# THE EXACT SOLUTION OF THE BORTZ APPROXIMATE EQUATION AND CONSTRUCTION OF THE QUATERNION ORIENTATION ALGORITHM OF STRAPDOWN INS ON ITS BASIS*

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***Abstract*—The exact solution of the Bortz approximate linear equation has made it possible to solve the problem of determining the quaternion of orientation of a rigid body for an arbitrary angular velocity and small angle of rotation of a rigid body with the help of quadratures. Proceeding from this solution, the following approach to the design of a new algorithm for computation of strapdown INS orientation is proposed.**

***Keywords—analytic solytion, algorithm, orientation, angular velocity, rigid body, strapdown INS, quaternion***

## I. INTRODUCTION

During operation of many strapdown inertial navigation systems (SINS) the orientation vector of a rigid body is periodically calculated by the method of approximate solution of the Bortz approximate linear differential equation (in the theory and practice of SINS construction, in ultra rapid cycles of algorithms for small angles of rotation, the nonlinear term in the Bortz equation is neglected). The angular velocity vector of a rigid body is the input quantity in the Bortz equation. Note that the full nonlinear Bortz equation for the orientation vector of a rigid body is an analog of the quaternion linear equation; the vector and the quaternion of the rigid body orientation are linked by known relations. The approximate linear vector differential Bortz equation in the literature is solved by various numerical methods, for example, by Picard's method, then the second iteration of this method in the practice of SINS can be taken for the final one. This term in the iteration formula of Picard's method is called a non-commutative rotation vector, or "coning". For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The study of non-commutative rotations (or "coning") as a kind of mechanical motion of bodies, separation of numerical algorithms for determining the orientation of a rigid body (SINS) for rapid and slow counting cycles are aimed at compensation for the effect of this phenomenon. Meanwhile, for some new angular velocity vector, which is obtained in determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector in unambiguous interchanges of variables in the motion equations for a rigid body, the approximate Bortz equation admits of an exact analytic solution. We will show this.

The problem is to determine the quaternion of orientation $\mathbf{\Lambda}$ of a rigid body with respect to an arbitrary given angular velocity vector $\boldsymbol{\omega}(t)$ and the initial angular position of a rigid body in space based on the quaternion kinematic equation known as the Darboux problem. Further, variables are replaced in accordance with the scheme $\mathbf{\Lambda}\rightarrow\mathbf{U}$, where $\mathbf{U}$ is the quaternion of the orientation of a certain introduced coordinate system, it is always possible to reverse the transition $\mathbf{U}\rightarrow\mathbf{\Lambda}$. These changes have the character of rotation transformations and reduce the initial problem of determining the orientation of a rigid body (quaternion $\mathbf{\Lambda}$) with an arbitrary variable angular velocity vector $\boldsymbol{\omega}(t)$ to the problem where the angular velocity vector $\mathbf{w}(t)$ of the introduced coordinate system, remaining generally variable in absolute value, performs a definite motion - rotates around one of the axes of the coordinate system. This motion is a generalized conical precession, which is in a good agreement with the known Poinsot's concept that any rotation of a rigid body about a fixed point can be represented as a conical motion. Finding an analytical solution of the quaternion differential equation obtained with respect to the new unknown quaternion $\mathbf{U}$ is still a difficult problem. However, the equation differing from this one only in the coefficient "1/2" in the right-hand side (i.e., with the angular velocity vector $\mathbf{w}(t)/2$) is solved in closed form. Moreover, we note that the quaternion differential equation is isomorphic to the homogeneous vector differential equation of Poisson. The resulting problem with the angular velocity vector $\mathbf{w}(t)$ and the unknown quaternion of orientation $\mathbf{U}$ is associated with the complete Bortz equation with respect to the unknown orientation vector $\boldsymbol{\varphi}$. The approximate linear Bortz equation, which is an inhomogeneous vector differential equation whose homogeneous part is equivalent to the Poisson equation with the vector coefficient **w**(t)/2, becomes analytically solvable and its solution $\boldsymbol{\varphi}^*$ is obtained in quadratures by the Lagrange method. The exact solution of the Bortz approximate linear equation has made it possible to solve the problem of determining the quaternion of orientation of a rigid body for an arbitrary angular velocity and small angle of rotation of a rigid body with the help of quadratures. Proceeding from this solution, the following approach to the design of a new algorithm for computation of SINS orientation has been implemented:

1) a new angular velocity **w**(t) of some new coordinate system was calculated using a set of components of the angular velocity of a rigid body on the basis of unambiguous replacements of the variables at each time point;

2) the exact solution of the Bortz approximate linear equation (vector of orientation) with a zero initial condition was found using the new angular velocity and the initial position of a rigid body with the help of quadratures;

3) the value of the quaternion orientation of a rigid body (SINS) was determined by the vector of orientation on the scheme $\boldsymbol{\varphi}^* \approx \boldsymbol{\varphi} \Leftrightarrow \mathbf{U} \rightarrow \boldsymbol{\Lambda}$.

During construction of the algorithm for SINS orientation at each subsequent step the replacement of the variables takes into account the previous step of the algorithm in such a way that each time the initial value of the vector orientation of a rigid body will be equal to zero. Since the proposed algorithm for the analytical solution of the approximate linear Bortz equation is exact, it has a regular character for all angular motions of a rigid body.

## II. Statement of the Problem of Determining the Orientation of a Rigid Body (SINS)

Consider the Cauchy problem for quaternion kinematic equation [1] with arbitrary given angular velocity vector-function **ω**(t), written in the following form (this problem is known as the Darboux problem):

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega}(t), \qquad (2.1)$$

$$\boldsymbol{\Lambda}(t_0) = \boldsymbol{\Lambda}_0. \qquad (2.2)$$

Here $\boldsymbol{\Lambda}(t) = \lambda_0(t) + \lambda_1(t)i_1 + \lambda_2(t)i_2 + \lambda_3(t)i_3$ is a quaternion describing the position of a rigid body in an inertial space; $\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{i}_1 + \omega_2(t)\mathbf{i}_2 + \omega_3(t)\mathbf{i}_3$ is the angular velocity vector of the rigid body specified by its projections onto body-fixed coordinate axes; $i_1, i_2, i_3$ – the units of the hypercomplex space (imaginary Hamiltonian units), which can be identified with the vectors of a three-dimensional vector space $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$; the symbol "$\circ$" stands for the quaternion product; $\boldsymbol{\Lambda}_0$ is the initial value of the quaternion $\boldsymbol{\Lambda}(t)$ at $t = t_0$, $t \in [t_0, \infty)$ ( $t_0$ set equal to 0). The problem is to find the quaternion $\boldsymbol{\Lambda}(t)$.

The problem of determining the orientation vector of a rigid body $\boldsymbol{\varphi}(t)$ [2] relative to an inertial space can also be posed by solving the exact differential Bortz equation

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\varphi} \times \boldsymbol{\omega} + \frac{1}{\varphi}\left(1 - \frac{\varphi \sin\varphi}{2(1-\cos\varphi)}\right)\boldsymbol{\varphi} \times (\boldsymbol{\varphi} \times \boldsymbol{\omega}), \qquad (2.3)$$

where "$\times$" means the vector product. In equation (2.3) the input quantity is the angular velocity vector $\boldsymbol{\omega}$. Note that the non-linear Bortz equation (2.3) for the orientation vector of a rigid body is an analogue of the quaternion linear equation (2.1); vector and quaternion are connected by the relations:

$$\boldsymbol{\varphi} = \varphi\mathbf{e}, \quad \mathbf{e} = e_1\mathbf{i}_1 + e_2\mathbf{i}_2 + e_3\mathbf{i}_3, \quad |\mathbf{e}| = \left(e_1^2 + e_2^2 + e_3^2\right)^{1/2} = 1,$$
$$\lambda_0 = \cos(\varphi/2), \quad \lambda_j = \sin(\varphi/2)e_j, \quad j = 1,2,3, \qquad (2.4)$$

where $\varphi$ is the angle of orientation of a rigid body and **e** - the Euler axis of rotation. In the practice of constructing of constructing SINS orientation algorithms by numerical solution of equation (2.3) on a time interval $t_{m-1} \le t < t_m$ the third member in this equation is neglected for small angles of rotation (it is the magnitude of the order $\varphi^2$ ). If the derived simplified (approximate) differential equation

$$\dot{\boldsymbol{\varphi}}^* = \boldsymbol{\omega} + \boldsymbol{\varphi}^* \times \boldsymbol{\omega}/2 \qquad (2.5)$$

is solved by Picard's iterative method, then the second iteration of this method is taken for the final one [2]:

$$\boldsymbol{\varphi}_m^* = \int_{t_{m-1}}^{t_m} (\boldsymbol{\omega}(t)dt + \boldsymbol{\alpha}(t) \times \boldsymbol{\omega}(t)/2)dt = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m,$$

$$\boldsymbol{\alpha}(t) = \int_{t_{m-1}}^{t_m} \boldsymbol{\omega}(\tau)d\tau, \quad \boldsymbol{\alpha}_m = \boldsymbol{\alpha}(t_m), \qquad (2.6)$$

$$\boldsymbol{\beta}(t) = \int_{t_{m-1}}^{t_m} \boldsymbol{\alpha}(\tau) \times \boldsymbol{\omega}(\tau)d\tau/2, \quad \boldsymbol{\beta}_m = \boldsymbol{\beta}(t_m),$$

where vector $\boldsymbol{\beta}$ is called a non-commutative rotation vector, or "coning". For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The study of non-commutative rotations (or "coning") as a kind of mechanical motion of bodies, separation of numerical algorithms for determining the orientation of a rigid body (SINS) for rapid and slow counting cycles are aimed at compensation for the effect of this phenomenon. Meanwhile, for some new angular velocity vector **w**(t), which is obtained in determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector **ω**(t) in unambiguous replacements of variables in the motion equations for a rigid body, the approximate Bortz equation admits of an exact analytic solution, which will be shown in what follows.

## III. The Exact Solution of the Bortz Approximate Equation and Design of the Algorithm for Determining SINS Orientation on its Basis

Let's write unambiguous replacements of variables in the problem (2.1), (2.2) [3] according to the scheme $\mathbf{\Lambda}\rightarrow\mathbf{U}$, where $\mathbf{U}$(t) is the quaternion of orientation of some introduced coordinate system (new variable), quaternion $\mathbf{V}$(t) is the generated transition operator, $\mathbf{K}$ is an arbitrary constant quaternion:

$$\mathbf{\Lambda}(t)=\mathbf{U}(t)\circ\mathbf{K}\circ\mathbf{V}(t),\quad \|\mathbf{K}\|=\|\mathbf{V}\|=1, \tag{3.1}$$

$$\mathbf{V}(t)=\left(-\mathbf{i}_1\sin \mathrm{N}(t)+\mathbf{i}_2\cos \mathrm{N}(t)\right)\circ \circ\exp(\mathbf{i}_3\mathrm{N}(t)/2)\circ\exp(\mathbf{i}_1\Omega_1(t)/2), \tag{3.2}$$

$$2\dot{\mathbf{U}}=\mathbf{U}\circ\mathbf{K}\circ\mathbf{w}(t)\circ\widetilde{\mathbf{K}}, \tag{3.3}$$

$$\mathbf{w}(t)=\mu(t)(-\mathbf{i}_1\sin \mathrm{N}(t)+\mathbf{i}_2\cos \mathrm{N}(t))-2\mathbf{i}_3\nu(t), \tag{3.4}$$

$$\begin{aligned}\mu(t)&=\omega_2(t)\cos\Omega_1(t)-\omega_3(t)\sin\Omega_1(t),\\ \nu(t)&=\omega_2(t)\sin\Omega_1(t)+\omega_3(t)\cos\Omega_1(t),\\ \mathrm{N}(t)&=\int_0^t\nu(\tau)d\tau,\quad \Omega_1(t)=\int_0^t\omega_1(\tau)d\tau,\end{aligned} \tag{3.5}$$

$$\mathbf{U}(0)=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\widetilde{\mathbf{K}}, \tag{3.6}$$

where (3.3)-(3.6) the new problem for determining the orientation of a rigid body with the new angular velocity vector $\mathbf{w}$(t), "$\|.\|$" means quaternion norm.

Finding an analytical solution to the resulting quaternion differential equation (3.3) remains a difficult task. However, the equation that differs from this one only **in** the coefficient “1/2 " on the right side (i.e. with the angular velocity vector $\mathbf{w}$(t)/2)

$$2\dot{\mathbf{\Psi}}=\mathbf{\Psi}\circ\mathbf{K}\circ\mathbf{w}(t)\circ\widetilde{\mathbf{K}}/2 \tag{3.7}$$

$$\mathbf{\Psi}(0)=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\widetilde{\mathbf{K}} \tag{3.8}$$

is solved in a closed form. Choose quaternion $\mathbf{K}$ in the form $\mathbf{K}=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)$ so that the initial conditions (3.6), (3.8) become **unit** $\mathbf{U}(0)=\mathbf{\Psi}(0)=1$. Note that this technique with quaternion $\mathbf{K}$ is important in the subsequent construction of the algorithm of SINS orientation. The solution of the Cauchy problem (3.7), (3.8) will be written as follows:

$$\mathbf{\Psi}=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\mathbf{\Phi}(t)\circ\mathbf{i}_2\circ\widetilde{\mathbf{\Lambda}}_0 \tag{3.9}$$

$$\begin{aligned}\mathbf{\Phi}(t)&=\exp(\mathbf{i}_2\mathrm{M}(t)/4)\circ\exp(-\mathbf{i}_3\mathrm{N}(t)/2),\\ \mathrm{M}(t)&=\int_0^t\mu(\tau)d\tau.\end{aligned} \tag{3.10}$$

On the basis of expressions of type (2.4) we associate the reduced quaternion problem of determining orientation (3.3)-(3.6) with the problem with a vector approximate Bortz equation of the type (2.5):

$$\begin{aligned}\dot{\boldsymbol{\varphi}}^*&=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\mathbf{w}(t)\circ\mathbf{i}_2\circ\widetilde{\mathbf{\Lambda}}_0+\\ &+\boldsymbol{\varphi}^*\times(\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\mathbf{w}(t)\circ\mathbf{i}_2\circ\widetilde{\mathbf{\Lambda}}_0)/2,\end{aligned} \tag{3.11}$$

$$\boldsymbol{\varphi}^*(0)=0 \tag{3.12}$$

We note that the homogeneous part of the vector linear differential equation (3.11) is equivalent to the solvable system (3.7) written in the form of a vector differential Poisson equation. From the Lagrange method of solving linear inhomogeneous differential systems of equations, the exact solution of the approximate Bortz equation (3.11) will have the form on the basis of (3.9), (3.10)

$$\boldsymbol{\varphi}^*=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2)\circ\widetilde{\mathbf{\Phi}}(t)\circ \circ\int_0^t\mathbf{\Phi}(\tau)\circ\mathbf{w}(\tau)\circ\widetilde{\mathbf{\Phi}}(\tau)d\tau\circ\mathbf{\Phi}(t)\circ\mathbf{i}_2\circ\widetilde{\mathbf{\Lambda}}_0. \tag{3.13}$$

Thus, the problem of determining the orientation of a rigid body (2.1)-(2.3) on the basis **of** (2.5) at small rotation angles is completely solved with the help of quadratures. We give the analytical algorithm for determining orientation of a rigid body (SINS) at arbitrary angles of rotation:

1) using the components of angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body, functions $\mu(t),\nu(t)$ are calculated at each moment of time *t* by the formulas:

$$\begin{aligned}\Omega_1(t)&=\int_0^t\omega_1(\tau)d\tau,\\ \mu(t)&=\omega_2(t)\cos\Omega_1(t)-\omega_3(t)\sin\Omega_1(t),\\ \nu(t)&=\omega_2(t)\sin\Omega_1(t)+\omega_3(t)\cos\Omega_1(t);\end{aligned}$$

2) vector $\mathbf{w}$(t) is determined by the calculated $\mu(t),\nu(t)$:

$$\begin{aligned}\mathrm{N}(t)&=\int_0^t\nu(\tau)d\tau,\\ \mathbf{w}(t)&=\mu(t)(-\mathbf{i}_1\sin \mathrm{N}(t)+\mathbf{i}_2\cos \mathrm{N}(t))-2\mathbf{i}_3\nu(t);\end{aligned}$$

3) the approximate value of the orientation vector of a rigid body $\boldsymbol{\varphi}^*$ is calculated using vector $\mathbf{w}$(t) and the initial position of rigid body $\mathbf{\Lambda}_0$:

$$\begin{aligned}\mathrm{M}(t)&=\int_0^t\mu(\tau)d\tau,\\ \mathbf{\Phi}(t)&=\exp(\mathbf{i}_2\mathrm{M}(t)/4)\circ\exp(-\mathbf{i}_3\mathrm{N}(t)/2),\\ \boldsymbol{\varphi}^*&=\mathbf{K}\circ\widetilde{\mathbf{\Phi}}(t)\circ\int_0^t\mathbf{\Phi}(\tau)\circ\mathbf{w}(\tau)\circ\widetilde{\mathbf{\Phi}}(\tau)d\tau\circ\mathbf{\Phi}(t)\circ\widetilde{\mathbf{K}},\\ \mathbf{K}&=\mathbf{\Lambda}_0\circ(-\mathbf{i}_2);\end{aligned}$$

4) the components of quaternion $\mathbf{U}$ are determined by the orientation vector $\boldsymbol{\varphi}^*$:

$$u_0 = \cos(\varphi/2),\ u_j = \sin(\varphi/2)e_j,\ j = 1, 2, 3,$$

$$\varphi = \left|\boldsymbol{\varphi}^*\right|,\ \ \mathbf{e} = \boldsymbol{\varphi}^* / \varphi,\ \ \varphi(t) \neq 0,\ \ \forall t;$$

5) an approximate value of quaternion of a rigid body (SINS) orientation $\boldsymbol{\Lambda}^{appox}$ is obtained

$$\boldsymbol{\Lambda}^{appox} = \mathbf{U}(t) \circ \mathbf{K} \circ \left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) \circ$$
$$\circ \exp\left(\mathbf{i}_3 \mathrm{N}(t)/2\right) \circ \exp\left(\mathbf{i}_1 \Omega_1(t)/2\right).$$

Quaternion $\mathbf{K}$ should be selected in the form $\mathbf{K}_m = \boldsymbol{\Lambda}_{m-1} \circ (-\mathbf{i}_2)$ when implementing the SINS orientation algorithm at each subsequent step $m$ of algorithm. Then the initial value of variable $\boldsymbol{\varphi}^*$ will be zero each time.

It should be noted that the differential equation with respect to the vector of the final rotation of a rigid body is also used in the theory and practice of SINS. Similar reasoning can be applied to it as well [4].

## References


[1] V.N. Branetz, I.P. Shmyglevskij, Primenenie kvaternionov v zadachah orientacii tverdogo tela (Application of quaternions to rigid body orientation problems). Nauka, Moscow, 1973

[2] P.G. Savage, Strapdown analytics. Strapdown Associates Inc., Maple Plan, Minnesota. 2007.

[3] A.V. Molodenkov, “On the solution of the Darboux problem,” Mechanics of Solids, vol. 42, no. 2, pp. 167-176, 2007.

[4] A.V. Molodenkov, Ya. G. Sapunkov, T.V. Molodenkova, “Analytical solution of the approximate equation for the vector of finite rotation of a rigid body and construction the algorithm for determining SINS orientation on its basis,” The Aerospace Instrument-Making Journal, no. 6. pp. 6-13, 2017.