[2] Engler, H., B.Kawohl and S.Luckhaus, *Gradient estimates for solutions of parabolic equations and systems* J. Math. Anal. Appl., 147 (1990), 309–329.

[3] Ladyzenskaya, O.A., V.A.Solonnikov and N.N.Ural'tseva, *Linear and quasilinear equations of parabolic type*, AMS, Providence, RI, (1968).

[4] Nakao, M., *L^p-estimate of solutions of some nonlinear degenerate diffusion equations*, J. Math. Soc. Japan, 37 (1985), 41–63.

[5] Nakao, M., *Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations*, Nonlinear Analysis T. M. A., 10 (1986), 299–314.

[6] Nakao, M. and Y.Ohara, *Gradient estimates for a quasilinear parabolic equation of the mean curvature type*, (to appear)

[7] Ohara, Y. *L^∞-estimates of solutions of some nonlinear degenerate parabolic equations*, Nonlinear Analysis T. M. A., 18 (1992), 413–426.

[8] Serrin, J. *The problem of Dirichlet for quasilinear elliptic differential equation with many independent variables*, Philos. Trans. Royal Soc. London, A. 264 (1969), 413–496.

[9] Véron, L. *Coercivité et proprietes regularisantes des semi-groupes non linêaires dans les espaces de Banach*, Facult des Sciences et Techniques, Université Francois Rabelais-tours, France, (1976).

Yasuhiro Ohara

Yatsushiro College of Technology
Yatsushiro Kumamoto 866, Japan

Advances in Mathematical Sciences and Applications
Gakkōtosho, Tokyo, Vol. 6, No. 2(1996), pp.541-558

THE STEFAN PROBLEM WITH HYSTERESIS-TYPE PHASE FUNCTION

A. Meirmanov and N. Shemetov

(Communicated by A. Visintin; Received June 1, 1994; Revised October 31, 1994)

Abstract. The Stefan problem in which the phase function is a simple hysteresis ope ator is considered. Existence and uniqueness of the weak solution for one-dimensional ca is proved.

1 Introduction

We will study the Stefan problem as a mathematical model for liquid-solid phase tran sitions in a pure material. This problem is characterized by the normalized temperature and the phase function χ, which satisfy the energy conservation law. It may be written in dimensionless form for the domain $\Omega = \{x \mid 0 < x < 1\}$, for $t > 0$

$$\theta_t + \chi_t = \theta_{xx}. \qquad (1$$

In accordance with the definition of the phase function: $0 \le \chi \le 1$, $\chi = 0$ in the solic phase and $\chi = 1$ in the liquid phase.

In the ordinary Stefan problem the phase function depends on the temperature

$$\chi \in H(\theta - \theta_*), \qquad (2)$$

where $H(\xi)$ is the Heaviside graph: $H(\xi) = 0$, $\xi < 0$; $H(\xi) \in [0,1]$, $\xi = 0$; $H(\xi) = 1$, $\xi > 0$, and $\theta_* = const$ is the melting temperature.

The Stefan problem (1), (2) is the equilibrium phase-transition model. For these models the liquid and solid parts of the medium always correspond to known ranges of the independent thermodynamic parameters. For the problem (1), (2) liquid always corresponds to the temperature θ which is greater than or equal to θ_*, and solid part always corresponds to the temperature θ which is less than or equal to θ_*. From the physical point of view it is more natural to use non-equilibrium phase transition models like the Stefan problem with surface tension or the Stefan problem with hysteresis-type phase function. Both of them describe, for example, supercooling or superheating phenomena.

In the Stefan problem with hysteresis-type phase function there are two values θ_*^- – *freezing* temperature and θ_*^+ – *melting* temperature with $\theta_*^- < \theta_*^+$. Thus $\chi = 0$ ($\chi = 1$) for $\theta < \theta_*^-$ ($\theta > \theta_*^+$) respectively and if $\theta_*^- < \theta < \theta_*^+$, the phase function may be 0 or 1 depending on the history of the temperature. Suppose, for example, that $\theta(x, t_0) < \theta_*^-$ and $\theta(x,t)$ continuously increases when $t > t_0$ increases. Then $\chi(x,t) = 0$ until the moment t_1, when $\theta(x,t_1) = \theta_*^+$, and $\chi(x,t) = 1$ for all $t \geq t_1$ (near t_1). So melting always takes place at the temperature θ_*^+. If $\theta(x,t_0) > \theta_*^+$ and $\theta(x,t)$ continuously decreases when $t > t_0$ increases, $\chi(x,t) = 1$ until the moment t_1, when $\theta(x,t_1) = \theta_*^-$ and $\chi(x,t) = 0$ for all $t \geq t_1$ (near t_1). Freezing always takes place at the temperature θ_*^-. This dependence may be represented by the formula

$$\chi(x,t) = F_{(\theta_*^-,\theta_*^+)}(\theta(x,\cdot))(t) \tag{3}$$

which can be interpreted as a two-position relay with "up" and "down" positions corresponding to $\{1\}$ and $\{0\}$ (Fig. 1).

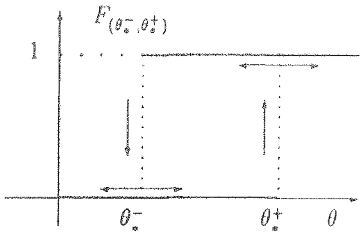

Fig. 1. Hysteresis operator 1

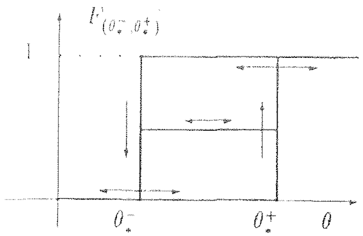

Fig. 2. Hysteresis operator 2

The purpose of the present paper is to study the problem (1), (3) with the Dirichlet boundary conditions

$$\theta(i,t) = g(i,t),\ i = 0, 1,\ t > 0 \tag{4}$$

and the initial conditions

$$\theta(x,0) = \theta_0(x),\ \chi(x,0) = \chi_0(x),\ x \in \Omega. \tag{5}$$

The operator $F_{(\theta_*^-,\theta_*^+)}$ (operator 1) transforms the space of continuous function $C(0,T)$ to the space of piecewise-constant functions. We say that θ and χ satisfy the hysteresis law in the usual sense if θ and χ satisfy hysteresis operator 1.

By analogy with the Stefan problem (1), (2) let us consider a mushy region

$$M_*^{\pm}(t) = \left\{ x \in \Omega \,\middle|\, \theta(x,t) = \theta_*^{\pm},\ \ 0 \leq \chi(x,t) \leq 1 \right\}. \tag{6}$$

How will $\chi(x,t)$ change when $t \geq t_0$, if for $t < t_0$ we have $0 < \chi(x,t) < 1$, $\theta(x,t) = \theta_*^{\pm}$? The law (3) doesn't lead to the fact that $\chi(x,t_0)$ equals 1 or 0, because the temperature doesn't have any dynamic history for $t \leq t_0$. Therefore if we allow an existence of mushy region, we have to extent the definition of the operator $F_{(\theta_*^-,\theta_*^+)}$. There are various possibilities. The first one is to form F as shown at Fig.2 (hysteresis operator 2). This operator was studied by A.Visintin. He is the first mathematician who has considered the Stefan problem with various types of hysteresis law for the temperature and the phase function [3 – 6]. The main feature of all these laws is the following: the phase function may achieve any value between 0 and 1, when $\theta_*^- < \theta < \theta_*^+$.

The second possibility is to extent formally operator 1 (Fig. 3).

And the last variant (hysteresis operator 4) allows phase transitions in both direction (Fig.4).

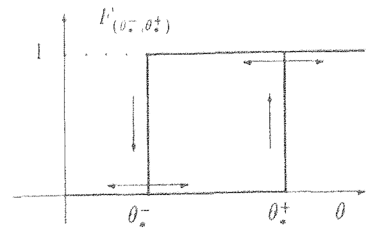

Fig. 3. Hysteresis operator 3

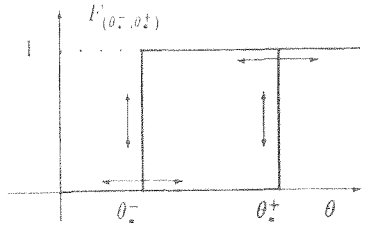

Fig. 4. Hysteresis operator 4

Therefore if we introduce the graphs H^-, H^+ :

$$H^-(\xi) = \begin{cases} 1, & \text{if } \xi > \theta_*^- \\ ]0,1], & \text{if } \xi = \theta_*^- \end{cases},\quad H^+(\xi) = \begin{cases} 0, & \text{if } \xi < \theta_*^+ \\ [0,1[, & \text{if } \xi = \theta_*^+ \end{cases},$$

the hysteresis law with operator 4 can be interpreted as: if $\chi(t_0) \in H^+(\theta(t_0))$ for a momen t_0, then $\chi(t)$ under increasing of t moves on the graph H^+ till a moment $t_1 > t_0$ when $\chi($ achieves the value 1 under $\theta(t_1) = \theta_*^+$. After that $\chi(t)$ will continue its motion on the grap H^- till a moment $t_2 > t_1$ when $\chi(t)$ takes the value 0 under $\theta(t_2) = \theta_*^-$ and then $\chi(t)$ retur on the graph H^+, and so on.

In the present paper we are going to consider hysteresis operator 4 by the next reason

1. Operator 4 permits existence of mushy region.

2. The mushy region for the Stefan problem with operator 4 behaves as the mushy regi for the ordinary Stefan problem, i.e. it has a finite time of existence. In the case of operat 2 for the same data the mushy region is stable.

Namely, let

$$\theta_*^- < g(i,t) < \theta_*^+ \text{ for } t > 0,\ i = 1, 2;$$

$$\theta_0(x) = \theta_*^-,\ 0 < \chi_0(x) < 1,\ x \in \Omega.$$

Then the problem (1), (3)-(5) with operator 2 gives a solution $\{\theta_2(x,t),\ \chi_2(x,t)\}$, whe θ_2 is a solution for the heat equation in Ω for $t > 0$ with the conditions (4), (5) ar $\chi_2(x,t) = \chi_0(x)$:

$$\theta_*^- < \theta_2(x,t) < \theta_*^+,\ \ 0 < \chi_2(x,t) < 1.$$

Therefore, the mushy region exists for all time and is stable.

On the other hand, the problem (1), (3)-(5) with operator 4 yields a solution $\{\theta_4(x,t),\ \chi_4(x,t)\}$ with decreasing mushy region. There exist two curves $x = R^-(t)$ (monotone increasing) and $x = R^+(t)$ (monotone decreasing) such that

$$\theta_4 = \theta_*^-, \quad \chi_4 = \chi_0(x), \text{ for } R^-(t) < x < R^+(t), \quad 0 < t < t_*;$$

$\theta_4(x,t)$ is a solution of the heat equation and $\chi_4 = 1$ in the domain

$$\left\{(x,t) : (0 < x < R^-(t)) \cup (R^+(t) < x < 1),\ 0 < t < t_*\right\}.$$

Moreover if $\int_0^1 \chi_0(x)dx$ is small enough, the mushy region disappears at a moment $t_* < \infty$ $(R^-(t_*) = R^+(t_*))$ and for $t > t_*$ we have liquid phase. So the mushy region is unstable.

3. The problem (1), (3)-(5) with operator 3 has no solution if the mushy region exists at initial time moment. A sketch of the proof for this fact is given in Appendix.

4. If the mushy region at initial time moment is a set of measure zero, then the solution for operator 4 coincides with the solution for operator 2 in Visintin's model.

We will use standard notations [1] for the functional spaces and the norms of their elements.

2 The main results

We assume that the data g, θ_0, χ_0 satisfy the following conditions (*)

$$\chi_0 \in graph H(\theta_0 - \theta_*^-) \cup graph H(\theta_0 - \theta_*^+)$$

$$\theta_0 \in W_2^1(\Omega), \quad g \in W_2^2(\Omega_T), \quad g(i,0) = \theta_0(i), \quad i = 1,2,$$

$$M_*^\pm(0) = \bigcup_{k=1}^{\infty} \left[\alpha_k^\pm, \beta_k^\pm\right] \cup M^\pm, \quad mes M^\pm = 0.$$

Theorem 1. *Under the conditions (*) there exists at least one weak solution*

$$\theta \in W_2^1(\Omega_T) \cap C(\overline{\Omega}_T)$$

for the problem (1), (3)-(5), such that the temperature θ and the phase function χ satisfy hysteresis operator 4. For almost all $x \in \Omega$ if $\chi(x,t_0) = 0,\ 1$, then $\theta(x,t)$, $\chi(x,t)$ satisfy the hysteresis law in the usual sense for all $t \geq t_0$.

Remark 1. If for all $t \geq 0 : M_*^\pm(0)$ is set of measure zero, then theorem 1 gives that θ, χ satisfy the hysteresis law in the usual sense for all $t \geq 0$.

Theorem 2. *If θ and χ satisfy the hysteresis law in the usual sense for all $t \geq 0$, then the problem (1), (3)-(5) has unique solution*

$$\theta \in W_2^{1,0}(\Omega_T) \cap C(\overline{\Omega}_T).$$

The idea of the proof of Theorem 2 was suggested by I.G.Götz and based upon the Hilpert's inequality [6].

Remark 2. Note that the proof of theorem 2 is also true for the case $\Omega \in \mathbf{R}^n$.

3 Proof of Theorem 1

We will prove this theorem using the standard approximations (see A.Visintin [...]) us consider the following sequence of problems.

Approximate problem (AP_N) :

To find the functions $\{\theta^n(x),\ \chi^n(x)\}$, $n = 1,2,...,N$ which are solutions for the equation

$$\theta^n + \chi^n - \tau\theta^n_{xx} = \theta^{n-1} + \chi^{n-1}, \quad x \in \Omega,$$

such that

$$\chi^n(x) \in graph\ H(\theta^n(x) - \theta_*^\pm) \text{ if } \chi^{n-1}(x) \in graph\ H(\theta^{n-1}(x) - \theta_*^\pm); \qquad (7)$$

$$\theta^n(i) = g(i,\tau n), \quad i = 0,1, \quad \tau = T/N;$$

$$\theta^0(x) = \theta_0(x), \quad \chi^0(x) = \chi_0(x), \quad x \in \Omega.$$

In (7) we mean that χ is a multi-valued function of θ and always

$$\chi \in graph\ H(\theta - \theta_*^+) \text{ if } \theta \leq \theta_*^- \text{ and } \chi = 0,$$

$$\chi \in graph\ H(\theta - \theta_*^-) \text{ if } \theta \geq \theta_*^+ \text{ and } \chi = 1.$$

The solution $\{\theta^n(x),\ \chi^n(x)\}$ of the problem (AP_N) gives for the problem (1), (3)-(5) the approximate solution $\left\{\theta^N(x,t),\ \chi^N(x,t)\right\}$ in the form:

$$\theta^N(x,t) = \theta^{n-1}(x) + \frac{(t-(n-1)\tau)}{\tau}(\theta^n(x) - \theta^{n-1}(x)),$$

$$\chi^N(x,t) = \chi^{n-1}(x) + \frac{(t-(n-1)\tau)}{\tau}(\chi^n(x) - \chi^{n-1}(x)),$$

when $(n-1)\tau < t \leq n\tau$, $x \in \Omega$, $n = 1,2,...,N$.

Lemma 1. *The problem (AP_N) has unique solution*

$$\theta^n \in W_p^1(\Omega), \quad n = 1,2,...,N$$

for any $p > 1$ and any N=1,2,...

Lemma 2. *Uniformly for all N=1,2,...*

$$\iint_{\Omega_T} \left|\theta_t^N(x,t)\right|^2 dxdt + \max_{0 \leq t \leq T} \int_{\Omega} \left|\theta_x^N(x,t)\right|^2 dx < C_0, \qquad (8)$$

$$0 \le \chi^N(x,t) \le 1, \quad (x,t) \in \Omega_T,$$

where the constant C_0 depends only on the data g, θ_0, χ_0.

For the case of one spatial variable the inequality (8) leads to the Holder continuity of $\theta^N(x,t)$ [1]

$$\left|\theta^N\right|_{\Omega_T}^{(\beta)} \le C_1(C_0)$$

with some positive $\beta = \beta(C_0)$. Therefore there exist such subsequences $\{\theta^{N_k}\}$, $\{\chi^{N_k}\}$ that

$$\theta^{N_k} \to \theta \text{ weakly in } W_2^1(\Omega_T) \text{ and stronly in } C(\overline{\Omega}_T)$$

$$\chi^{N_k} \to \chi \text{ weakly in } L_2(\Omega_T)$$

and

$$\theta \in W_2^1(\Omega_T) \cap C(\overline{\Omega}_T), \quad 0 \le \chi(x,t) \le 1, \quad (x,t) \in \Omega_T.$$

We can easily conclude that θ meets the boundary conditions (4) and the initial condition (5) in the usual sense and the equation (1) is satisfied in the weak sense:

$$\iint_{\Omega_T} \{(\theta+\chi)\cdot\varphi_t - \theta_x \cdot \varphi_x\}\, dx dt = 0$$

for any smooth function φ with sup $\varphi \subset \Omega_T$.

Finally we have to check whether θ and χ satisfy the hysteresis law (3) and χ satisfies the initial condition (5). Note that we have to study the behavior of $\chi(x,t)$ only for the points (x_0,t_0) where $\theta_*^- < \theta(x_0,t_0) < \theta_*^+$, because if $\theta(x_0,t_0) < \theta_*^-$ $(\theta(x_0,t_0) > \theta_*^+)$ then $\chi(x_0,t_0) = 0$ $(\chi(x_0,t_0) = 1)$. Really if $\theta(x_0,t_0) < \theta_*^-$ $(\theta(x_0,t_0) > \theta_*^+)$,then due to uniform convergence of the continuous functions $\theta^N(x,t)$ to the continuous function $\theta(x,t)$ in some small neighborhood of (x_0,t_0), $\theta^N(x,t) < \theta_*^-$ $(\theta^N(x,t) > \theta_*^+)$ for all $N > N_0$ (using renotation, we may assume $\theta^N \to \theta$, $\chi^N \to \chi$). But according with the construction of $\chi^N(x,t)$ we obtain $\chi^N(x,t) = 0$ $(\chi^N(x,t) = 1)$ in the mentioned neighborhood, so $\chi(x_0,t_0) = \lim\limits_{N\to\infty} \chi^N(x_0,t_0) = 0$ $(=1)$.

The behavior of $\chi(x_0,t_0)$ when $\theta_*^- < \theta(x_0,t_0) < \theta_*^+$ is based upon the following statements.

Lemma 3. *If*

$$\theta_*^- < \theta(x_0,t) < \theta_*^+, \quad 0 \le t_1 < t < \bar{t}^*$$

and

$$\theta(x_0,t) = \theta_*^- \quad (\theta(x_0,t) = \theta_*^+), \quad \bar{t}^* \le t \le \bar{t},$$

then

$$\chi(x_0,t) = 0 \quad (\chi(x_0,t) = 1), \quad \bar{t}^* \le t \le t_0,$$

for almost all $x_0 \in \Omega$ and for all $t_0 \le T$ such that

$$\theta(x_0,t) < \theta_*^+ \quad (\theta(x_0,t) > \theta_*^-), \quad \bar{t} \le t \le t_0.$$

(8)

Lemma 4. *Let*

$$\theta_*^- < \theta(x_0,t) < \theta_*^+, \quad \bar{t} < t \le t_0$$

and

$$\theta(x_0,t) = \theta_*^\pm, \quad 0 \le t \le \bar{t}.$$

Then for almost all $x_0 \in M_*^\pm(0)$

$$\chi(x_0,t_0) = 0 \text{ or } \chi(x_0,t_0) = 1$$

and

$$\chi(x_0,t) = \chi_0(x_0), \quad 0 \le t \le t_*(x_0) \le \bar{t}.$$

Suppose now that

$$\theta_*^- < \theta(x_0,t_0) < \theta_*^+.$$

Then there are several alternatives:

1) there exist such moments $0 < t_1 < \bar{t}^* \le \bar{t} < t_0$ that

$$\theta(x_0,t) > \theta_*^- \quad (\theta(x_0,t) < \theta_*^+), \quad t_1 < t < \bar{t}^*,$$

$$\theta(x_0,t) = \theta_*^- \quad (\theta(x_0,t) = \theta_*^+), \quad \bar{t}^* \le t \le \bar{t},$$

and

$$\theta_*^- < \theta(x_0,t) < \theta_*^+, \quad \bar{t} < t \le t_0;$$

2) $\theta_*^- < \theta(x_0,t) < \theta_*^+$ for all t, $0 \le t \le t_0$;

3) there exist such moments $0 < t_1 < \bar{t}^* \le \bar{t} < t_0$ that

$$\theta(x_0,t) < \theta_*^- \quad (\theta(x_0,t) > \theta_*^+), \quad t_1 \le t < \bar{t}^*,$$

$$\theta(x_0,t) = \theta_*^- \quad (\theta(x_0,t) = \theta_*^+), \quad \bar{t}^* \le t \le \bar{t},$$

and

$$\theta_*^- < \theta(x_0,t) < \theta_*^+, \quad \bar{t} < t \le t_0;$$

4) there exists such moment $\bar{t} \ge 0$ that

$$\theta(x_0,t) = \theta_*^- \quad (\theta(x_0,t) = \theta_*^+), \quad 0 \le t \le \bar{t}$$

and

$$\theta_*^- < \theta(x_0,t) < \theta_*^+, \quad \bar{t} < t < t_0.$$

In case 1) we can apply lemma 3: for almost all $x_0 \in \Omega$ $\chi(x_0,t_0) = 0$ $(\chi(x_0,t_0) = 1)$. I case 2) due to continuity of the temperature we have $\chi(x_0,t_0) = \chi_0(x_0)$. The same reaso in case 3) yields $\chi(x_0,t_0) = 0$ $(\chi(x_0,t_0) = 1)$, because $\theta(x_0,t_1) < \theta_*^-$ $(\theta(x_0,t_1) > \theta_*^+)$ an $\theta_*^- < \theta(x_0,t) < \theta_*^+$ for $\bar{t} < t \le t_0$. In case 4) we use lemma 4: $\chi(x_0,t_0) = 0$ or $\chi(x_0,t_0) = 1$ All this means that θ and χ satisfy the hysteresis law (3) in the usual sense.

Due to the continuity of the temperature, lemma 3 and lemma 4 we can obtain for almos all $x \in \Omega$

$$\chi(x,t) = \chi_0(x)$$

in some small neighborhood of x for sufficiently small $t > 0$. Therefore

$$\lim_{t\to 0} \chi(x,t) = \chi_0(x)$$

for almost all $x \in \Omega$.

4 Proof of theorem 2

Suppose that there exist two different solutions $\{\theta_1, \chi_1\}$, $\{\theta_2, \chi_2\}$. Let's put

$$\overline{\theta} = \theta_1 - \theta_2, \quad \overline{\chi} = \chi_1 - \chi_2$$

By virtue of the continuity of θ_1, θ_2, the functions χ_1, χ_2, being regarded as functions of time t, have a finite number of points of discontinuity for each fixed point $x \in \Omega$. So the function $\overline{\chi}_t$ is measure, and the integral

$$I(x, t_*, \varphi) = \int_0^{t_*} \varphi(x,t) \cdot \overline{\chi}_t(x,t)dt, \quad \varphi \in L_\infty(0,T)$$

is defined for almost all $x \in \Omega$. By the standard procedure with $\varphi = sgn\overline{\theta}$ as the test function we get the inequality

$$\int_\Omega \left|\overline{\theta}(x, t_*)\right| dx + \int_\Omega I(x, t_*, \varphi)dx \le 0, \; 0 < t_* \le T. \tag{9}$$

Our aim is to prove that

$$I(x, t_*, \varphi) \ge 0 \text{ a.e. in } \Omega. \tag{10}$$

In this case we obtain from (9)

$$\overline{\theta}(x,t) \equiv 0, \quad (x,t) \in \Omega_T,$$

and from this equality and the equation (1)

$$\overline{\chi}(x,t) \equiv 0, \text{ for almost all } (x,t) \in \Omega_T.$$

So in order to prove theorem 2 it is necessary to demonstrate only the inequality (10).

Note that we can exclude the set of measure zero composed of points, where the hysteresis law fails for χ_1 and χ_2. Let $0 < t_1 < .... < t_m \le t_*$ be all points of discontinuity of $\overline{\chi}(x,t)$ for a fixed point $x \in \Omega$. Then

$$I(x, t_*, \varphi) = \sum_{k=1}^{m} sgn\overline{\theta}(x, t_k) \cdot [\overline{\chi}(x, t_k)] \equiv \sum_{k=1}^{m} I_k(x),$$

where $[\overline{\chi}(x, t_k)] = \overline{\chi}(x, t_k + 0) - \overline{\chi}(x, t_k - 0)$.

We are going to show that if $I_k(x) < 0$ then

$$k > 1 \text{ and } I_k(x) + I_{k-1}(x) \ge 0.$$

This statement proves the inequality (17) and theorem 2.

Let t_k be a point where $I_k(x) < 0$. Without loss of generality we can assume that t_k is a point of discontinuity of χ_1. Suppose that $\theta_1(x, t_k) = \theta_*^-$. Then

$$\chi_1(x, t_k + 0) = 0, \quad \chi_1(x, t_k - 0) = 1$$

and

$$I_k(x) = sgn(\theta_*^- - \theta_2) \cdot (-1 - [\chi_2]).$$

For $[\chi_2]$ there are only two possibilities: $[\chi_2] = 0$ or $[\chi_2] = 1$, and for θ_2-only one possibility: $\theta_2(x, t_k) < \theta_*^-$. So $\chi_2(x, t_k \pm 0) = 0$ and $I_k(x) = -1$.

Let's consider now $I_{k-1}(x)$. There are several alternatives:

1) $k = 1 : \chi_1(x,t) = 1$, $\theta_1(x,t) > \theta_*^-$ and $\chi_2(x,t) = 0$, $\theta_2(x,t) < \theta_*^+$ for $0 \le t < t_1$. But at $t = 0$ these equalities contradict the initial condition $\chi_i(x, 0) = \chi_0(x)$, $i = 1, 2$. So $k > 1$.

2) t_{k-1} is a point of discontinuity of χ_1:

$$\theta_1(x, t_{k-1}) = \theta_*^+, \quad \chi_1(x, t_{k-1} - 0) = 0, \quad \chi_1(x, t_{k-1} + 0) = 1.$$

It is easy to get that $\theta_2(x, t_{k-1}) < \theta_*^+$, because of the continuity χ_2 over the interval (t_{k-1}, t_k). It should be mentioned that $\chi_2(x, t_k - 0) = 0$, so if $\theta_2(x, t_{k-1}) = \theta_*^+$ then $\chi_2(x, t_{k-1} + 0) = 1$ and there exists at least one point of discontinuity of χ_2 that is greater than t_{k-1}. Therefore

$$I_{k-1}(x) = sgn(\theta_*^+ - \theta_2) \cdot (1 - [\chi_2]) = 1 - [\chi_2].$$

If χ_2 is continuous at $t = t_{k-1}$, then $[\chi_2] = 0$ and $I_{k-1}(x) = 1$. If t_{k-1} is a point of discontinuity of χ_2, then there exists only one possibility:

$$\chi_2(x, t_{k-1} - 0) = 1, \quad \chi_2(x, t_{k-1} + 0) = 0, \quad I_{k-1}(x) = 2.$$

So in both cases

$$I_{k-1}(x) + I_k(x) \ge 0.$$

3) t_{k-1} is a point of discontinuity of χ_2:

$$\theta_2(x, t_{k-1}) = \theta_*^-, \quad [\chi_2] = -1, \quad [\chi_1] = 0,$$

because the case $[\chi_1] \ne 0$ has already been considered. These equalities together with the inequality $\theta_1(x, t_{k-1}) > \theta_*^-$ (χ_1 is continuous over the interval (t_{k-1}, t_k)) yields the equality

$$I_k(x) = sgn(\theta_1 - \theta_*^-) \cdot ([\chi_1] - [\chi_2]) = sgn(\theta_1 - \theta_*^-) = 1,$$

or

$$I_{k-1}(x) + I_k(x) \ge 0.$$

Thus theorem 2 is proved.

5 Appendix

5.1 Proof of lemma 1

Suppose that we have already known the functions $\theta^{n-1}(x)$ and $\chi^{n-1}(x)$. To find $\theta^n(x)$ and $\chi^n(x)$ let us define the approximate problem $(AP_{N,\epsilon})$, where H_ϵ is substituted to (7) instead of Heaviside graph H :

$$\chi^n(x) = H_\epsilon(\theta^n(x) - \theta_*^\pm) \text{ if } \chi^{n-1}(x) \in H(\theta^{n-1}(x) - \theta_*^\pm). \tag{11}$$

Here $H_\epsilon(\xi)$ is a simple approximation of $H(\xi)$: $H_\epsilon(\xi) = 0$, $\xi < 0$; $H_\epsilon(\xi) = \xi/\epsilon, 0 \le \xi \le \epsilon$; $H_\epsilon(\xi) = 1$, $\xi > \epsilon$. In (11)

$$\chi(x) \in H(\theta(x) - \theta_*^+) \text{ if } \theta(x) \le \theta_*^-, \ \chi(x) = 0,$$

and

$$\chi(x) \in H(\theta(x) - \theta_*^-) \text{ if } \theta(x) \ge \theta_*^+, \ \chi(x) = 1.$$

Denote the operator (11) as $\chi^n = H_\epsilon(\theta^n, x)$.

We look for the solution of problem $(AP_{N,\epsilon})$ with the help of the Shauder's fixed point theorem. Namely, let $\Re$ be the set of all continuous functions $u(x)$ in Ω such that

$$|u|_\Omega^{(0)} \le K_0,$$

where the constant K_0 is to be defined later. For each fixed $u \in \Re$ the linear equation

$$v + H_\epsilon(u, x) = \tau v_{xx} + f, \quad x \in \Omega,$$

$$f(x) = \theta^{n-1}(x) + \chi^{n-1}(x),$$

with the boundary conditions

$$v(i) = g(i, n\tau), \quad i = 0, 1,$$

has unique solution $v \in W_p^1(\Omega)$ for any $p > 1$ and

$$\|v\|_p^{(2)} \le K(p, f, g). \tag{12}$$

Here the constant $K(p, f, g)$ doesn't depend on the function $u \in \Re$ and the parameter ϵ, because for all $u \in \Re$ and $\epsilon > 0 : 0 \le H_\epsilon(u, x) \le 1$. From the embedding theorem [1] it follows that if $p > 2$ then $v \in C^{1+\alpha}(\overline{\Omega})$ with some $\alpha > 0$ and

$$|v|_\Omega^{(\alpha)} \le C \cdot K(p, f, g),$$

where C is some absolute constant. So the function v is operator on the function $u \in \Re$: $v = \Psi(u)$. This operator maps the set $\Re$ into its compact part if we put $K_0 = C \cdot K(p, f, g)$. The continuity of Ψ follows from

$$|H_\epsilon(u_1(x), x) - H_\epsilon(u_2(x), x)| \le \frac{1}{\epsilon}|u_1(x) - u_2(x)|.$$

Therefore there exists at least one fixed point $u^\epsilon(x)$ of Ψ, which is at the same time the solution for the nonlinear equation

$$u^\epsilon + H_\epsilon(u^\epsilon, x) = \tau u^\epsilon_{xx} + f.$$

For $p > 2$ the estimate (12) allows to choose such subsequences $\{u^{\epsilon_k}\}$, $\{H_{\epsilon_k}(u^{\epsilon_k}, x)\}$ that

$$u^{\epsilon_k} \to \theta^n \text{ strongly in } C(\overline{\Omega}) \text{ and weakly in } W_2^2(\Omega),$$

$$H_{\epsilon_k}(u^{\epsilon_k}, x) \to \chi^n \text{ almost everywhere in } \Omega,$$

because for each fixed $x \in \Omega$ the function $H_\epsilon(u, x)$ coincides either with $H_\epsilon(u - \theta_*^-)$ or with $H_\epsilon(u - \theta_*^+)$. It is obvious that $\{\theta^n, \chi^n\}$ is a solution of the problem (AP_N).

The uniqueness is proved standardly if we take into account the inequality

$$(\theta_1^n - \theta_2^n) \cdot (\chi_1^n - \chi_2^n) \ge 0.$$

5.2 Proof of lemma 2

To prove lemma let us multiply each equation for θ^n by the expression

$$(\theta^n(x) - \theta^{n-1}(x)) - (g(x, n\tau) - g(x, (n-1)\tau)),$$

integrate with respect to spatial variable and take sum with respect to n from $n = 1$ till $n = N_0 \le N$. By the construction of θ^n, χ^n the term

$$(\theta^n - \theta^{n-1}) \cdot (\chi^n - \chi^{n-1})$$

is nonnegative and we neglect it. Finally fulfilling integration by parts with respect to time, we get the inequality

$$\sum_{n=1}^{N_0} \tau \int_\Omega \frac{1}{\tau^2}(\theta^n - \theta^{n-1})^2 dx + \int_\Omega \left|\theta_x^{N_0}\right|^2 dx \le$$

$$\le \sum_{n=1}^{N_0} \tau \int_\Omega |\theta_x^n|^2 dx + C_1.$$

Together with some analogy of Gronwall's inequality for differences it gives the desired estimate.

5.3 Proof of lemma 3

Let us consider the set Q^- of all points (x_0, t_0) which satisfy the conditions of lemma with $\theta(x_0, t) = \theta_*^-$:

$$\theta_*^- < \theta(x_0, t) < \theta_*^+, \quad 0 < t_1(x_0) \le t < \overline{t}^*(x_0),$$

$$\theta(x_0, t) = \theta_*^-, \quad \overline{t}^*(x_0) \le t \le \overline{t}(x_0),$$

$$\theta(x_0, t) < \theta_*^+, \quad \overline{t}(x_0) \le t \le t_0.$$

Moreover we treat only the subset

$$Q_m^- = Q^- \cap \{(x, t)\, |x - x^m| < \delta, \ |t - t^m| < \delta\}$$

with some small $\delta > 0$. Here $(x^m, t^m) \in Q^-$ and $Q^- = \bigcup_{m=1}^{\infty} Q_m^-$. Without loss of generality we can assume that

$$\theta_*^- \le \theta(x_0, t) < \theta_*^+ \text{ for } \overline{t}(x_0) \le t \le t_0.$$

In the interval $|x - x^m| < \delta$ we consider the set N^- of all points $\overline{x}$ which have no any small neighborhood $|\overline{x} - x| < \lambda$ such that

$$\chi(x, t) = 0 \text{ for } |\overline{x} - x| < \lambda, \ \overline{t}^*(\overline{x}) \le t \le t^m.$$

To prove lemma it remains only to show that N^- is a set of measure zero. The structure of N^- is described by

Statement A. *For all $\overline{x} \in N^-$ there exists either a right-hand side interval $O(\overline{x}) = (\overline{x}, \overline{x}+\gamma)$ or a left-hand side interval $O(\overline{x}) = (\overline{x}-\gamma, \overline{x})$, such that for any point $x \in O(\overline{x})$ there exists a time-moment $\tilde{t} < \tilde{t}^*(\overline{x})$ where $\theta(x, \tilde{t}) < \theta_*^-$.*

Due to continuity of the temperature this statement means that $\chi(x,t) = 0$ for $x \in O(\overline{x})$, $\tilde{t}^*(\overline{x}) \le t \le t^m$.

Put

$$O = \cup O(\overline{x}), \quad x \in N^-.$$

The set O is open therefore

$$O = \bigcup_{p=1}^{\infty} (\alpha_p, \beta_p), \quad (\alpha_p, \beta_p) \cap (\alpha_q, \beta_q) = \emptyset, \quad p \ne q.$$

On the other hand

$$O = \bigcup_{k=1}^{\infty} O(\overline{x}_k).$$

Let $\overline{x} \in N^-$, $\overline{x} \ne \overline{x}_k, \alpha_p, \beta_p$. Because $O(\overline{x}) \subset O$, there exists interval (α_p, β_p) such that $O(\overline{x}) \subset (\alpha_p, \beta_p)$ and $\overline{x} \in (\alpha_p, \beta_p)$. Therefore $\overline{x} \in O$ and there exists at least one point $\overline{x}_k$ such that $\overline{x} \in O(\overline{x}_k)$. The statement A implies the equality $\chi(x,t) = 0$ for $x \in O(\overline{x}_k)$ and $\tilde{t}^*(\overline{x}_k) \le t \le t^m$. So $\tilde{t}^*(\overline{x}) < \tilde{t}^*(\overline{x}_k)$. But by construction of $O(\overline{x}_k)$ for $\overline{x} \in O(\overline{x}_k)$ there exists a moment $\tilde{t}(\overline{x}) < \tilde{t}^*(\overline{x}_k)$ such that $\theta(\overline{x}, \tilde{t}(\overline{x})) < \theta_*^-$. Obviously $\tilde{t}(\overline{x}) < \tilde{t}^*(\overline{x})$. Thus $\chi(x,t) = 0$ in some small neighborhood of $\overline{x}$ for $\tilde{t}^*(\overline{x}) \le t \le t^m$. The previous statement contradicts to supposition $\overline{x} \in N^-$. So N^- is composed only of the points $\overline{x}_k$, α_p, β.

The case $\theta(x_0, t) = \theta_*^+$ for $\tilde{t}^*(x_0) \le t \le \tilde{t}(x_0)$ is considered by analogy to the first one.

Proof of statement A

Let us assume the contrary: for every $\gamma > 0$ there exist points $x_1, x_2, \overline{x}-\gamma < x_1 < x_2 < \overline{x}+\gamma$ such that

$$\theta_*^- \le \theta(x_i, t), \quad t_1 \le t \le \tilde{t}^*, \quad i = 1, 2..$$

If the parameter γ is chosen from the condition

$$\theta_*^- < \theta(x, t_1), \quad x_1 \le x \le x_2$$

then $\theta_*^- \le \theta(x,t)$ both at the initial moment and on the boundary of the domain

$$Q = \left\{(x,t) \mid x_1 < x < x_2, \quad t_1 < t < \tilde{t}^*\right\},$$

and $\theta(x,t) = \theta_*^-$ at the interior point $(\overline{x}, \tilde{t}^*) \in Q$.

If we examined the exact heat equation, we might conclude that it is impossible due to the maximum principle. We are going to show that for the approximate equation the case is the same. Let us consider the approximate solutions $\theta^N(x,t)$. Without loss of generality we can assume $N = 2^m$, $m = 1, 2, ..., t_1 = k \cdot \tau$ and

$$\theta^k(x) = \theta^N(x, t_1) > \theta_*^-, \quad x_1 \le x \le x_2, \quad N > N_0.$$

We begin with proving that

$$\chi^N(x, n\cdot\tau) \le \chi^N(x, (n-1)\cdot\tau), \quad x_1 \le x \le x_2, \quad n = k+1, ..., \left[\tilde{t}^*/\tau\right] + 1. \qquad (13)$$

This inequality is based upon the following two statements

Statement B. *Let $u(x)$ be a solution for the problem*

$$u - \tau \cdot u_{xx} = f, \quad a < x < b; \quad u(a) = u(b) = 0.$$

Then $u(x) \equiv 0$ if $f = 0$ a.e. in (a,b) and $u(x) > 0 (< 0)$ if $f(x) \ge 0 (\le 0)$ and

$$mes\{x \mid f(x) > 0 (< 0)\} > 0.$$

Statement C. *Let*

$$\theta^N(x, (n-1)\cdot\tau) > \theta_*^- \quad \text{for} \quad a < x < b.$$

Then there isn't any maximal interval $(\alpha, \beta) \subset (a,b)$ where

$$\theta^N(x, n\cdot\tau) < \theta_*^-.$$

The proof of statement B follows from the exact formula for the solution.

To prove statement C suppose that there exists an maximal interval (α, β) where $\theta^N(x, n\cdot\tau) < \theta_*^-$ and $\theta^N(\alpha, n\cdot\tau) = \theta^N(\beta, n\cdot\tau) = 0$. By the construction of χ^N, $\chi^n(x) = 0$ in (α, β). So from statement B for $u = \theta^n - \theta_*^-$, $f = \theta^{n-1} - \theta_*^- + \chi^n > 0$ it follows $\theta^n(x) > \theta_*^-$ in (α, β). But this inequality contradicts to the initial supposition.

Let us consider the moment $t = t_1 + \tau$. As follows from statement C there isn't any inner maximal interval where $\theta^{k+1}(x) < \theta_*^-$. So for $\theta^{k+1}(x)$ we have the unique structure: there exists an interval (a_1, b_1) where $\theta^{k+1}(x) > \theta_*^-$ and $\theta^{k+1}(x) < \theta_*^-$ in $J_1 = (x_1, a_1) \cup (b_1, x_2)$ (note that $a_1 > x_1$ if $\theta^{k+1}(x_1) < \theta_*^-$ and $b_1 < x_2$ if $\theta^{k+1}(x_2) < \theta_*^-$). For $x \in J_1 : \chi^{k+1}(x) = 0$ and $\chi^{k+1}(x) \le \chi^k(x)$. For $x \in (a_1, b_1)$ $\chi^{k+1}(x) = \chi^k(x)$ by the construction of χ^N. So the inequality (13) is valid for $n = k+1$. Because $\chi^N(x,t) = 0$ for $t_1 + \tau \le t \le \tilde{t}^*$ and $x \in J_1$ we can now examine only the interval (a_1, b_1). Step by step we get (13) for all $n \le \left[\tilde{t}^*/\tau\right] + 1$.

Let us choose the functions $\overline{\theta}^n(x)$, $n = k, ..., N$ as solutions for the problem

$$\overline{\theta}^{n+1} - \tau \cdot \overline{\theta}^{n+1}_{xx} = \overline{\theta}^n; \quad \overline{\theta}^{n+1}(i) = \theta^n(i), \quad i = 0, 1; \quad \overline{\theta}^k(x) = \theta^k(x).$$

We put into discussion the function $\overline{\theta}^N(x,t)$ that is the piecewise linear interpolation in t of the functions $\overline{\theta}^n(x)$. From (13) and statement B for $u = \theta^n - \overline{\theta}^n$ it follows $\overline{\theta}^N(x,t) \le \theta^N(x,t)$. Therefore

$$\overline{\theta}(x,t) = \lim_{N\to\infty} \overline{\theta}^N(x,t) \le \lim_{N\to\infty} \theta^N(x,t) = \theta(x,t).$$

It is easy to see that $\overline{\theta}(x,t)$ is a solution for the heat equation in Q and achieves its minimum θ_*^- at the interior point $(\overline{x}, \tilde{t}^*)$. This fact contradicts to the maximum principle. The statement is proved.

5.4 Proof of lemma 4

Let $\overline{x} \in M_*^-(0)$. It will be enough to pay attension to the set

$$Q = \{(x,t)\,|x - x_0| < \delta,\ |t - t_0| < \delta\},$$

where $x_0 \neq \alpha_k^-, \beta_k^-$; $x_0 \notin M^-$. The parameter δ is chosen from conditions

$$J = (x_0 - \delta, x_0 + \delta) \subset (\alpha_k^-, \beta_k^-) \text{ for some } k,$$

$$\theta_*^- < \theta(x,t) < \theta_*^+ \text{ for } (x,t) \in Q.$$

If $\chi_0(x) = 0$ for $x \in (a,b) \subset J$, then $\chi(x,t) = 0$ for $x \in (a,b)$ and $0 \le t \le t_0$. It is known that there exists at most countable set of such intervals: $J_0 = \bigcup\limits_{m=1}^{\infty} (a_m, b_m)$. Thus if $\overline{x} \notin [a_m, b_m]$, $m = 1, 2, \dots$, then for any sufficiently small $\gamma > 0$

$$I^-(\gamma) = \int\limits_{\overline{x}-\gamma}^{\overline{x}} \chi_0(x)dx > 0,\ I^+(\gamma) = \int\limits_{\overline{x}}^{\overline{x}+\gamma} \chi_0(x)dx > 0. \tag{14}$$

On the other hand for any $\overline{x} \in J$ there are only two alternatives:

(i) there exists an interval $O(\overline{x})$ ($(\overline{x} - \gamma, \overline{x})$-left- or $(\overline{x}, \overline{x} + \gamma)$-right-hand side) where for all $x \in O(\overline{x})$ $\theta(x, t') < \theta_*^-$ for some $t' = t'(x)$, $0 < t' < t_0$;

(ii) for any sufficiently small $\gamma > 0$ there exist $x_0^\pm$, $x_0^- < \overline{x} < x_0^+$, $|x_0^\pm - \overline{x}| < \gamma$ such that

$$\theta(x_0^\pm, t) \ge \theta_*^- \text{ for } 0 \le t \le t_0.$$

Determine

$$O = \cup O(\overline{x}),\quad \overline{x} \in J,$$

as it was mentioned above in lemma 3

$$O = \bigcup_{q=1}^{\infty} O(\overline{x}_q) = \bigcup_{p=1}^{\infty} (c_p, d_p)$$

where (c_p, d_p), $p = 1, 2, \dots$, are the maximal intervals.

In according with the construction, for almost all $\overline{x} \in J$ ($\overline{x} \neq \overline{x}_q, c_p, d_p$) either $\chi(\overline{x}, t_0) = 0$ (because $\overline{x} \in (a_m, b_m)$ or $\overline{x} \in O(\overline{x}_q)$) or for a sufficiently small $\gamma > 0$ the alternative (ii) realizes and (14) takes place:

$$I^\pm(\gamma) > 0, \text{ and } \theta(x_0^\pm, t) \ge \theta_*^- \text{ for } 0 \le t \le t_0. \tag{15}$$

Now we have to prove that in (15): $\chi(\overline{x}, t_0) = 0$ or $\chi(\overline{x}, t_0) = 1$. Let the parameter γ be determined from the condition

$$\theta(x, t_0) > \theta_*^- \text{ for } |x - \overline{x}| < \gamma.$$

The continuous functions $\theta^N(x,t)$ uniformly converge to the continuous function $\theta(x,t)$. So for every $\epsilon > 0$ there exists $N_0(\epsilon)$ such that for $N \ge N_0$

$$\theta^N(x, t_0) > \theta_*^- \text{ for } |x - \overline{x}| < \gamma,$$

$$\theta^N(x_0^\pm, t) \ge \theta_*^- - \epsilon \text{ for } 0 \le t \le t_0.$$

Due to condition (15) we can choose ϵ to satisfy the condition

$$I_0^\pm(\gamma) = \left| \int\limits_{x_0^\pm}^{\overline{x}} (x - x_0^\pm)\chi_0(x)dx \right| > 2 \cdot t_0 \cdot \epsilon. \tag{16}$$

The main statement follows from the inequalities

$$\theta^N(x,t) \ge \theta_*^-,\quad \chi^N(x,t) \ge \chi_0(x) \text{ for } N \ge N_0 \tag{17}$$

in some small neighborhood $\hat{Q}$ of $(\overline{x}, t_0)$ where $\theta^N(x,t) > \theta_*^-$.

The fact is that by the construction $\chi^N(x,t) = 0$ if $\chi_0(x) = 0$, and $\chi^N(x,t) = 1$ if $\chi_0(x) > 0$. So $\chi^N(x,t) = sgn\ \chi_0(x)$ in $\hat{Q}$ for all $N \ge N_0$ and

$$\lim_{N \to \infty} \chi^N(x,t) = sgn\ \chi_0(x).$$

We will prove (17) using the barrier functions $\overline{\theta}^N$, that satisfy the problem $AP(N)$ in the domain $(x_0^-, x_0^+) \times (0, t_0)$ with the following boundary and initial conditions:

$$\overline{\theta}^N(x_0^\pm, n \cdot \tau) = \overline{\theta}^n(x_0^\pm) = \theta_*^- - \epsilon,\ n = 1, 2, \dots, N;$$

$$\overline{\theta}^0(x) = \theta_*^-,\ \overline{\chi}^0(x) = \chi_0(x),\ x_0^- < x < x_0^+.$$

As we have already proved in lemma 1, there exists unique solution $\left\{\overline{\theta}^N, \overline{\chi}^N\right\}$. Let us show that this solution has very simple structure

$$\overline{\chi}^n(x) = 0,\ \overline{\theta}^n(x) < \theta_*^- \text{ for } x_0^- < x < x_n^-,\ x_n^+ < x < x_0^+,$$

$$\overline{\chi}^n(x) = \chi_0(x),\ \overline{\theta}^n(x) = \theta_*^- \text{ for } x_n^- < x < x_n^+.$$

Moreover $x_n^- \le x_{n+1}^- \le x_{n+1}^+ \le x_n^+$ and

$$x_n^- \le \overline{x} - \gamma_0,\quad \overline{x} + \gamma_0 \le \tag{18}$$

for some small $\gamma_0 > 0$ for all $n = 1, 2, \dots, N$, $N \ge N_0$.

As usual we suppose $\overline{\theta}^{n-1}, \overline{\chi}^{n-1}$ to have the described structure. Then $\overline{\theta}^n(x)$ hasn't any inner maximal interval (α, β) where $\overline{\theta}^n(x) > \theta_*^-$. Really for such intervals always $\overline{\chi}^{n-1}(x) \le \overline{\chi}^n(x)$ (if $\overline{\chi}^{n-1}(x) = 0$, then $\overline{\chi}^n(x) = 0$; if $\overline{\chi}^{n-1}(x) > 0$, then $\overline{\chi}^n = 1$), and furthermore, in these intervals the function $v = \overline{\theta}^n - \theta_*^-$ satisfies the equation

$$v - \tau \cdot v_{xx} = f, \tag{19}$$

with $f = \overline{\chi}^{n-1} - \overline{\chi}^n + \overline{\theta}^{n-1} - \theta_*^- \le 0$. Therefore due to statement B $v \le 0$ or $\overline{\theta}^n \le \theta_*^-$ over the interval (α, β). So the unique possibility for $\overline{\theta}^n$ is to have the structure described above. Note that $x_{n-1}^- \le x_n^-$, $x_n^+ \le x_{n-1}^+$. In the contrary case the solution v of the equation (19) with non positive f over the intervals (x_0^-, x_n^-), (x_n^+, x_0^+) achieves its maximum at the boundary points x_n^-, x_n^+, where $\overline{\theta}_x^n(x_n^- - 0) > 0$, $\overline{\theta}_x^n(x_n^+ + 0) < 0$. These inequalities contradict to the continuity of $\overline{\theta}_x^n(x)$ on the interval (x_0^-, x_0^+).

Now we will prove inequalities

$$\theta^n(x) \ge \overline{\theta}^n(x), \ \chi^n(x) \ge \overline{\chi}^n(x) \tag{20}$$

for $x \in (x_0^-, x_0^+)$ and $n = 1, 2, ..., [t_0/\tau] + 1$.

As usual, let us suppose (20) to be fulfilled for $k < n$ and consider the difference $u = \theta^n - \overline{\theta}^n$ over the interval (x_0^-, x_0^+) :

$$u - \tau \cdot u_{xx} = f - \chi^n + \overline{\chi}^n, \quad u(x_0^\pm) \ge 0,$$

where $f = \theta^{n-1} - \overline{\theta}^{n-1} + \chi^{n-1} - \overline{\chi}^{n-1} \ge 0$ as it was supposed. If $u < 0$ at some internal point, then there exists at least one maximal interval (α, β) where $u < 0$ or $\theta^n(x) < \overline{\theta}^n(x) \le \theta_*^-$. Therefore $\chi^n(x) = 0$ and $f - \chi^n + \overline{\chi}^n \ge 0$ over the interval (α, β). It contradicts to statement B. So (20) is valid for all $n = 1, 2, ..., [t_0/\tau] + 1$. Taking into account (18) we have $\overline{\theta}^n(x) = \theta_*^-$, $\overline{\chi}^n(x) = \chi_0(x)$ for $|\overline{x} - x| < \gamma_0$. Desired inequalities (17) follow from (20) if $\hat{Q} = (\overline{x} - \gamma_0, \overline{x} + \gamma_0) \times (t_0 - \delta, t_0 + \delta)$.

To complete the proof of the lemma we have to prove now (18). After multiplication the equations for $(\theta^k - \theta_*^-)$ by $(x - x_0^-)$, integration by parts over the interval (x_0^-, x_0^+) and summing over $k = 1, 2, ..., n$, we obtain

$$\int_{x_0^-}^{x_n^-} (x - x_0^-) \cdot \chi_0(x) dx + \int_{x_0^-}^{x_n^-} (x - x_0^-) \cdot \left|\theta^n - \theta_*^-\right| dx = \tau \cdot n \cdot \epsilon \le t_0 \cdot \epsilon.$$

Because

$$\lim_{\xi \to \overline{x}} \int_{x_0^-}^{\xi} (x - x_0^-) \cdot \chi_0(x) dx > 2 \cdot t_0 \cdot \epsilon,$$

there exists $\gamma_0 > 0$ such that

$$x_n^- < \overline{x} - \gamma_0 \text{ for } n = 1, 2, ..., N; \ N \ge N_0(\epsilon).$$

The same argument applies to point x_n^+.

We can repeat above reasons, and prove that for all $\overline{x} \in J$ and γ_0 responding to the condition (14) there exists sufficiently small moment $t_* > 0$ such that

$$\theta^N \le \theta_*^-, \ \chi^N \le \chi_0 \text{ for } 0 < t < t_*, \ |\overline{x} - x| < \gamma_0.$$

Using barrier functions by the same way we show the contrary inequalities:

$$\theta^N \ge \theta_*^-, \ \chi^N \ge \chi_0 \text{ for } 0 < t < t_*, \ |\overline{x} - x| < \gamma_0.$$

So $\theta^N(x,t) = \theta_*^-$, $\chi^N(x,t) = \chi_0(x)$ for $0 < t < t_*$, $|\overline{x} - x| < \gamma_0$, $N \ge N_0$, and for $\theta(x,t)$, $\chi(x,t)$ we have the same equalities in the mentioned neighborhood.

The case $\overline{x} \in M_*^+(0)$ is considered by analogy.

5.5 Nonexistence for a solution of problem (1), (3)-(5) ... erator 3

Let us consider the problem (1), (3)-(5) where

$$\theta_*^- < g(i,t) < \theta_*^+, \ t > 0, \ i = 1, 2,$$

$$\theta_0(x) = \theta_*^-, \ 0 < \chi_0(x) < 1, \ x \in \Omega.$$

Suppose that there exists at least one solution $\{\theta_3, \chi_3\}$ with continuous temperature $\theta_3(x,t)$. By the definition of operator 3 we have $\frac{\partial}{\partial t}(\chi_3) \le 0$ and obviously

$$\theta_*^- \le \theta_3(x,t) \le \theta_*^+.$$

Let $t = h(x)$ be the first time moment such that

$$\theta_3(x,t) = \theta_*^-, \ 0 \le t \le h(x), \ \theta_3(x,t) > \theta_*^-, \ t > h(x).$$

Almost analogously to [2] we can prove the next statements: $h(x)$ is a continuous ... $h(x)$ strictly increases on some interval $(0, x^-)$; $h(x) = T$ $(0 < t < T)$ on the interval (x^-, x^+), if $h(x^-) = T$; $h(x)$ strictly decreases on interval $(x^+, 1)$. For this function ... exists on the interval $(0, x^-)$ inverse function $R^-(t)$ such that

$$R^-(h(x)) = x, \ 0 \le \frac{dR^-}{dt}(t) < \infty \text{ almost everywhere},$$

$$\theta > \theta_*^-, \ \theta_t = \theta_{xx}, \ 0 < x < R^-(t), \ 0 < t < T,$$

$$\theta(R^-(t), t) = 0, \ 0 < t < T.$$

Using the maximum principle with the barrier function $k \cdot (R^-(t_0) - x)$ for $\frac{1}{2}t_0 \le t \le ...$ the inequalities

$$0 \ge \theta_x(R^-(t_0) - 0, t_0) \ge -k(t_0) \ge -\infty.$$

can be proved. Therefore the standard technique may be applied to (1) and we obtain the Stefan condition

$$\theta_x(R^-(t_0) - 0, t_0) = \chi_0(R^-(t_0)) \cdot \frac{dR^-}{dt}(t_0),$$

that is impossible, because $\chi_0(R^-(t_0)) > 0$, $\frac{dR^-}{dt}(t_0) \ge 0$.

Acknowledgements

We would like to thank A.Visintin and I.Götz for fruitful discussions.

References.

[1] Ladyzhenskaya O. A., Solonnikov V. A., Uraltseva N. N. *Linear and quasilinear equations of parabolic type*. Trans. Math. Monographs, vol. 23, AMS, Providence (1984).

[2] Meirmanov A. *The Stefan problem*. Walter De Gruyter, Berlin-New York (1992)

[3] Visintin A. *A phase transition problem with delay*. Control and Cybernetics, 1-2, 5-18 (1982)

above. Note that $x_{n-1}^- \le x_n^-$, $x_n^+ \le x_{n-1}^+$. In the contrary case the solution v of the equation (19) with non positive f over the intervals (x_0^-, x_n^-), (x_n^+, x_0^+) achieves its maximum at the boundary points x_n^-, x_n^+, where $\overline{\theta}_x^n(x_n^- - 0) > 0, \overline{\theta}_x^n(x_n^+ + 0) < 0$. These inequalities contradict to the continuity of $\overline{\theta}_x^n(x)$ on the interval (x_0^-, x_0^+).

Now we will prove inequalities

$$\theta^n(x) \ge \overline{\theta}^n(x), \ \chi^n(x) \ge \overline{\chi}^n(x) \tag{20}$$

for $x \in (\ x_0^-\ ,\ x_0^+)$ and $n = 1, 2, ..., [t_0/\tau] + 1$.

As usual, let us suppose (20) to be fulfilled for $k < n$ and consider the difference $u = \theta^n - \overline{\theta}^n$ over the interval (x_0^-, x_0^+) :

$$u - \tau \cdot u_{xx} = f - \chi^n + \overline{\chi}^n, \quad u(x_0^\pm) \ge 0,$$

where $f = \theta^{n-1} - \overline{\theta}^{n-1} + \chi^{n-1} - \overline{\chi}^{n-1} \ge 0$ as it was supposed. If $u < 0$ at some internal point, then there exists at least one maximal interval (α, β) where $u < 0$ or $\theta^n(x) < \overline{\theta}^n(x) \le \theta_*^-$. Therefore $\chi^n(x) = 0$ and $f - \chi^n + \overline{\chi}^n \ge 0$ over the interval (α, β). It contradicts to statement B. So (20) is valid for all $n = 1, 2, ..., [t_0/\tau] + 1$. Taking into account (18) we have $\overline{\theta}^n(x) = \theta_*^-$, $\overline{\chi}^n(x) = \chi_0(x)$ for $|\overline{x} - x| < \gamma_0$. Desired inequalities (17) follow from (20) if $\hat{Q} = (\overline{x} - \gamma_0, \overline{x} + \gamma_0) \times (t_0 - \delta, t_0 + \delta)$.

To complete the proof of the lemma we have to prove now (18). After multiplication the equations for $(\theta^k - \theta_*^-)$ by $(x - x_0^-)$, integration by parts over the interval (x_0^-, x_0^+) and summing over $k = 1, 2, ..., n$, we obtain

$$\int_{x_0^-}^{x_n^-} (x - x_0^-) \cdot \chi_0(x) dx + \int_{x_0^-}^{x_n^-} (x - x_0^-) \cdot \left|\theta^n - \theta_*^-\right| dx = \tau \cdot n \cdot \epsilon \le t_0 \cdot \epsilon.$$

Because

$$\lim_{\xi \to \overline{x}} \int_{x_0^-}^{\xi} (x - x_0^-) \cdot \chi_0(x) dx > 2 \cdot t_0 \cdot \epsilon,$$

there exists $\gamma_0 > 0$ such that

$$x_n^- < \overline{x} - \gamma_0 \text{ for } n = 1, 2, ..., N; \ N \ge N_0(\epsilon).$$

The same argument applies to point x_n^+.

We can repeat above reasons, and prove that for all $\overline{x} \in J$ and γ_0 responding to the condition (14) there exists sufficiently small moment $t_* > 0$ such that

$$\theta^N \le \theta_*^-, \ \chi^N \le \chi_0 \text{ for } 0 < t < t_*, \ |\overline{x} - x| < \gamma_0.$$

Using barrier functions by the same way we show the contrary inequalities:

$$\theta^N \ge \theta_*^-, \ \chi^N \ge \chi_0 \text{ for } 0 < t < t_*, \ |\overline{x} - x| < \gamma_0.$$

So $\theta^N(x,t) = \theta_*^-$, $\chi^N(x,t) = \chi_0(x)$ for $0 < t < t_*$, $|\overline{x} - x| < \gamma_0$, $N \ge N_0$, and for $\theta(x,t)$, $\chi(x,t)$ we have the same equalities in the mentioned neighborhood.

The case $\overline{x} \in M_*^+(0)$ is considered by analogy.

5.5 Nonexistence for a solution of problem (1), (3)-(5) [illegible]erator 3

Let us consider the problem (1), (3)-(5) where

$$\theta_*^- < g(i,t) < \theta_*^+, \ t > 0, \ i = 1, 2,$$

$$\theta_0(x) = \theta_*^-, \ 0 < \chi_0(x) < 1, \ x \in \Omega.$$

Suppose that there exists at least one solution $\{\theta_3, \chi_3\}$ with continuous [illegible] $\theta_3(x,t)$. By the definition of operator 3 we have $\frac{\partial}{\partial t}(\chi_3) \le 0$ and obviously

$$\theta_*^- \le \theta_3(x,t) \le \theta_*^+.$$

Let $t = h(x)$ be the first time moment such that

$$\theta_3(x,t) = \theta_*^-, \ 0 \le t \le h(x), \ \theta_3(x,t) > \theta_*^-, \ t > h(x).$$

Almost analogously to [2] we can prove the next statements: $h(x)$ is a continuous [illegible] $h(x)$ strictly increases on some interval $(0, x^-)$; $h(x) = T$ $(0 < t < T)$ on the [illegible] (x^-, x^+), if $h(x^-) = T$; $h(x)$ strictly decreases on interval $(x^+, 1)$. For this function [illegible] exists on the interval $(0, x^-)$ inverse function $R^-(t)$ such that

$$R^-(h(x)) = x, \ 0 \le \frac{dR^-}{dt}(t) < \infty \text{ almost everywhere},$$

$$\theta > \theta_*^-, \ \theta_t = \theta_{xx}, \ 0 < x < R^-(t), \ 0 < t < T,$$

$$\theta(R^-(t), t) = 0, \ 0 < t < T.$$

Using the maximum principle with the barrier function $k \cdot (R^-(t_0) - x)$ for $\frac{1}{2} t_0 \le$ [illegible] the inequalities

$$0 \ge \theta_x(R^-(t_0) - 0, t_0) \ge -k(t_0) \ge -\infty.$$

can be proved. Therefore the standard technique may be applied to (1) and we obtain [illegible] Stefan condition

$$\theta_x(R^-(t_0) - 0, t_0) = \chi_0(R^-(t_0)) \cdot \frac{dR^-}{dt}(t_0),$$

that is impossible, because $\chi_0(R^-(t_0)) > 0$, $\frac{dR^-}{dt}(t_0) \ge 0$.

Acknowledgements

We would like to thank A.Visintin and I.Götz for fruitful discussions.

References.

[1] Ladyzhenskaya O. A., Solonnikov V. A., Uraltseva N. N. *Linear and quasilinear equations of parabolic type*. Trans. Math. Monographs, vol. 23, AMS, Providence (1984).

[2] Meirmanov A. *The Stefan problem*. Walter De Gruyter, Berlin-New York (1992)

[3] Visintin A. *A phase transition problem with delay*. Control and Cybernetics, 1-2, 5-18 (1982)

[4] Visintin A. *On the Preisach model for hysteresis*. Nonlinear Anal. 9, 977-996, (1984)

[5] Visintin A. Hysteresis and semigroups. In: Models of hysteresis (A.Visintin, ed.) Longman, Harlow, 192-206,(1993).

[6] Visintin A. *Differential models of hysteresis*. Springer, Berlin, 1995.

A. Meirmanov, N. Shemetov
Departamento de Matematica
Universidade da Beira Interior
R. Marquês d' Avila e Bolama
6200 Covilha - Portugal

Advances in Mathematical Sciences and Applications
Gakkōtosho, Tokyo, Vol. 6, No. 2(1996), pp.559-587

THE N-DIMENSIONAL QUASISTATIC PROBLEM OF THERMOELASTIC CONTACT WITH BARBER'S HEAT EXCHANGE CONDITIONS

XIANGSHENG XU

(Communicated by J. R. Ockendon; Received March 18, 1994; Revised December 9, 1994

ABSTRACT: The existence of a weak solution is established for the quasistatic problem of unilateral frictionless contact of a thermoelastic body with a rigid foundation in which the Barber heat exchange condition is assumed for the temperature.

KEY WORDS AND PHRASES: Thermoelastic contact, Barber's heat exchange condition, penalty method.

1. Introduction.

This paper is a continuation of the work initiated in [SS], where the authors study the evolution of temperature and displacement of an elastic body that is subject to external forces and heat sources, and that may come in contact with a rigid foundation. The acceleration of the elastic body i assumed to be negligible and the contact frictionless. The resulting mathematical problem consists of an elliptic system with the time variable appearing as a parameter coupled with a parabolic equation of energy, along with Signorini's contact condition for the displacement and Dirichlet's hea exchange condition for the temperature. In [SS] this problem is formulated as a variational inequality, and the existence of a "strong-weak" solution is established under the assumption that the coefficient of thermal expansion is sufficiently small. However, as is pointed out in [ASSW], the use of the Dirichlet heat exchange condition for the temperature is not appropriate from the point of view of applications. In this paper, we shal consider a more realistic modeling of the heat exchange based upon the work of Barber [B]. A detailed discussion of this will be presented in Section 2 The main purpose of this paper is to show that the N-dimensional quasistati thermoelastic contact problem with Barber's heat exchange condition has a