$ is an union of finite number of open intervals with disjoint closures, and the union of the boundary points of the intervals coincides with the set $l_{\mathbf{a}}(\mathbf{x}) \cap \partial^* E$.*

Since the space $LD^p(\Omega)$, $p \geq 1$, is a subspace of the space of bounded deformation $BD(\Omega)$, we can apply the result of [34] (see also [33]), that the trace of functions in $LD^p(\Omega)$ is well defined. The same result was also described carrefully in Proposition 4.1. of [6].

Proposition 1.8. *Let Ω be a set with finite perimeter and let $\partial^*\Omega$ be its essential boundary. If $\mathbf{u}(\mathbf{x}) \in LD^p(\Omega)$, then for $\mathcal{H}^{N-1}-$a.e. $\mathbf{x} \in \partial^*\Omega$ there exist a vector function $\gamma\mathbf{u}(\mathbf{x}) \in \mathbb{R}^N$, such that*

$$\lim_{\rho\to 0^+} \frac{2}{\omega_N(\rho)} \int_{B_\rho(\mathbf{x},\boldsymbol{\nu})} |\mathbf{u}(\mathbf{y}) - \gamma\mathbf{u}(\mathbf{x})|\, d\mathbf{y} = 0 \tag{3}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x}) \in \mathbb{S}^{N-1}$ is the internal normal at $\mathbf{x} \in \partial^*\Omega$ and the half ball $B_\rho(\mathbf{x},\boldsymbol{\nu})$ is defined as

$$B_\rho(\mathbf{x},\boldsymbol{\nu}) = \{\mathbf{y} \in \mathbb{R}^N : \quad |\mathbf{y}-\mathbf{x}| < \rho, \quad (\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\nu} > 0\}. \tag{4}$$

We are able to show the embedding result $LD^2(\Omega) \hookrightarrow L^2(\partial\Omega)$ for any domain Ω with Lipschitz boundary, using the same approach of Theorem 1.1, the page 117 of [33] and Theorem 3.2 of [6] (see also Theorem and Example given on the pages 224-227 of [35]). Nevertheless of it such type of embedding result is not valid for the cuspidal domains. Let us explain it, returning to the example 1.2.

Example 1.9. *As we have shown in this example 1.2, the function* (2) *belongs to $LD^2(V(x^2))$ for any given $s < 1 + \frac{3}{2}$. Let us fix a real parameter*

$$s \in [\frac{3}{2}, 1 + \frac{3}{2}).$$

Then the boundary integral of $\mathbf{w}$ on $0 < x < 1, \quad y = 0$ in L^2 is equal to

$$\int_0^1 x^{-2(s-1)} dx = +\infty,$$

that is $\mathbf{w} \notin L^{2+\varepsilon}(\partial V(x^2))$ and the inclusion

$$LD^2(\Omega) \hookrightarrow L^2(\partial\Omega)$$

is not valid for the cuspidal domain Ω.

Hence the introduction of the following space $LD^p_\gamma(\Omega)$ is natural.

Definition 1.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a finite perimeter. Let $LD^p_\gamma(\Omega)$ be a space of functions $\mathbf{u} \in L^p(\Omega)$ and $\mathbf{u}$ has the trace value $\gamma\mathbf{u}$ on the essential boundary $\partial^*\Omega$ (in the sense of* (3)*), which is $p-$power integrable on $\partial^*\Omega$ with respect of Haussdorf measure $\mathcal{H}^{N-1}$. The norm in the space $LD^2_\gamma(\Omega)$ is defined as*

$$\|\mathbf{u}\|^p_{LD^p_\gamma(\Omega)} = \int_\Omega |\mathbb{D}\mathbf{u}|^p\, d\mathbf{x} + \int_{\partial^*\Omega} |\gamma\mathbf{u}|^p\, d\mathcal{H}^{N-1}(\mathbf{x}).$$

Remark 1.11. *In fact the space $LD^2_\gamma(\Omega)$ appears under the construction of the weak solutuion for the motion of rigid body in a viscous fluid by a priori estimate (2.8) deduced in Theorem 2.1 of* [9] *(see also a priori estimate (4.5) of Theorem 1 in* [14]*).*

Let us introduce the following notations. For given $\mathbf{a} \in \mathbb{S}^{N-1}$ we define the section

$$\Omega_{\mathbf{y}} = \{t \in \mathbb{R} : \mathbf{y} + t\mathbf{a} \in \Omega\}$$

of Ω corresponding to a point $\mathbf{y} \in P_{\mathbf{a}}$. If $\Omega_{\mathbf{y}}$ is empty, we set

$$\int_{\Omega_{\mathbf{y}}} f(\mathbf{y} + t\mathbf{a})\, dt = 0$$

for any Lebesgue integrable function $f : \Omega \to \mathbb{R}$. Then the Fubini-Tonelli theorem implies

$$\int_{\Omega} f(\mathbf{x})\, d\mathbf{x} = \int_{P_{\mathbf{a}}} \left(\int_{\Omega_{\mathbf{y}}} f(\mathbf{y} + t\mathbf{a})\, dt \right) d\mathcal{L}^{N-1}(\mathbf{y}). \tag{5}$$

The following result of absolute continuity on lines is an analogue of Theorems 7.13 and 10.35 of [23].

Lemma 1.12. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $\mathbf{a}_k \in \mathbb{R}^N$, $k = 1, ..., N$, be arbitrary independent vectors.*

For given $\mathbf{u} \in LD^p(\Omega)$ there exists a representative $\overline{\mathbf{u}}$ of $\mathbf{u}$, such for each $k = 1, ..., N$ and $\mathcal{L}^{N-1}$- a.e. $\mathbf{y} \in P_{\mathbf{a}_k}$ the function

$$v_k(t) = \mathbf{a}_k \cdot \overline{\mathbf{u}}(\mathbf{y} + t\mathbf{a}_k)$$

is absolutely continuous on $t \in \Omega_{\mathbf{y}}$ and the following formula

$$v_k(t) = v_k(t') + \int_{t'}^{t} \mathbf{a}_k \mathbb{D}\mathbf{u}(\mathbf{y} + s\mathbf{a}_k) \cdot \mathbf{a}_k\ ds \tag{6}$$

is valid for any $[t', t] \subset \Omega_{\mathbf{y}}$.

Proof. Let us consider a sequence of standard mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$ (see C.4, the pages 552-560, of [23]) and for every $\varepsilon > 0$ define

$$\mathbf{u}^\varepsilon = \mathbf{u} * \varphi_\varepsilon \in \Omega_\varepsilon = \{\mathbf{x} \in \Omega :\ dist(\mathbf{x}, \partial\Omega) > \varepsilon\}.$$

By the same approach as in Lemma 10.16 of [23], we have

$$\lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} (|\mathbf{u}^\varepsilon - \mathbf{u}|^p + |\mathbb{D}\mathbf{u}^\varepsilon - \mathbb{D}\mathbf{u}|^p)\, d\mathbf{x} = 0.$$

Using (5), we have

$$\begin{aligned} \int_{P_{\mathbf{a}_k}} \left(\int_{\Omega_{\mathbf{y}}} |\mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2\, dt \right) d\mathcal{L}^{N-1}(\mathbf{y}) &< \infty, \\ \lim_{\varepsilon \to 0^+} \int_{P_{\mathbf{a}_k}} \left(\int_{(\Omega_\varepsilon)_{\mathbf{y}}} |\mathbb{D}\mathbf{u}^\varepsilon(\mathbf{y} + t\mathbf{a}_k) - \mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2\, dt \right) d\mathcal{L}^{N-1}(\mathbf{y}) &= 0. \end{aligned}$$

Therefore there exists a subsequence $\{\varepsilon_n\}$, such that for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} \in P_{\mathbf{a}_k}$, $k = 1, ..., N,$ we have

$$\int_{\Omega_{\mathbf{y}}} |\mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2 \, dt \quad < \quad \infty,$$

$$\lim_{\varepsilon_n \to 0^+} \int_{(\Omega_{\varepsilon_n})_{\mathbf{y}}} |\mathbb{D}\mathbf{u}^{\varepsilon_n}(\mathbf{y} + t\mathbf{a}_k) - \mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2 \, dt \quad = \quad 0. \tag{7}$$

Let us put $\mathbf{u}^n = \mathbf{u}^{\varepsilon_n}$ and

$$E = \{\mathbf{x} \in \Omega : \lim_{n \to \infty} \mathbf{u}^n(\mathbf{x}) \ \text{ exists in } \ \mathbb{R}^N\}.$$

This set E is well-defined, since for every $\mathbf{x} \in \Omega$ we have $\mathbf{x} \in \Omega_{\varepsilon_n}$ for all n sufficiently large (depending on $\mathbf{x}$) and, thus $\mathbf{u}_n(\mathbf{x})$ is well-defined for all n sufficiently large. Let us define

$$\overline{\mathbf{u}}(\mathbf{x}) = \begin{cases} \lim_{n \to \infty} \mathbf{u}^n(\mathbf{x}), & \text{if } \mathbf{x} \in E; \\ 0, & \text{if } \mathbf{x} \in \Omega \backslash E. \end{cases}$$

By Theorem C.19 and Corrollary B.122 of [23] the sequence$\{\mathbf{u}_n\}$ converges point-wise to $\mathbf{u}$ for $\mathcal{L}^N-$a.a. points of Ω. Therefore we conclude that $\mathcal{L}^N(\Omega \backslash E) = 0$ and the function $\overline{\mathbf{u}}$ is one of representatives of $\mathbf{u}$. The Fubini theorem implies that

$$\int_{P_{\mathbf{a}_k}} \left(\mathcal{L}^1(\{t \in \mathbb{R} : \quad \mathbf{y} + t\mathbf{a}_k \notin E\})|^2 \, dt \right) \, d\mathcal{L}^{N-1}(\mathbf{y}) = 0,$$

and, thus we have that for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} \in P_{\mathbf{a}_k}$

$$\mathbf{y} + t\mathbf{a}_k \in E \quad \text{for} \quad \mathcal{L}^1 - \text{a.a.} \ \ t \in \mathbb{R}, \qquad \forall k = 1, ..., N, \tag{8}$$

Let $\mathbb{P}$ be a N-dimensional rectangle with the edges parallel to the vectors $\mathbf{a}^1, ..., \mathbf{a}^N,$ that can be described as

$$\mathbb{P} = \left\{ \mathbf{x} = \sum_{i,j=1}^{N} t_k \, \mathbf{a}_k : \quad t_k \in [c_k, d_k] \subset \mathbb{R}, \quad \forall k = 1, ..., N \right\}.$$

Let us consider the rectangle $\mathbb{P} \subset \Omega$ with c_k, d_k $(k = 1, ..., N)$ being rationals. For $\varepsilon > 0$ sufficiently small, we have that $\mathbb{P} \subset \Omega_\varepsilon$, then by (7) we have

$$\int_{c_k}^{d_k} |\mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2 \, dt \quad < \quad \infty,$$

$$\lim_{n \to \infty} \int_{c_k}^{d_k} |\mathbb{D}\mathbf{u}^n(\mathbf{y} + t\mathbf{a}_k) - \mathbb{D}\mathbf{u}(\mathbf{y} + t\mathbf{a}_k)|^2 \, dt \quad = \quad 0 \tag{9}$$

for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} \in P_{\mathbf{a}_k}$ and all $k = 1, ..., N$.

We define $v_k^n(t) = \mathbf{a}_k \cdot \mathbf{u}^n(\mathbf{y} + t\mathbf{a}_k), \qquad t \in [c_k, d_k]$. Using (8) we choose $t' \in [c_k, d_k]$, such that $\mathbf{y} + t'\mathbf{a}_k \in E$. Then there exists the limit

$$v_k^n(t') \to v_k(t') = \mathbf{a}_k \cdot \overline{\mathbf{u}}(\mathbf{y} + t'\mathbf{a}_k) \in \mathbb{R}. \tag{10}$$

Since $v_n \in C^\infty([c_k, d_k])$, we have

$$\begin{aligned} v_k^n(t) &= v_k^n(t') + \int_{t'}^{t} \frac{d}{ds}\left(v_k^n(s)\right)\, ds \\ &= v^n(t') + \int_{t'}^{t} \sum_{i,j=1}^{N} a_i a_j d_{ij}(\mathbf{u}^n(\mathbf{y}+s\mathbf{a}_k))\, ds \quad \text{for all} \quad t \in [c_k, d_k]. \end{aligned}$$

Hence (9)-(10) imply the existence of the limit

$$\lim_{n\to\infty} v^n(t) = \mathbf{a}_k \cdot \overline{\mathbf{u}}(\mathbf{y}+t'\mathbf{a}_k) + \int_{t'}^{t} \mathbf{a}_k \mathbb{D}\mathbf{u}(\mathbf{y}+s\mathbf{a}_k\,)\cdot \mathbf{a}_k\, ds \quad \text{for all} \quad t \in [c_k, d_k].$$

The definitions of E and $\overline{\mathbf{u}}$ give that

$$\{\mathbf{y} + t\mathbf{a}_k : \quad t \in [c_k, d_k]\} \subset E \tag{11}$$

and the functions $v_k(t) = \mathbf{a}_k \cdot \overline{\mathbf{u}}(\mathbf{y} + t\mathbf{a}_k\,)$, $k = 1, ..., N$, fulfil

$$v_k(t) = v_k(t') + \int_{t'}^{t} \mathbf{a}_k \mathbb{D}(\mathbf{u}(\mathbf{y} + s\mathbf{a}_k)) \cdot \mathbf{a}_k\, ds \quad \text{for all} \quad t \in [c_k, d_k]. \tag{12}$$

Hence each function $v_k{=}v_k(t)$ is absolutely continuous on $[c_k, d_k]$, such that $v'(t) = \mathbf{a}_k \mathbb{D}(\mathbf{u}(\mathbf{y}+t\mathbf{a}_k)) \cdot \mathbf{a}_k$ for $\mathcal{L}^1$–a.e. $t \in [c_k, d_k]$ by Lemma 3.31 of [23].

Now if $\widetilde{\mathbb{P}} \subset \Omega$ is another rectangle, such that

$$[c_k, d_k] \cap [\widetilde{c}_k, \widetilde{d}_k] \neq \varnothing, \quad \forall k = 1, ..., N,$$

then taking $\mathbf{y} \in P_{\mathbf{a}_k}$ which is admissible for both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ and $t' \in [c_k, d_k] \cup [\widetilde{c}_k, \widetilde{d}_k]$, it follows from (11) and (12) that v is absolutely continuous in $[c_k, d_k] \cup [\widetilde{c}_k, \widetilde{d}_k]$.

Since Ω can be written as a countable union of closed rectangles of this type and since the union of countably many sets of $\mathcal{L}^{N-1}$-measure zero still has $\mathcal{L}^{N-1}$-measure zero, using (11), (12) we conclude that for $\mathcal{L}^{N-1}$-a.e. $\mathbf{y} \in P_{\mathbf{a}_k}$, the function $v_k(t)$ is absolutely continuous on any connected component of $\Omega_{\mathbf{y}}$. ■

Let us formulate the following result.

Proposition 1.13. *Let Ω be a set with finite perimeter. Let $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x}) \in \mathbb{S}^{N-1}$ be the internal normal at $\mathbf{x} \in \partial^*\Omega$ and $B_1(\mathbf{x}, \boldsymbol{\nu})$ be the half ball defined by* (4)*.*

Let $\mathbf{a} \in \mathbb{S}^{N-1} \cap B_1(\mathbf{x}, \boldsymbol{\nu})$ be arbitrary fixed vector and

$$\lambda_{\mathbf{a}}(E) = \mathcal{L}^{N-1}(\pi_{\mathbf{a}} E) \tag{13}$$

for any measurable (Borel) set $E \subset \mathbb{R}^N$ on the plane $P_{\mathbf{a}}$ and $\pi_{\mathbf{a}} E$ is the projection of the set $E \subset \mathbb{R}^N$ on $P_{\mathbf{a}}$ (the properties of $\lambda_{\mathbf{a}}$ are given on the pages 235-236 of [36]*).*

For a given function $\mathbf{u} \in LD^p_\gamma(\Omega)$, there exists the limit

$$\gamma_{\mathbf{a}} \mathbf{u}(\mathbf{x}) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbf{u}(\mathbf{x} + \varepsilon s\mathbf{a})\, ds \qquad \textit{for} \quad \lambda_{\mathbf{a}} - \textit{a.e.} \quad \mathbf{x} \in \partial^*\Omega,$$

such that

$$\gamma_{\mathbf{a}}\mathbf{u}(\mathbf{x}) = \gamma\mathbf{u}(\mathbf{x}) \qquad \textit{for} \quad \lambda_{\mathbf{a}} - a.e. \quad \mathbf{x} \in \partial^*\Omega.$$

The proof of this Proposition is absolutely the same as the proof of Theorem of 11.2, the pages 243-245, of [36] without some changes. Since the proof of Theorem, subsection 11.2, of [36] is based on the structure of the set with finite perimeter and the existence of the trace values $\gamma\mathbf{u}$ for a given function $\mathbf{u}$. In our case when $\mathbf{u} \in LD^p_\gamma(\Omega)$ the existence of $\gamma\mathbf{u}$ is guaranted by Proposition 1.8. By these reasons we omit the proof Proposition 1.13.

Corollary 1.14. *Under the assumptions of Proposition 1.12 and the assumption that $\Omega \subset \mathbb{R}^N$ is a open set with finite perimeter, then the formula* (11) *is valid for any $t, t' \in \overline{\Omega_{\mathbf{y}}}$.*

Proof. We have that for $\mathcal{L}^{N-1}$- a.e. $\mathbf{y} \in P_{\mathbf{a}_k}$ the relation (11) fulfills

$$v_k(t) = v_k(t') + \int_{t'}^{t} \mathbf{a}_k \mathbb{D}\mathbf{u}(\mathbf{y} + s\mathbf{a}_k) \cdot \mathbf{a}_k \; ds \qquad \text{for any} \quad [t', t] \in \Omega_{\mathbf{y}}.$$

Let us use the structure of sets with finite perimeter. By 3) of Proposition 1.7 there exist $t_0, t_0' \in \mathbb{R}$, such that

$$[t_0', t_0] = \overline{\Omega_{\mathbf{y}}} \quad \text{and} \quad \mathbf{y} + t_0\mathbf{a}_k, \; \mathbf{y} + t_0'\mathbf{a}_k \in \partial^*\Omega.$$

If we integrate this equality (11) over $t' \in (t_0', , t_0' + \varepsilon)$, divide on ε and take the limit transition for $\varepsilon \to 0$, then Proposition 1.13 and $(9)_1$ imply the validity of this equality for $s = t_0'$. By the same way we can demonstrate the validation of (11) for the point $t = t_0$. ■

We are now ready to prove our main result. Let us formulate a preliminary lemma, which proof can be find in [33], the page 128-129, Lemma 1.1. For a vector $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and each $i \in \{1, \ldots, N\}$, we denote the vectors

$$\widehat{\xi_i} = (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, ..., \xi_N) \in \mathbb{R}^{N-1}. \tag{14}$$

Lemma 1.15. *Let $\theta_i = \theta_i(\widehat{\xi_i})$ be non-negative integrable functions in $\mathbb{R}^{N-1}$, $i = 1, \ldots, N$. Then*

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \theta_i \right)^{\frac{1}{N-1}} d\xi \le \prod_{i=1}^{N} \left(\int_{\mathbb{R}^{N-1}} \theta_i(\widehat{\xi_i}) \, d\widehat{\xi_i} \right)^{\frac{1}{N-1}}. \tag{15}$$

The following is the main result of this work.

Theorem 1.16. *Let Ω be a bounded open set with finite perimeter. Then if $\mathbf{u} \in LD^p_\gamma(\Omega)$, then $\mathbf{u} \in L^{\frac{pN}{(N-1)}}(\Omega)$ and there exists a positive constant C, depending only on N, p and the diameter of the domain Ω, such that*

$$\|\mathbf{u}\|_{L^{\frac{pN}{(N-1)}}(\Omega)} \leq C\,\|\mathbf{u}\|_{LD^p_\gamma(\Omega)}\,. \tag{16}$$

Proof. We follow closely the proof in Theorem 1.2, page 117, of [33] and Theorem 6.95, the pages 333-336, of [10], combining with the approach developed in Theorem, section 5, the pages 218-220, of [35]. This approach was adapted to the case at hand of sets with finite perimeter.

(I) To explain our proof let us start from the simplest situation, considering the 2−dimensional case and $p = 2$, as warm up for the general case. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the euclidean basis of $\mathbb{R}^2$. We denote a point in $\mathbb{R}^2$ with $\mathbf{x} = (x_1, x_2)$ and a vector field with $\mathbf{u} = (u, v)$.

1st step: Since the set Ω has a finite perimeter, then by 3) of Proposition 1.7 for $\mathcal{L}^1$-a.a. $x_2 \in \mathbb{R}$ the intersection

$$\Omega(x_2) = l_{\mathbf{e}_1}((0, x_2)) \cap \Omega$$

consists of a finite number $M_2(x_2)$ of open intervals with disjoint closures

$$\Omega(x_2) = \cup_{l=1}^{M_2(x_2)} \triangle_{2,k}, \quad \text{such that} \quad \overline{\triangle_{2,l}} \cap \overline{\triangle_{2,m}} = \varnothing, \quad \forall l \neq m,$$

where $\triangle_{2,l} = (\mathbf{c}_{2,l}(x_2), \mathbf{d}_{2,l}(x_2))$ is a straight line connecting the points

$$\mathbf{c}_{2,l}(x_2) = (c_{2,l}(x_2), x_2), \quad \mathbf{d}_{2,l}(x_2) = (d_{2,l}(x_2), x_2) \in \partial^*\Omega.$$

Consequently, Corollary 1.14 implies that for such admissible $x_2 \in \mathbb{R}$ and arbitrary chosen $x_1 \in \Omega(x_2)$, there exists an index $k \in \{1, \ldots, M_2(x_2)\}$, such that $x_1 \in \triangle_{2,k}$ and

$$u(\mathbf{x}) = u(x_1, x_2) = \gamma u(\mathbf{c}_{2,k}(x_2)) + \int_{c_{2,k}(x_2)}^{x_1} \partial_{x_1} u(s, x_2)\; ds.$$

It follows that

$$\begin{aligned} u^2(\mathbf{x}) \;&\leq\; 2\left[|\gamma u(\mathbf{c}_{2,k}(x_2))|^2 + (d_{2,k}(x_2) - c_{2,k}(x_2))\int_{c_{2,k}(x_2)}^{d_{2,k}(x_2)} |\partial_{x_1} u(s, x_2)|^2\; ds\right] \\ &\leq\; C\sum_{l=1}^{M_2(x_2)}\left[|\gamma \mathbf{u}(\mathbf{c}_{2,l}(x_2))|^2 + \int_{c_{2,l}(x_2)}^{d_{2,l}(x_2)} |d_{11}(\mathbf{u})(s, x_2)|^2\; ds\right] = f_2(x_2) \qquad (17) \end{aligned}$$

with the constant C depending only on the diameter of Ω.

In the same fashion, for $\mathcal{L}^1$-a.a. $x_1 \in \mathbb{R}$ the intersection

$$\Omega(x_1) = l_{\mathbf{e}_2}((x_1, 0)) \cap \Omega$$

consists of a finite number $M_1(x_1)$ of open intervals with disjoint closures

$$\Omega(x_1) = \cup_{l=1}^{M_1(x_1)} \triangle_{1,l}, \quad \text{such that} \quad \overline{\triangle_{1,l}} \cap \overline{\triangle_{1,m}} = \varnothing, \quad \forall l \neq m,$$

where $\triangle_{1,l} = (\mathbf{c}_{1,l}(x_1), \mathbf{d}_{1,l}(x_1))$ is a straight line connecting the points

$$\mathbf{c}_{1,l}(x_1) = (x_1, c_{1,l}(x_1)), \quad \mathbf{d}_{1,l}(x_1) = (x_1, d_{1,l}(x_1)) \in \partial^*\Omega.$$

For admissible $x_1 \in \mathbb{R}$ and arbitrary chosen $x_2 \in \Omega(x_1)$, there exists an index $k \in \{1, \ldots, M_1(x_1)\}$, such that $x_2 \in \triangle_{1,k}$ and

$$v(\mathbf{x}) = v(x_1, x_2) = \gamma v(\mathbf{c}_{1,k}(x_1)) + \int_{c_{1,k}(x_1)}^{x_2} \partial_{x_2} v(x_1, s)\ ds.$$

Hence

$$v^2(\mathbf{x}) \leq C \sum_{l=1}^{M_1(x_1)} \left[|\gamma \mathbf{u}(\mathbf{c}_{1,l}(x_1))|^2 + \int_{c_{1,l}(x_1)}^{d_{1,l}(x_1)} |d_{22}(\mathbf{u})(x_1, s)|^2\ ds \right] = f_1(x_1). \tag{18}$$

Multiplying (17) with (18) and integrating over Ω, by Lemma 1.15 (or simply by the Fubini-Tonelli theorem) we obtain

$$\int_\Omega u^2(\mathbf{x}) v^2(\mathbf{x})\ d\mathbf{x} \leq \int_\Omega f_1(x_1) f_2(x_2)\ d\mathbf{x} \leq \int_{I_1} f_1(x_1)\ dx_1 \int_{I_2} f_2(x_2)\ dx_2,$$

where I_i, $i = 1, 2$, are the projections of Ω on the x_i-coordinate axis. We have

$$\left.\begin{array}{l} \int_{I_1} \sum_{l=1}^{M_1(x_1)} |\gamma \mathbf{u}(\mathbf{c}_{1,l}(x_1))|^2\ dx_1 \\ \\ \int_{I_2} \sum_{l=1}^{M_2(x_2)} |\gamma \mathbf{u}(\mathbf{c}_{2,l}(x_2))|^2\ dx_2 \end{array}\right\} \leq \int_{\partial^*\Omega} |\gamma(\mathbf{u})|^2\, d\mathcal{H}^{N-1}(\mathbf{x})$$

by the properties of the measure $\lambda_{\mathbf{a}}$ given on the pages 235-236, section 7, of [36]. Therefore

$$\begin{aligned} \int_\Omega u^2(\mathbf{x}) v^2(\mathbf{x})\, d\mathbf{x} &\leq C \int_{I_2} f_1(x_1)\ dx_1 \int_{I_1} f_2(x_2)\ dx_2 \\ &\leq C \left(\int_{\partial^*\Omega} |\gamma(\mathbf{u})|^2\, d\mathbf{x} + \int_\Omega |\mathbb{D}\mathbf{u}|^2\, d\mathbf{x} \right)^2 = C \left\| \mathbf{u} \right\|^4_{LD^2_\gamma(\Omega)}. \end{aligned} \tag{19}$$

2nd step: Now we consider the basis $\mathbf{a}_1 = \frac{1}{\sqrt{2}}(1, 1)$, $\mathbf{a}_2 = \frac{1}{\sqrt{2}}(-1, 1)$. We denote the coordinates of $\mathbf{x}$ in the basis $(\mathbf{a}_1, \mathbf{a}_2)$ by (ξ_1, ξ_2), that is

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) = \xi_1 \mathbf{a}_1 + \xi_2 \mathbf{a}_2.$$

Again, for $\mathcal{L}^1$- a.e. $\mathbf{y}_2 = \xi_2 \mathbf{a}_2 \in P_{\mathbf{a}_1}$, that is for $\mathcal{L}^1$- a.e. $\xi_2 \in \mathbb{R}$, the intersection of lines parallel to $\mathbf{a}_1$ with the domain Ω, crossing through $\mathbf{y}_2$,

$$\Omega(\xi_2) = l_{\mathbf{a}_1}(\mathbf{y}_2) \cap \Omega$$

consists of a finite number $M_2(\xi_2)$ of open intervals with disjoint closures, such that for $\mathbf{x} \in \Omega(\xi_2)$, there exists an interval

$$(\mathbf{c}_{2,k}(\xi_2), \mathbf{d}_{2,k}(\xi_2)) \subset \Omega(\xi_2) \qquad \text{with} \quad \mathbf{c}_{2,k}(\xi_2), \mathbf{d}_{2,k}(\xi_2) \in \partial^*\Omega.$$

For simplicity of notations we assume that this interval, being a part of $l_{\mathbf{a}_1}(\mathbf{y}_2)$, is described as

$$(\mathbf{c}_{2,k}(\xi_2), \mathbf{d}_{2,k}(\xi_2)) = \left\{ \mathbf{y} = \mathbf{y}_2 + s\mathbf{a}_1 \in \mathbb{R}^N :\ s \in (c_{2,k}(\xi_2), d_{2,k}(\xi_2)) \right\}$$

Applying Corollary 1.14 for the function

$$v_2(\xi_1) = \mathbf{a}_1 \cdot \mathbf{u}(\mathbf{y}_2 + \xi_1 \mathbf{a}_1)$$

and proceeding as in (17)-(18), we obtain

$$\begin{aligned} v_2^2(\xi_1) &\leq 2\Big[|\mathbf{a}_1 \cdot \gamma \mathbf{u}(\mathbf{c}_{2,k}(\xi_2))|^2 \\ &\qquad +(d_{2,k}(\xi_2) - c_{2,k}(\xi_2)) \int_{c_{2,k}(\xi_2)}^{d_{2,k}(\xi_2)} |\mathbf{a}_2 \mathbb{D}\mathbf{u}(\mathbf{y}_2 + s\mathbf{a}_1) \cdot \mathbf{a}_2|^2 \, ds\Big] \\ &\leq C \sum_{l=1}^{M_2(\xi_2)} \left[|\gamma \mathbf{u}(\mathbf{c}_{2,l}(\xi_2))|^2 + \int_{c_{2,l}(\xi_2)}^{d_{2,l}(\xi_2)} |\mathbb{D}\mathbf{u}(\mathbf{y}_2 + s\mathbf{a}_1)|^2 \, ds \right] = f_2(\xi_2). \end{aligned} \tag{20}$$

Also for $\mathcal{L}^1$- a.e. $\mathbf{y}_1 = \xi_1 \mathbf{a}_1 \in P_{\mathbf{a}_2}$, that is or $\mathcal{L}^1$- a.e. $\xi_1 \in \mathbb{R}$, the intersection of the line parallel to $\mathbf{a}_2$ with Ω, crossing through $\mathbf{y}_1$,

$$\Omega(\xi_1) = l_{\mathbf{a}_2}(\mathbf{y}_1) \cap \Omega$$

is a finite number $M_1(\xi_1)$ of open intervals with disjoint closures, then for $\mathbf{x} \in \Omega(\xi_1)$ there exists an interval, such that

$$(\mathbf{c}_{1,k}(\xi_1), \mathbf{d}_{1,k}(\xi_1)) \subset \Omega(\xi_1) \qquad \text{with} \quad \mathbf{c}_{1,k}(\xi_1), \mathbf{d}_{1,k}(\xi_1) \in \partial^* \Omega.$$

Defining

$$v_1(\xi_2) = \mathbf{a}_1 \cdot \mathbf{u}(\mathbf{y}_1 + \xi_2 \mathbf{a}_2),$$

Corollary 1.14 gives

$$v_1^2(\xi_2) \leq C \sum_{l=1}^{M_1(\xi_1)} \left[|\gamma \mathbf{u}(\mathbf{c}_{1,l}(\xi_1))|^2 + \int_{c_{1,l}(\xi_1)}^{d_{1,l}(\xi_1)} |\mathbb{D}\mathbf{u}(\mathbf{y}_2 + s\mathbf{a}_1)|^2 \, ds \right] = f_1(\xi_1).$$

Multiplying this inequality with (20), integrating over $(\xi_1, \xi_2) \in \Omega$ and proceeding as it was done under the deduction of (19), we obtain the inequality

$$\int_\Omega^2 v_2(\xi_1)^2 v_1(\xi_2)^2 \, d\xi_1 d\xi_2 \leq C \|\mathbf{u}\|^2_{LD^2_\gamma(\Omega)}.$$

Observing that

$$\begin{aligned} \int_\Omega v_2(\xi_1)^2 v_1(\xi_2)^2 \, d\xi_1 d\xi_2 &= \int_\Omega (\mathbf{a}_1 \cdot \mathbf{u}(\mathbf{x}))^2 (\mathbf{a}_2 \cdot \mathbf{u}(\mathbf{x}))^2 \, d\mathbf{x} \\ &= \frac{1}{4} \int_\Omega (u-v)^2 (u+v)^2 \, d\mathbf{x}, \end{aligned}$$

we obtain

$$\int_\Omega (u^4 - 2u^2 v^2 + v^4) \, d\mathbf{x} \leq C \|\mathbf{u}\|^2_{LD^2_\gamma(\Omega)}.$$

Therefore, combining this estimate with estimate (19) we conclude

$$\|\mathbf{u}\|^4_{L^4(\Omega)} = \int_\Omega (u^4 + v^4) \, d\mathbf{x} \leq C \|\mathbf{u}\|^4_{LD^2_\gamma(\Omega)}.$$

which coincides with (16) for $N = 2$ and $p = 2$.

(II) We now turn to the general N-dimensional case. In the sequel we follow closely the proof of Theorem 6.95, the pages 333-336 of [10].

In what follows we use the following notations. Let us denote the euclidian basis of $\mathbb{R}^N$ by $\{\mathbf{e}_i\}_{i=1}^N$. Given a vector $\mathbf{a} \in \mathbb{S}^{N-1}$ and a point $\mathbf{x} \in \Omega$ we denote by $\mathbf{y} = Proj_{\mathbf{a}}\mathbf{x} \in P_{\mathbf{a}}$ the projection of $\mathbf{x}$ on the plane $P_{\mathbf{a}}$ and

$$\Omega_{\mathbf{a}}(\mathbf{y}) = l_{\mathbf{a}}(\mathbf{y}) \cap \Omega.$$

the intersection of Ω with the line parallel to $\mathbf{a}$ and crossing $\mathbf{y}$ (and $\mathbf{x}$).

Since Ω is, by hypothesis, a set of finite perimeter, for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} = Proj_{\mathbf{a}}\mathbf{x} \in P_{\mathbf{a}}$, $\Omega_{\mathbf{a}}(\mathbf{y})$ is a finite number $M_{\mathbf{a}}(\mathbf{y})$ of open intervals with disjoint closures. Consequently, for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} = Proj_{\mathbf{a}}\mathbf{x} \in P_{\mathbf{a}}$, the point $\mathbf{x}$ belongs to one of these intervals and its endpoints, which we denote by $\mathbf{c}_k(\mathbf{x})$, $\mathbf{d}_k(\mathbf{x})$ are on the essential boundary $\partial^*\Omega$ of Ω. For simplicity of notations we assume that this interval is described as

$$(\mathbf{c}_{\mathbf{a},k}(\mathbf{y}), \mathbf{d}_{\mathbf{a},k}(\mathbf{y})) = \left\{ \mathbf{x} \in \mathbb{R}^N : \ \mathbf{x} = \mathbf{y} + t\mathbf{a}, \qquad t \in (c_{\mathbf{a},k}(\mathbf{y}), d_{\mathbf{a},k}(\mathbf{y})) \right\}.$$

If we consider the function

$$v_{\mathbf{a}}(\mathbf{x}) = \mathbf{a} \cdot \mathbf{u}(\mathbf{x}) = \sum_{i=1}^N a_i u_i(\mathbf{x}),$$

then Corollary 1.14 implies that

$$|v_{\mathbf{a}}(\mathbf{x})| \leq |\gamma(v_{\mathbf{a}})(\mathbf{c}_{\mathbf{a},k}(\mathbf{y}))| + \int_{c_{\mathbf{a},k}(\mathbf{y})}^{d_{\mathbf{a},k}(\mathbf{y})} |\mathbf{a}\mathbb{D}\mathbf{u}(\mathbf{y} + s\mathbf{a}) \cdot \mathbf{a}| \ ds.$$

From this inequality, it follows that

$$\begin{aligned} |v_{\mathbf{a}}(\mathbf{x})|^p &\leq C \sum_{l=1}^{M_{\mathbf{a}}(\mathbf{y})} \left(|\gamma\mathbf{u}(\mathbf{c}_{\mathbf{a},l}(\mathbf{y}))|^p + \int_{c_{\mathbf{a},l}(\mathbf{y})}^{d_{\mathbf{a},l}(\mathbf{y})} |\mathbb{D}(\mathbf{u})(\mathbf{y} + s\,\mathbf{a})|^p \ ds \right) \\ &= H_N(\mathbf{u})(\mathbf{y}) \end{aligned} \tag{21}$$

for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y} = Proj_{\mathbf{a}}\mathbf{x} \in P_{\mathbf{a}}$. Here and below C are constants depending only on N, p and the diameter of Ω. Above, we have used the elementary inequalities

$$|\gamma(v_{\mathbf{a}})| \leq C|\gamma\mathbf{u}|, \qquad |\mathbf{a}\mathbb{D}\mathbf{u}(\mathbf{y} + s\mathbf{a}) \cdot \mathbf{a}| \leq C|\mathbb{D}(\mathbf{u})|.$$

Let us introduce normalized orthogonal projections

$$\mathbf{h}_k = \frac{\mathbf{a} - a_k\,\mathbf{e}_k}{|\mathbf{a} - a_k\,\mathbf{e}_k|} \qquad \text{for each } k = 1, 2, ..., N-1$$

of the vector $\mathbf{a}$ onto coordinates hyperplanes, identified canonically with $\mathbb{R}^{N-1}$. By the same way as it was shown (21), for a fixed $k \in \{1, ..., N-1\}$ the function $v_{\mathbf{h}_k}(\mathbf{x}) = \mathbf{h}_k \cdot \mathbf{u}(\mathbf{x})$ satisfies the inequality

$$\begin{aligned} |v_{\mathbf{h}_k}(\mathbf{x})|^p &\leq C \sum_{l=1}^{M_{\mathbf{h}_k}(\mathbf{y}')} \left(|\gamma\mathbf{u}(\mathbf{c}_{\mathbf{h}_k,l}(\mathbf{y}'))|^p + \int_{c_{\mathbf{h}_k,l}(\mathbf{y}')}^{d_{\mathbf{h}_k,l}(\mathbf{y}')} |\mathbb{D}(\mathbf{u})(\mathbf{y}' + s\,\mathbf{h}_k)|^p \ ds \right) \\ &= I_k(\mathbf{u})(\mathbf{y}') \end{aligned} \tag{22}$$

for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y}' = Proj_{\mathbf{h}_k}\mathbf{x} \in P_{\mathbf{h}_k}$ and the function $v_{\mathbf{e}_k}(\mathbf{x}) = \mathbf{e}_k \cdot \mathbf{u}(\mathbf{x})$ fulfills

$$\begin{aligned} |v_{\mathbf{e}_k}(\mathbf{x})|^p &\leq C \sum_{l=1}^{M_{\mathbf{e}_k}(\mathbf{y}'')} \left(|\gamma \mathbf{u}(\mathbf{c}_{\mathbf{e}_k,l}(\mathbf{y}''))|^p + \int_{c_{\mathbf{e}_k,l}(\mathbf{y}'')}^{d_{\mathbf{e}_k,l}(\mathbf{y}'')} |\mathbb{D}(\mathbf{u})(\mathbf{y}'' + s\,\mathbf{e}_k)|^p \, ds \right) \\ &= J_k(\mathbf{u})(\mathbf{y}'') \end{aligned} \tag{23}$$

for $\mathcal{L}^{N-1}-$a.a. $\mathbf{y}'' = Proj_{\mathbf{e}_k}\mathbf{x} \in P_{\mathbf{e}_k}$. Keeping k fixed, it follows that,

$$v_{\mathbf{a}}(\mathbf{x}) = \sum_{i=1}^{N} a_i u_i(\mathbf{x}) = v_{\mathbf{h}_k}(\mathbf{x}) + a_k v_{\mathbf{e}_k}(\mathbf{x})$$

Consequently

$$|v_{\mathbf{a}}(\mathbf{x})|^p \leq C \left[I_k(\mathbf{u})(\mathbf{y}') + J_k(\mathbf{u})(\mathbf{y}'') \right]. \tag{24}$$

We next use estimates (21)-(23) to bound

$$|v_{\mathbf{a}}(\mathbf{x})|^{pN} \leq C\, H_N(\mathbf{u}) \prod_{k=1}^{N-1} \left[I_k(\mathbf{u}) + J_k(\mathbf{u}) \right].$$

So that, calculating this product and accounting the trivial inequality

$$(\alpha_1 + ... + \alpha_n)^{1/(N-1)} \leq n^{1/(N-1)} (\alpha_1^{1/(N-1)} + ... + \alpha_n^{1/(N-1)}),$$

which is valid for any positive $\alpha_1, ..., \alpha_n$ and $n \in \mathbb{N}$ (in particular for $n = 2^{N-1}$). Therefore the term $|v_{\mathbf{a}}(\mathbf{x})|^{pN/(N-1)}$ is bounded by a linear combination of 2^{N-1} terms of the form

$$I_\sigma = (H_1 \dots H_N)^{1/(N-1)}, \tag{25}$$

where H_k denotes either I_k or J_k.

Each of the terms H_k in the product above depends on $N-1$ variables, and hence we can apply Lemma 1.15. To see this fact, we introduce an adapted basis $\{\mathbf{E}_k\}_{k=1}^N$ as follows. For each index $k \in \{1, \dots, N-1\}$, we set a vector $\mathbf{E}_k$ belonging to $\{\mathbf{h}_k, \mathbf{e}_k\}$ and for $k = N$, we set $\mathbf{E}_N = \mathbf{a}$. If all components of the vector $\mathbf{a}$ are non zero, then it is easy to see that

$$\{\mathbf{E}_k\}_{k=1}^{N-1} \quad \text{is a basis of } \mathbb{R}^{N-1} \qquad \text{and} \qquad \{\mathbf{E}_k\}_{k=1}^{N} \quad \text{is a basis of } \mathbb{R}^N.$$

The proof of this fact are given in Lemma 6.96, the page 334-335 of [10]. We let ξ_j, $j = 1, ..., N$, denote the coordinates of $\mathbf{x} \in \mathbb{R}^N$ in the basis $\mathbf{E}_1, ..., \mathbf{E}_N$, that is,

$$\mathbf{x} = \sum_{j=1}^{N} x_i \mathbf{e}_j = \sum_{j=1}^{N} \xi_j\, \mathbf{E}_j$$

and identify $\mathbf{x}$ with the vector $\xi = \sum_{j=1}^{N} \xi_j\, \mathbf{E}_j$.

Then, each term I_σ can be rewritten as

$$(I_\sigma(\mathbf{x}(\boldsymbol{\xi})))^{N-1} = \prod_{k=1}^{N} \theta_k(\widehat{\xi}_k) \qquad \text{with } \theta_k(\widehat{\xi}_k) = H_k(Proj_{\mathbf{E}_k}\mathbf{x}(\xi)), \qquad k = 1, ..., N.$$

Proceeding as it was done for the deduction of (19), we have

$$\int_{\mathbb{R}^{N-1}} \theta_k(\widehat{\xi}_k)\, d\widehat{\xi}_k \leq \|\mathbf{u}\|^p_{LD^p_\gamma(\Omega)}.$$

By Lemma 1.15 it follows that

$$\int_\Omega I_\sigma(\mathbf{x})\, d\mathbf{x} \leq C(\sigma) \prod_{i=1}^N \left(\int_{\mathbb{R}^{N-1}} \theta_k\, d\widehat{\xi}_k \right)^{\frac{1}{N-1}} \leq C\ \|\mathbf{u}\|^{\frac{pN}{N-1}}_{LD^p_\gamma(\Omega)}, \tag{26}$$

where the dependence on σ in the constant C comes from the Jacobian of the change of variables from $\mathbf{x}$ to ξ, being a constant. Then, the integration over Ω of $|v_{\mathbf{a}}(\mathbf{x})|^{2N/(N-1)}$, being the linear combination of 2^{N-1} terms of (25), yields

$$\int_\Omega |v_{\mathbf{a}}(\mathbf{x})|^{2N/N-1}\, d\mathbf{x} \leq C\ \|\mathbf{u}\|^{\frac{pN}{N-1}}_{LD^2_\gamma(\Omega)}.$$

Lastly, we observe that, since $\mathbf{a}$ can be chosen arbitrarily away from the coordinate planes, by varying $\mathbf{a}$ we can bound $\|u_i\|_{L^{2N/N-1}}$ for each component u_i of $\mathbf{u}$ as exemplified in the two-dimensional case. For example, choosing $\mathbf{a} = \frac{1}{\sqrt{N}}(1, \ldots, 1)$ first and the $\bar{\mathbf{a}} = \frac{1}{\sqrt{N}}(1, \ldots, -1, \ldots, 1)$, where -1 is in the i-th component, gives a bound on

$$\|u_i\|_{L^{pN/N-1}(\Omega)} = \frac{\sqrt{N}}{2} \|v_{\mathbf{a}} - v_{\bar{\mathbf{a}}}\|_{L^{pN/N-1}(\Omega)}.$$

We conclude that estimate (16) holds. □

Remark 1.17. *The obtained embeeding result of Theorem 1.16 is an analog of the embeeding result* $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ *for* $q = \frac{pN}{(N-p)}$*, which is valid for arbitrary open set (see Theorem 4.1.1., the page 177, of* [38]*). In our embeeding result instead of zero boundary values we have used the boundness of the non zero boundary values and some regularity of the boundary of the domain, considering the domain with finite perimeter. Theorem 1.16 shows that the space* $LD^p_\gamma(\Omega)$ *has less regularity properties than* $W^{1,p}_0(\Omega)$*, which is natural, since the Korn inequality is not valid in domains with finite perimeter (see, example of* [3] *for cuspidal domain).*

References

[1] Adams R.. Sobolev spaces. *Boston, MA: Academic Press,* 1975.

[2] Acosta G., Armentano M. G., Durán R. G., Lombardi A. L.. Non-homogeneous Neumann problem for the Poisson equation in domains with an external cusp. *J. Math. Anal. Appl.,* **310** (2005) 397-411.

[3] Acosta G., Durán R. G., López García F., Korn inequality and divergence operator: counterexamples and optimality of weighted estimates. *Proc. Amer. Math. Soc.,* **141**, 1 (2013) 217—232.

[4] Ambrosio L., Fusco N., Pallara D.. Functions of bounded variation and free discontinuity problems. *Oxford Science publications, Clarendon press, Oxford,* 2000.

[5] Ambrosio L., S. Mortola, Tortorelli V. M.. Functionals with linear growth defined on vector valued BV functions. *J. Math. Pures et Appt.* 70 (1991), 269- 323.

[6] Babadjian J.F.. Traces of functions of bounded deformation. *ArXiv preprint, https://arxiv.org/abs/1308.5497.* (2013) 1-12.

[7] Besov O.V.. Integral estimates for differentiable functions on irregular domains. *Doklady Mathematics,* **1** (2010) 87–90 (published in *Doklady Academii Nauk,* **430**, 5 (2010) 583-585).

[8] Bost C., Cottet G.-H., Maitre E., Convergence analysis of a penalization method for the three-dimensional motion of a rigid body in an incompressible viscous fluid. *SIAM J. Numer. Anal.,* **48**(4) (2010) 1313-1337.

[9] Chemetov N.V., Nečasov´a Š.. The motion of the rigid body in viscous fluid including collisions. Global solvability result. *Nonlinear Analysis: Real World Applications.* **34** (2017) 416–445.

[10] Demengel F., Demengel F.. Functional spaces for the theory of elliptic partial differential equations. *CRC Press,* 1991.

[11] Evans L.C., Gariepy R.E.. Measure theory and fine properties of functions. *CRC Press,* 1991.

[12] Federer H., Geometric measure theory. *Springer,* 1969.

[13] Feireisl E., Hillairet M., Nečasov´a Š.. On the motion of several rigid bodies in an incompressible non-Newtonian fluid. *Nonlinearity,* **21** (2008) 1349–1366.

[14] Gérard-Varet D., Hillairet M.. Existence of weak solutions up to collision for viscous fluid-solid systems with slip. *Comm. Pure Appl. Math.,* **67**(12) (2014) 2022–2075.

[15] Gérard–Varet D., Hillairet M., Wang C.. The influence of boundary conditions on the contact problem in a 3D Navier-Stokes flow. *J. Math. Pures Appl.,* **103**(1) (2015) 1–38.

[16] Gunzburger M.D., Lee H.-C., Seregin G.A.. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. math. fluid mech.,* **2** (2000) 219–266.

[17] Giusti E.. Minimal surfaces and functions of bounded variation. *Birkhttuser,* 1984.

[18] Grisvard P.. Problemes aux limites dans des domaines avec points de rebroussement. *Ann. Fac. Sci. Toulouse,* **4** (1995) 561-578.

[19] Hesla T.I.. Collision of smooth bodies in a viscous fluid: A mathematical investigation. 2005. PhD Thesis - Minnesota.

[20] Hillairet M.. Lack of collision between solid bodies in a 2D incompressible viscous flow. *Comm. Partial Differential Equations,* **32**, 7-9 (2007) 1345–1371.

[21] Hoffmann K.-H., Starovoitov V. N.. On a motion of a solid body in a viscous fluid. Two dimensional case. *Adv. Math. Sci. Appl.,* **9** (1999) 633–648.

[22] Kilpelainen T., Maly J., Sobolev inequalities on sets with irregular boundaries. *Z. Anal. Anwendungen,* **19**, 2 (2000) 369–380. The correction of the proof in ”A correction to: Sobolev inequalities on sets with irregular boundaries”.

[23] Leoni G.. A first course in Sobolev spaces. *Graduate studies in mathematics, vol. 105, AMS, Providence, Rhode Island,* 2009.

[24] Labutin D.A., Definitiveness of Sobolev inequalities for a class of irregular domains. *Proc. Steklov Inst. Math.,* **232** (2001) 211–215.

[25] Maz'ya V. G., Classes of domains and embedding theorems for function spaces. *Soviet Math. Dokl.,* **133**, 1 (1960) 882–885.

[26] Maz'ya V. G., Poborchi S. V.. Differentiable functions on bad domains. *World Scientific Publishing Co., River Edge, NJ,* 1997.

[27] Neustupa J., P. Penel P.. Existence of a weak solution to the Navier-Stokes equation with Navier's boundary condition around striking bodies. *Comptes Rendus Mathematique,* **347**, 11-12 (2009) 685–690.

[28] Neustupa J., Penel P.. A weak solvability of the Navier-Stokes equation with Navier's boundary condition around a ball striking the wall. In the book: *Advances in Mathematical Fluid Mechanics: Dedicated to Giovanni Paolo Galdi, Springer-Verlag Berlin,* (2010) 385–408.

[29] Planas G., Sueur F.. On the "viscous incompressible fluid + rigid body" system with Navier conditions. *Annales de l'I.H.P. Analyse non linÃ©aire,* **31**(1) (2014) 55–80.

[30] San Martin J.A., Starovoitov V., Tucsnak M., Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.,* **161** (2002) 93–112.

[31] Starovoitov V.N.. Behavior of a rigid body in an incompressible viscous fluid near boundary. In the book: *International Series of Numerical Mathematics,* **147** (2003) 313–327.

[32] Takahashi T.. Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. *Advances in Differential Equations,* **8**(12) (2003) 1499–1532.

[33] Temam R.. Problèmes mathématique en plasticité' . *Gauthier-Villars. Bordas, Paris,* 1983.

[34] Temam R., Strang G.. Functions of bounded deformation. *Archive for Rational Mechanics and Analysis,* **75**, 1 (1980) 7-21.

[35] Vol'pert A.I., Hudjaev S.I.. Analysis in classes of discontinuous functions and equations of mathematical physics. *Martinus Nijhoff Publushers,* 1985.

[36] Vol'pert A.I.. The spaces BV and quasilinear equations. *Mat. Sb.,* **73**, 115 (1967) 255-302.

[37] Weck N., Local compactness for linear elasticity in irregular domains. *Math. Meth. Appl. Sci.*, **17**, (1994) 107–113.

[38] Ziemer W.. Weakly differentiable functions. *Springer,* 1989.

[39] Judakov N.V.. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. (Russian) *Dinamika Splošn. Sredy Vyp.,* **18** (1974) 249–253.

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