Physics of Atomic Nuclei, Vol. 63, No. 11, 2000, pp. 1949–1955. Translated from Yadernaya Fizika, Vol. 63, No. 11, 2000, pp. 2041–2047.

ELEMENTARY PARTICLES AND FIELDS
Theory

Annihilation of a Neutrino Pair into a Muon–Positron Pair in a Magnetic Field

A. V. Borisov, V. A. Guseinov[1)], and N. B. Zamorin
Moscow State University, Vorob'evy gory, Moscow, 119899 Russia
Received July 21, 1999; in final form, November 15, 1999

Abstract—The cross section for the process $\nu_\mu + \bar{\nu}_e \longrightarrow \mu^- + e^+$ in a constant magnetic field is calculated with allowance for muon and positron polarizations. The asymptotic behavior of the cross section as a function of the kinematical and the field parameter is investigated in the case where a high-energy neutrino (antineutrino) is scattered by a low-energy antineutrino (neutrino). The effect of a weak field is especially important near the threshold for the free process. The spectrum and the total cross section for the process in a strong field differ markedly from the corresponding characteristics of the free process. Possible astrophysical applications are considered. *© 2000 MAIK "Nauka/Interperiodica".*

1. Investigation of neutrino–lepton processes makes it possible to deduce detailed information about the structure of weak currents in a pure form not complicated by strong-interaction effects [1]. These processes play an important role in astrophysics [2]. The inverse muon decay $\nu_\mu + e^- \longrightarrow \mu^- + \nu_e$, a purely leptonic process that is caused by the charged weak current (W-boson exchange), has been investigated experimentally since the late 1970s [3, 4]. Within the Standard Model, the cross section for this process was calculated in [5] with allowance for one-loop radiative corrections. For astrophysical applications, it is of great interest to take into account the effect of strong external electromagnetic fields on electroweak processes. For example, magnetic fields of neutron stars can be as large as $H \gtrsim H_0 = m_e^2 c^3/e\hbar = 4.41 \times 10^{13}$ G [6]. Fields of $H \sim 10^{15}$–10^{17} G are generated in supernova explosions (see, for example, [7]). We note that, even in laboratory experiments with beams of high-energy particles traversing single crystals, it is necessary to take into account strong internal electrical fields ($E \lesssim 10^{-4}H_0$) [8]. Inverse muon decay in a constant crossed field ($\mathbf{E} \cdot \mathbf{H} = \mathbf{E}^2 - \mathbf{H}^2 = 0$) was investigated in [9, 10]; the case of a magnetic field was considered in [11, 12]. In the present study, we calculate the cross section for the process

$$\nu_\mu + \bar{\nu}_e \longrightarrow \mu^- + e^+. \tag{1}$$

In a constant crossed field, this process is related to muon decay by the crossing-symmetry equation. Here, we study characteristic polarization effects associated with the direction specified by the external field and with the weak-current structure. Presently, various processes induced by the inelastic scattering of ultrahigh-energy cosmic (anti)neutrinos on low-energy relic (anti)neutrinos in the Milky Way Galaxy are considered as possible sources of high-energy cosmic rays (see, for example, [13]).

2. By using the four-fermion approximation of the Weinberg–Salam Standard Model and the Fierz identity [1], the amplitude of the process in (1) can be represented in the form

$$S_{fi} = \frac{4G_F}{\sqrt{2}} \frac{\bar{v}(k')\gamma_L^\alpha u(k)}{2L^3(\omega\omega')^{1/2}} J_\alpha(q), \tag{2}$$

where G_F is the Fermi constant; $u(k)$ and $v(k')$ are the bispinors of the massless neutrinos ν_μ and antineutrinos $\bar{\nu}_e$ with 4-momenta $k = (\omega, \mathbf{k})$ and $k' = (\omega', \mathbf{k}')$ ($k^2 = k'^2 = 0$), respectively; $q = k + k' = (E, \mathbf{q})$; $\gamma_L^\alpha = \gamma^\alpha(1 + \gamma^5)/2$ are the left components of the Dirac matrices; $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$; and L is the normalization length. The charged-lepton current is given by

$$\begin{aligned} J^\alpha(q) &= \int d^4x e^{-iqx} \overline{\psi_{n'}^{(+)}}(x)\gamma_L^\alpha \psi_n^{(-)}(x) \\ &= 2\pi\delta(\varepsilon' + \varepsilon - E) j^\alpha(\mathbf{q}), \end{aligned} \tag{3}$$

where the muon wave functions $\psi_{n'}^{(+)}$ and the positron (negative-frequency electron) wave functions $\psi_n^{(-)}$ are exact solutions to the Dirac equation in a constant magnetic field and where the delta function expresses the energy-conservation law in a time-independent field. We use the pseudo-Euclidean metric with signature (+ – – –) and the system of units where $\hbar = c = 1$.

By using Eqs. (2) and (3), we represent the cross section for the process in the general form (compare

[1)] Nakhichevan State University, Nakhichevan, Azerbaijan.

1063-7788/00/6311-1949$20.00 © 2000 MAIK "Nauka/Interperiodica"

with [12])

$$\sigma = \frac{8\pi G_F^2}{L^3 k'k}\sum_{f',f}\delta(\varepsilon' + \varepsilon - E)[(k'j^*)(kj) + (k'j)(kj^*) - (k'k)(j^*j) - i\varepsilon^{\alpha\beta\mu\nu}j^*_\alpha j_\beta k'_\mu k_\nu], \quad (4)$$

where summation is performed over the sets of four muon quantum numbers $f' = (n', p'_z, s', \zeta')$ and four positron quantum numbers $f = (n, p_z, s, \zeta)$. These numbers have the following meaning [14]: $n = 0, 1, 2, \ldots$ is the principal quantum number (number of the Landau level); $-\infty < p_z < \infty$ is the projection of the 3-momentum onto the direction of the magnetic field aligned with the z axis; $s = 0, 1, 2, \ldots$ is the radial quantum number that corresponds to the axisymmetric gauge of the 4-potential of the magnetic field,

$$A^\mu = \left(0, -\frac{1}{2}yH, \frac{1}{2}xH, 0\right); \quad (5)$$

and $\zeta = \pm 1$ is the spin quantum number specifying the particle polarization (see below). The energy spectrum of a particle in a magnetic field is degenerate in s and ζ,

$$\begin{aligned}\varepsilon' &= (m_\mu^2 + 2eHn' + p_z'^2)^{1/2},\\ \varepsilon &= (m_e^2 + 2eHn + p_z^2)^{1/2},\end{aligned} \quad (6)$$

where $e > 0$ is the positron charge.

By using the explicit form of the wave functions for charged leptons in a magnetic field in the gauge specified by Eq. (5) (see [14]), we find that the components of the current in (3) can be represented as

$$\begin{pmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{pmatrix} = 2\exp\left[i(n-n')\left(\varphi + \frac{\pi}{2}\right)\right] \times \frac{2\pi}{L}\delta(p_z + p'_z - q_z)I_{ss'}\begin{pmatrix} F_0 \\ F_1\cos\varphi + iF_2\sin\varphi \\ F_1\sin\varphi - iF_2\cos\varphi \\ F_3 \end{pmatrix}, \quad (7)$$

$$\begin{aligned}\begin{pmatrix} F_0 \\ F_3 \end{pmatrix} &= l'_2 l_2 I_{nn'} \pm l'_1 l_1 I_{n-1, n'-1},\\ \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= l'_1 l_2 I_{n, n'-1} \pm l'_2 l_1 I_{n-1, n'},\end{aligned} \quad (8)$$

where $I_{ss'}$ and $I_{nn'}$ are known Laguerre functions [14] of the argument

$$x = \frac{q_\perp^2}{2eH}, \quad q_\perp^2 = q_x^2 + q_y^2, \quad (9)$$

while φ is the azimuthal angle of the vector $\mathbf{q}$ ($\cos\varphi = q_x/q_\perp$, $\sin\varphi = q_y/q_\perp$). The quantities l_k ($k = 1, 2$) in (8) are expressed in terms of the spin coefficients C_i ($i = \overline{1, 4}$) in the positron (more precisely, negative-frequency-electron) wave functions,

$$l_1 = \frac{1}{2}(C_1 - C_3), \quad l_2 = \frac{1}{2}(C_2 - C_4), \quad (10)$$

while l'_k refer to a muon. The explicit expressions for them depend on the choice of the lepton polarization operator (integral of the motion in a given external field); the wave function is an eigenfunction of this operator; the spin number ζ is a normalized eigenvalue [14];

$$\begin{aligned}\mu_3\psi &= \frac{\varepsilon_\perp}{m}\zeta\psi,\\ h\psi &= \zeta p\psi.\end{aligned} \quad (11)$$

Here, the transverse-polarization operator

$$\mu_3 = \Sigma_3 + i\gamma^0\gamma^5(\mathbf{\Sigma}\times\mathbf{P})_z = \frac{\varepsilon}{m}\gamma^0(\Sigma_3 + \gamma^5 p_z) \quad (12)$$

determines the projection of the lepton spin onto the direction of the magnetic field $\mathbf{H}$, $\varepsilon_\perp = (\varepsilon^2 - p_z^2)^{1/2}$, and $\mathbf{P} = -i\nabla + e\mathbf{A}$ is the momentum operator.

The longitudinal polarization is associated with the helicity operator

$$h = \mathbf{\Sigma}\cdot\mathbf{P} = \gamma^5(m\gamma^0 - \varepsilon), \quad (13)$$

and $p = (\varepsilon^2 - m^2)^{1/2}$. We note that the operator equalities (12) and (13) are valid in the class of functions ψ satisfying the Dirac equation (the general theory of polarization operators in external fields is developed in [14, 15]). The interactions of the lepton anomalous magnetic moment representing the radiative correction to the Dirac moment destroy the longitudinal polarization (quite fast under the actual conditions in storage rings), but the operator in (12) remains an integral of the motion [14]. For this reason, we consider below only transverse polarization.

3. We restrict ourselves to the case where a neutrino and an antineutrino approach each other from opposite directions in the plane orthogonal to the field $\mathbf{H}$. Since the problem in an external field possesses axial symmetry, the choice of the x axis along the collision axis imposes no constraints on the generality of our consideration. Accordingly, the 4-momenta in the neutrino pair are then taken to be

$$k = \omega(1, 1, 0, 0), \quad k' = \omega'(1, -1, 0, 0), \quad (14)$$

in which case $q_y = q_z = 0$ and the angle φ in (7) is

$$\varphi = \begin{cases} 0, & \omega > \omega', \\ \pi, & \omega' > \omega. \end{cases} \quad (15)$$

Taking into account (14), (15), and (7), we express the cross section (4) in terms of the functions in (8) as

$$\sigma^{(\pm)} = \frac{8}{\pi} G_{\mathrm{F}}^2 eH \sum_{\zeta,\zeta'} \sum_{n,n'} \int dp_z \delta(\varepsilon + \varepsilon' - E)(F_3 \pm F_2)^2, \quad (16)$$

where the upper and the lower sign in the superscript refer to the cases of $q_x = \omega - \omega' > 0$ and $q_x < 0$, respectively. In deriving (16), we have used the conservation of the z component of the momentum [see Eq. (7)] and the known formula for summation over radial quantum numbers [14],

$$\sum_{s,s'} I_{ss'}^2 = \frac{eH}{2\pi} L^2.$$

The case of a collision between a high-energy neutrino (antineutrino) and a low-energy antineutrino (neutrino) is of interest for astrophysical applications. Suppose that the energy of the $\nu_\mu \bar{\nu}_e$ pair, $E = \omega + \omega'$, and the momentum transfer $q_\perp = |\omega - \omega'|$ are both much greater than m_μ and that the magnetic-field strength satisfies the condition $H \ll H_\mu = m_\mu^2/e$. The main contribution to the total cross section for process (16) then comes from the final-lepton states having large quantum numbers, n, $n' \gg 1$ (high Landau levels); for the Laguerre functions $I_{nn'}$ in (8), we can therefore use the well-known semiclassical asymptotic form in terms of the relativistic parameter $\gamma^{-1} = m_\mu/E \ll 1$ [14],

$$\begin{gathered}\begin{pmatrix} I_{nn'}(x) \\ dI_{nn'}(x)/dx \end{pmatrix} \equiv \begin{pmatrix} I \\ I' \end{pmatrix} \simeq \frac{(-1)^{n'}}{\sqrt{\pi}} \gamma^{-1} (v\bar{v})^{-1/2} \begin{pmatrix} w^{1/3}\Phi(y) \\ \gamma^{-1} w^{2/3} \Phi'(y) \end{pmatrix}, \\ I_{n,n'-1} \simeq -I - \frac{1}{\bar{v}} I', \quad I_{n-1,n'} \simeq I + \frac{1}{v} I', \\ I_{n-1,n'-1} \simeq -I - \frac{1}{v\bar{v}} I', \\ w = 2\kappa v \bar{v}, \quad \bar{v} = 1 - v, \end{gathered} \quad (17)$$

where

$$\Phi(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty dt \cos\left(yt + \frac{t^3}{3}\right) \quad (18)$$

is the Airy function of the argument

$$y = (2\kappa v \bar{v})^{-2/3} (v + \delta^2 \bar{v} - \lambda v \bar{v} + \tau^2), \quad (19)$$

and $\Phi'(y) = d\Phi(y)/dy$. In (19), we have introduced the field, the kinematical, and the mass parameter (κ, λ, and δ, respectively)

$$\begin{gathered}\kappa = \frac{e}{m_\mu^3} [-(F_{\alpha\beta} q^\beta)^2]^{1/2} = \frac{q_\perp}{m_\mu} \frac{H}{H_\mu} \simeq \frac{eHE}{m_\mu^3}, \\ \lambda = \frac{q^2}{m_\mu^2} = \frac{4\omega\omega'}{m_\mu^2}, \quad \delta = \frac{m_e}{m_\mu}, \end{gathered} \quad (20)$$

and the spectral and the angular variable (v and τ, respectively)

$$v = \frac{\chi}{\chi + \chi'} \simeq \frac{\varepsilon}{\varepsilon + \varepsilon'}, \quad \tau = \frac{eq^\alpha \tilde{F}_{\alpha\beta} p^\beta}{m_\mu^4 (\chi + \chi')} \simeq \frac{p_z}{m_\mu}, \quad (21)$$

where

$$\chi = \frac{e}{m_\mu^3} [-(F_{\alpha\beta} p^\beta)^2]^{1/2} \simeq \frac{eH\varepsilon}{m_\mu^3}, \quad \chi' = \chi(p \longrightarrow p'),$$

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the strength tensor of an external magnetic field, and $\tilde{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\lambda\sigma} F^{\lambda\sigma}$ is its dual counterpart. We note that $v \in (0, 1)$ and that, in the ultrarelativistic approximation adopted here ($\gamma = E/m_\mu \gg 1$), $\tau \in (-\infty, \infty)$.

Since the motion of leptons is semiclassical, summation over the quantum numbers n and n' in (16) can be replaced by integration according to the relation [see (6)]

$$dndn' = \frac{\varepsilon\varepsilon' d\varepsilon d\varepsilon'}{(eH)^2}.$$

After that, the integral with respect to ε' is removed by the delta function. Further, we go over to the variables v and τ (21). As a result, we derive the cross section for the process at fixed lepton polarizations ζ' and ζ in the form

$$\sigma^{(\pm)}(\zeta', \zeta) = \frac{8}{\pi} G_{\mathrm{F}}^2 \frac{m_\mu^2}{\kappa} \gamma^4 \int_0^1 dv\, v \bar{v} \int_{-\infty}^{\infty} d\tau (\tilde{F}_3 \pm \tilde{F}_2)^2, \quad (22)$$

where the upper and the lower sign in the superscript refer to the kinematical conditions $\omega \gg \omega'$ and $\omega \ll \omega'$, respectively, and $\tilde{F}_i$ is the semiclassical asymptotic form of the function F_i.

The expressions for $\tilde{F}_i$ in (22) can be obtained by substituting (17) into (8) and retaining only the first, linear, terms of the expansion in the small parameter γ^{-1} in the spin coefficients C_i and C_i' [see Eqs. (8) and (10)]. In doing this, it should be considered that $\omega\omega' \lesssim m_\mu^2$ ($\lambda \sim \gamma^0$) and that

$$\frac{m_e}{\varepsilon} = \gamma^{-1} \frac{\delta}{v}, \quad \frac{m_\mu}{\varepsilon'} = \gamma^{-1} \frac{1}{\bar{v}},$$

$$\frac{p_z}{\varepsilon} = \gamma^{-1} \frac{\tau}{v}, \quad \frac{p_z'}{\varepsilon'} = -\gamma^{-1} \frac{\tau}{\bar{v}},$$

the main contribution to the total cross section (22) coming from the region specified by inequalities $v \lesssim 1$ and $|\tau| \lesssim 1$. Further, we note that, in fact, the mass parameter [see (20)] is small, $\delta \simeq 4.8 \times 10^{-3}$, and that it is on the same order of magnitude as radiative corrections (about $\alpha/\pi \sim 10^{-3}$), which are disregarded in this study. To the precision adopted here, we therefore set this parameter to zero below.

Finally, we find that, in terms of the spectral and the angular variable, the cross sections for the production of transversely polarized leptons [see (12)] are given by

$$\begin{pmatrix}\sigma^{(+)}(\zeta', \zeta)\\ \sigma^{(-)}(\zeta', \zeta)\end{pmatrix} = \frac{G_F^2}{\pi^2} m_\mu^2 \int_0^1 d\nu \begin{pmatrix}\bar\nu/\nu\\ \nu/\bar\nu\end{pmatrix} \int_{-\infty}^{\infty} d\tau \Big[(2\tilde\kappa)^{-1/3}$$
$$\times \begin{pmatrix}\tau^2\\ 1+\tau^2+2\zeta'\tau\end{pmatrix} \Phi^2(y) + (2\tilde\kappa)^{1/3} \begin{pmatrix}1\\ 1\end{pmatrix} \Phi'^2(y) \tag{23}$$
$$+ 2\begin{pmatrix}-\tau\\ \zeta'+\tau\end{pmatrix} \Phi(y)\Phi'(y)\Big],$$

where $\tilde\kappa = \kappa\nu\bar\nu$ and the argument of the Airy functions is determined by (19) at $\delta = 0$.

The cross sections (23), which are expressed in terms of the invariant parameters (20) and the variables in (21), is applicable (in the ultrarelativistic approximation) in an arbitrary constant field $F_{\mu\nu}$ of strength F much less than $H_\mu = m_\mu^2/e$ [see (12)], the spin numbers ζ' and ζ being eigenvalues of the corresponding invariant spin operator (see [12, 15]). The conditions under which the above generalization to the case of two-body processes in an external field is applicable are analyzed in [16] (see also Section 6 below).

4. The integrands in (23) determine the differential cross sections $d^2\sigma^{(\pm)}/d\nu d\tau$. The asymmetric dependence on the angular variable τ and the spin variables ζ and ζ' is due to P and C nonconservation in weak interactions and to the choice of kinematical conditions (compare with [12])—an ultrarelativistic muon and an ultrarelativistic positron are emitted at small angles (not greater than γ^{-1}) with respect to the direction of the high-energy-(anti)neutrino momentum.

In order to investigate the spectral distribution $d\sigma^{(\pm)}/d\nu$, we perform integration with respect to the variable τ by using the relations

$$\int_{-\infty}^{\infty} d\tau \Phi^2(y) = \sqrt{\pi}\cdot 2^{-2/3} b^{-1/2} \Phi_1(z),$$
$$\int_{-\infty}^{\infty} d\tau \tau^2 \Phi^2(y) = \sqrt{\pi}\cdot 2^{-5/3} b^{-3/2} [\Phi'(z) + z\Phi_1(z)], \tag{24}$$
$$\int_{-\infty}^{\infty} d\tau \Phi'^2(y) = -\sqrt{\pi}\cdot 2^{-7/3} b^{-1/2} [3\Phi'(z) + z\Phi_1(z)],$$

where the arguments are $y = x + a\tau^2$, $z = 2^{2/3}x$, and $b = 2^{2/3}a$ and where $\Phi_1(z) = \int_z^\infty dt\Phi(t)$. The relations in (24) were derived with the aid of the well-known relations for the Airy functions {see [17, ch. 5, formulas (46), (48), (58)]}.

From expressions (23), (24), and (19), we obtain

$$\begin{pmatrix}\sigma^{(+)}(\zeta', \zeta)\\ \sigma^{(-)}(\zeta', \zeta)\end{pmatrix} = \frac{G_F^2}{\pi^{3/2}} m_\mu^2 \int_0^1 d\nu \begin{pmatrix}\bar\nu/\nu\\ \nu/\bar\nu\end{pmatrix}$$
$$\times \Big[\begin{pmatrix}-(\nu/2)(1-\lambda\bar\nu)\\ (\bar\nu/2)(1+\lambda\nu)\end{pmatrix} \Phi_1(z) - \tilde\kappa^{2/3} \begin{pmatrix}1\\ 1\end{pmatrix} \Phi'(z) \tag{25}$$
$$+ \tilde\kappa^{1/3} \begin{pmatrix}0\\ -\zeta'\end{pmatrix} \Phi(z)\Big].$$

The integrands in expression (25) represent the spectral distributions $d\sigma^{(\pm)}/d\nu$, the argument of the Airy functions being

$$z = (\kappa\nu\bar\nu)^{-2/3} \nu(1-\lambda\bar\nu). \tag{26}$$

For the free process (in the absence of an external field, $\kappa = 0$), we find the spectral distributions in the form

$$\begin{pmatrix}d\sigma_0^{(+)}(\zeta', \zeta)/d\nu\\ d\sigma_0^{(-)}(\zeta', \zeta)/d\nu\end{pmatrix} = \frac{G_F^2}{2\pi} m_\mu^2 \begin{pmatrix}-\bar\nu(1-\lambda\bar\nu)\\ \nu(1+\lambda\nu)\end{pmatrix} \theta(\lambda\bar\nu - 1). \tag{27}$$

These expressions follow from (25) for $\kappa \longrightarrow 0$ with allowance for the weak limit

$$\lim_{A\to\infty} \Phi_1(Ax) = \sqrt{\pi}\theta(-x),$$

where $\theta(x) = (1 + \operatorname{sgn}x)/2$ is the Heaviside step function. The range of the spectral variable ν in (27) is determined from the condition $\lambda\bar\nu - 1 \ge 0$, which yields

$$0 \le \nu \le \nu_1 = 1 - 1/\lambda. \tag{28}$$

From (28), it can be seen that, in the absence of a field, reaction (1) has a threshold; that is, the kinematically allowed region is

$$\lambda > 1. \tag{29}$$

According to the general theory developed in [16], the external-field effect on the process allowed in the absence of a field as well is determined by the parameter

$$\eta = \kappa/|\lambda - 1|. \tag{30}$$

Let us investigate the qualitative characteristics of the spectra given by (25) in the limiting cases of $\eta \ll 1$ and $\eta \gg 1$ by using the known properties of the Airy functions [17].

For $\eta \ll 1$ (weak field) and $\lambda > 1$, oscillations are superimposed on the smooth free spectra (27) in the region determined by (28), and these oscillations grow as v approaches its boundaries. In the region $v_1 \le v \le 1$, which is forbidden at $\eta = 0$, the differential cross section $d\sigma^{(\pm)}/dv$ decreases fast as we go away from the point v_1 [for $v \longrightarrow 1$, it is proportional to $\exp(-2/3\kappa\bar{v})$].

At $\eta \gg 1$ (strong field) and $\lambda > 1$, the region of oscillations is comparatively narrow.

At $\lambda < 1$, the free process (1) is forbidden, and there are no oscillations in the spectra. In a weak field, the spectra are exponentially small in this case over the entire interval $0 \le v \le 1$.

The above features of the spectra are typical of all processes proceeding in the absence of external fields as well—in particular, of inverse muon decay [12] and of the Compton effect in a magnetic field [16].

5. Let us consider the asymptotic behavior of the total cross section for the process.

At $\kappa \ll 1$ and $\lambda > 1$, it follows from (26) that we can use the weak asymptotic expansions of the Airy functions (see, for example, [10, 12]),

$$\begin{aligned} \Phi(Ax) &= \sqrt{\pi}A^{-1}\delta(x) + O(A^{-4}), \\ \Phi'(Ax) &= \sqrt{\pi}A^{-2}\delta'(x) + O(A^{-5}), \\ \Phi_1(Ax) &= \sqrt{\pi}\left[\theta(-x) + \frac{1}{3}A^{-3}\delta''(x)\right] + O(A^{-6}), \end{aligned} \tag{31}$$

where $A \gg 1$ and $\delta(x) = d\theta(x)/dx$ is a delta function. In our case, $A = \kappa^{-2/3}$ and

$$x(v) = (v\bar{v})^{-2/3}v(1-\lambda\bar{v}). \tag{32}$$

Let us substitute (31) into (25) and perform integration with respect to v by using the relations

$$\int_0^1 dv f(v)\theta(-x) = \int_0^{v_1} dv f(v),$$

$$\int_0^1 dv\,\delta^{(n)}(x) f(v) = (-1)^n \frac{d^n}{dx^n}[v' f(v)]\Bigg|_0^{v_1},$$

where $v' = dv(x)/dx$; $v = v(x)$ is the function inverse to $x(v)$ (32); and $\delta^{(n)}(x) = d^n\delta(x)/dx^n$, $n = 1, 2$. In the above relations, we have considered that $x'(0) < 0$ and $x'(v_1) > 0$. The derivatives $d^n v/dx^n$ ($n = 1, 2, 3$) are calculated by differentiating, with respect to x, the left- and the right-hand side of the equation $x = x(v)$, which determines the function $v(x)$ implicitly.

To terms of order κ^2 inclusive, we eventually obtain the asymptotic expressions

$$\begin{aligned} \begin{pmatrix} \sigma^{(+)}(\zeta', \zeta) \\ \sigma^{(-)}(\zeta', \zeta) \end{pmatrix} &= \frac{G_F^2}{4\pi}m_\mu^2\left[F_0(\lambda)\begin{pmatrix}1\\1\end{pmatrix}\right. \\ &\left. + 4\frac{\kappa}{\lambda-1}\begin{pmatrix} 0 \\ -\zeta'(1-1/\lambda)^2\end{pmatrix} + \left(\frac{\kappa}{\lambda-1}\right)^2\begin{pmatrix}F_+(\lambda)\\F_-(\lambda)\end{pmatrix}\right], \end{aligned} \tag{33}$$

where

$$\begin{aligned} F_0(\lambda) &= \frac{1}{3}(2\lambda+1)\left(1-\frac{1}{\lambda}\right)^2, \\ F_+(\lambda) &= -\frac{8}{3}\left(1+\frac{3}{\lambda^2}-\frac{2}{\lambda^3}\right), \\ F_-(\lambda) &= -\frac{8}{3}\left(1-\frac{2}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)^2. \end{aligned} \tag{34}$$

From (33), it can be seen that, in accord with (30), the external-field effect on the process is described by the parameter $\kappa/(\lambda-1)$. This effect is stronger for polarized particles than for unpolarized particles, being of the first and of the second order in κ, respectively. In a relatively weak field ($\kappa \ll 1$) such that the relation $\kappa \gtrsim \lambda - 1$ nevertheless holds, the cross section for the process differs markedly from the cross section for the free process, the latter being very small near the threshold [$F_0 = 0$ at $\lambda = 1$; see (34)]. From (33), it follows that, for $\omega \gg \omega'$ ($\omega' \gg \omega$), we predominantly have the generation of muons (positrons) whose spins are aligned with (opposite to) the direction of the magnetic field **H**—that is, $\zeta = +1$ ($\zeta' = -1$). This effect is similar to the Sokolov–Ternov effect, the radiative polarization of electrons and positrons in a magnetic field due to synchrotron radiation [14].

For $\lambda < 1$, the free process (at $\kappa = 0$) is forbidden. In a weak field ($\eta \ll 1$), the cross section for process (1) is exponentially small, which is characteristic of all processes that have a threshold in the absence of a field [17]. In this case, the argument of the Airy function is very large ($z \gg 1$), so that we asymptotically have

$$\Phi(z) \simeq \frac{1}{2}z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right).$$

By using this asymptotic expression and the method of steepest descent and taking into account a finite value of the mass parameter ($\delta \ll 1$), we can easily obtain

$$\sigma \sim \kappa\exp\left(-\sqrt{3}\delta\frac{1-\lambda}{\kappa}\right).$$

For $\eta \gg 1$ and $\kappa \gg 1$ (strong field), the main contribution to the integrals in (25) comes from the region $|z| \ll 1$. If only the leading terms are retained in the asymptotic expressions in κ, the integrands can be simplified by setting there

$$\Phi(z) \simeq \Phi(0) = 3^{-1/6}\frac{\Gamma(1/3)}{2\sqrt{\pi}},$$

$$\Phi'(z) \simeq \Phi'(0) = -3^{1/6}\frac{\Gamma(2/3)}{2\sqrt{\pi}}.$$

Upon evaluating the remaining standard integrals with respect to v, we obtain the strong-field asymptotic expressions for the cross sections (25) in the form

$$\begin{pmatrix}\sigma^{(+)}(\zeta', \zeta)\\ \sigma^{(-)}(\zeta', \zeta)\end{pmatrix} = \frac{G_F^2}{4\pi^3}m_\mu^2\left[c_2(3\kappa)^{2/3}\begin{pmatrix}1\\1\end{pmatrix} + c_1(3\kappa)^{1/3}\begin{pmatrix}0\\-\zeta'\end{pmatrix}\right],$$

$$c_2 = \frac{15}{14}\Gamma^4\left(\frac{2}{3}\right), \quad c_1 = \frac{2}{5}\Gamma^4\left(\frac{1}{3}\right). \tag{35}$$

As in the case of a weak field [see Eq. (16)], we see that, for $\omega \gg \omega'$ ($\omega' \gg \omega$), there is a predominant production of positrons (muons) polarized in the direction parallel (antiparallel) to the external field **H**.

6. In the above analysis [see Eq. (2)], we have made use of the four-fermion approximation for the amplitude of process (1). As in the absence of an external field, the relative smallness of the momentum transfer—$|q^2| \ll m_W^2$, where m_W is the W-boson mass—is the necessary condition of its applicability. Taking into account (20), we obtain the kinematical constraint

$$\lambda \ll (m_W/m_\mu)^2 \simeq 6\times 10^5. \tag{36}$$

In an external field, however, there arises an additional condition: changes in the particle momentum over the formation length for the process, l_f, must be much less than m_W. In a strong field ($\kappa \gg 1$), this length is independent of the particle mass and is given by [17, 18]

$$l_f \sim [-(eF_{\alpha\beta}q^\beta)^2]^{1/6}/eH.$$

As a result, we find that the field parameter κ [see Eq. (20)] must be constrained as

$$\kappa \ll (m_W/m_\mu)^3 \simeq 4.4\times 10^8. \tag{37}$$

In our case, the condition in (36) is obviously satisfied since the semiclassical asymptotic expressions used for the Laguerre functions $I_{nn'}(x)$ (17) are valid [14] when the argument x is close to the transition point $x_0 = (\sqrt{n} + \sqrt{n'})^2 = (p_\perp + p'_\perp)^2/2eH$,

$$x/x_0 - 1 = O(\gamma^{-2})$$

or [see Eqs. (9), (6), and (14)]

$$q^2 = q_0^2 - q_\perp^2 \lesssim \gamma^{-2}E^2 = m_\mu^2;$$

that is, we have [see Eq. (20)] $\lambda \lesssim 1$.

Following [16], we will now show that the basic results deduced here under the kinematical conditions chosen in a special way [see Eq. (14)] can be used in a more general case. The cross section for the two-body process (1) in an arbitrary constant field $F_{\alpha\beta}$ for arbitrary directions of the neutrino momenta depends on eight independent invariant parameters (for unpolarized particles). These are the quantities κ and λ, which were already defined in (20), and

$$f_1 = \frac{e}{m_\mu^2}[|F_{\alpha\beta}F^{\alpha\beta}|/2]^{1/2}, \quad f_2 = \frac{e}{m_\mu^2}[|F_{\alpha\beta}\tilde{F}^{\alpha\beta}|/2]^{1/2},$$

$$f_3 = \frac{e}{m_\mu^4}|k_\alpha F^{\alpha\beta}k'_\beta|, \quad f_4 = \frac{e}{m_\mu^4}|k_\alpha \tilde{F}^{\alpha\beta}k'_\beta|, \tag{38}$$

$$f_5 = \frac{e}{m_\mu^3}[-(F^{\alpha\beta}k'_\beta)^2]^{1/2}, \quad f_6 = \frac{e}{m_\mu^3}[|k^\alpha F_{\alpha\beta}F^{\beta\gamma}k'_\gamma|]^{1/2}.$$

We note that, in a purely magnetic field, $f_1 = H/H_\mu$ and $f_2 = 0$. Suppose that the parameters in (38) satisfy the conditions

$$f_i \ll \kappa, \quad f_i \ll \lambda, \quad f_i \ll 1 \quad (i = 1\text{–}6). \tag{39}$$

They are fulfilled for field strengths $F \ll H_\mu$, high energies of a neutrino pair (for markedly different energies of its components), and at not overly small angles between the ultrarelativistic-(anti)neutrino 3-momentum **k** and the field-strength vectors **E** and **H**. By virtue of (39), the general expression for the cross section can be approximated by the simpler two-parameter formula

$$\sigma(\kappa, \lambda, f_1, \ldots, f_6) \simeq \sigma(\kappa, \lambda, 0, \ldots, 0).$$

Thus, our results expressed in terms of the invariant parameters κ and λ are applicable not only to the case of kinematics specified by (14) but also in the rather general case specified by (39).

Let us consider the possibilities for observing the external-field effect on process (1). Muon neutrinos of energy $\omega \simeq 20$ GeV are used in experimental investigations of inverse muon decay [4]. Our results are applicable if the energies ω' of electron antineutrinos obey the condition

$$\omega' \ll m_\mu^2/\omega \lesssim 1 \text{ MeV}$$

at $\omega \gtrsim 10$ GeV. Such energies ω' correspond to the lower limit on the reactor- and solar-(anti)neutrino energies recorded by conventional methods [2]. Let us set $H = 10^8$ G (this can be pulsed magnetic fields or effective single-crystal fields [8]) and $E = \omega + \omega' \simeq \omega = 20$ GeV. The field parameter is then given by [see (20)]

$$\kappa \simeq \frac{\omega}{m_\mu}\frac{H}{H_\mu} \simeq 10^{-8},$$

where we have used the value of $H_\mu = m_\mu^2/e \simeq 1.9\times 10^{18}$ G. From (30), it follows that, in this case, the external-field effect becomes sizable in a narrow region of λ values lying near the free-process threshold: $\lambda \simeq 1$, $|\lambda - 1| \lesssim 10^{-8}$ [see (33), (34)]. However, the observation of the effect under laboratory conditions is complicated by a relatively low density of neutrino beams and by small dimensions of the interaction region. For this reason, we will focus on astrophysical conditions.

As was indicated in Section 1, much attention is being given at present to the possibility that high-energy cosmic rays are generated in the annihilation of ultrahigh-energy neutrinos on low-energy galactic relic antineutrinos (see [13] and references therein). Assuming that relic (anti)neutrinos are massless, we estimate their energy (that is, the temperature of relic radiation) at $\omega' \sim 2$ K $\simeq 1.7 \times 10^{-4}$ eV. Our results are valid in the region of cosmic-neutrino energies,

$$\omega \lesssim m_\mu^2/\omega' \lesssim 10^{20} \text{ eV}.$$

For the field parameter, we then have

$$\kappa \lesssim 10^{-7} \, (H/1 \text{ G}). \tag{40}$$

The mean-galactic magnetic field is overly weak: $H \sim 10^{-6}$ G and $\kappa \lesssim 10^{-13}$. However, compact objects in the Milky Way Galaxy can develop strong fields [19]. The surface fields of white dwarfs take values of $H \lesssim 10^9$ G, in which case expression (40) yields $\kappa \lesssim 10^2$. The dipole fields of neutron stars are $H \lesssim 10^{13}$ G. For these fields, $\kappa \lesssim 10^6$ [the condition in (37) is satisfied in this case], and the cross section for process (1) in a magnetic field is much larger than the cross section for the free process owing to the factor [see Eq. (35)]

$$F(\kappa) = (3\kappa)^{2/3} \lesssim 10^4.$$

Thus, process (1) can be a source of high-energy charged leptons; in the vicinity of strongly magnetized stars, their spectral distributions and total fluxes can differ considerably [see Eq. (25)] from the corresponding values in the regions where the field can be disregarded [see (27)]. We note that, despite the relatively small dimensions of neutron stars, they may modify sizably the energy spectra of cosmic rays owing to the large-scale pulsar-wind effect (see, for example, [20]).

ACKNOWLEDGMENTS

We are grateful to a reviewer for constructive criticism.

REFERENCES

1. L. B. Okun, *Leptons and Quarks* (Nauka, Moscow, 1990; North-Holland, Amsterdam, 1984).
2. J. N. Bahcall, *Neutrino Astrophysics* (Cambridge Univ. Press, Cambridge, 1989; Mir, Moscow, 1993).
3. N. Armenise *et al.*, Phys. Lett. B **84**, 137 (1979).
4. P. Vilain *et al.*, Preprint No. CERN-PPE/96-01 (Geneva, 1996).
5. D. Yu. Bardin and V. A. Dokuchaeva, Nucl. Phys. B **287**, 839 (1987).
6. V. M. Lipunov, *Astrophysics of Neutron Stars* (Nauka, Moscow, 1987).
7. G. S. Bysnovatyi-Kogan, Astron. Astrophys. Trans. **3**, 287 (1993).
8. V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko, Itogi Nauki Tekh., Ser.: Puchki Zaryazhennykh Chastits Tverd. Tela **4**, 57 (1992).
9. V. A. Lyul'ka, Yad. Fiz. **39**, 680 (1984) [Sov. J. Nucl. Phys. **39**, 431 (1984)].
10. A. V. Borisov, L. V. Morozova, and M. K. Nanaa, Izv. Vyssh. Uchebn. Zaved., Fiz., No. 12, 106 (1992).
11. A. V. Borisov and V. A. Guseinov, Yad. Fiz. **57**, 496 (1994) [Phys. At. Nucl. **57**, 466 (1994)].
12. A. V. Borisov, V. A. Guseinov, and O. S. Pavlova, Yad. Fiz. **61**, 103 (1998) [Phys. At. Nucl. **61**, 94 (1998)].
13. D. Fargion and B. Mele, astro-ph/9902024; J. J. Blanco-Pillado, R. A. Vázquez, and E. Zas, astro-ph/9902266.
14. A. A. Sokolov and I. M. Ternov, *Radiation from Relativistic Electrons* (Nauka, Moscow, 1983; AIP, New York, 1986).
15. V. G. Bagrov, D. M. Gitman, I. M. Ternov, *et al.*, *Exact Solutions of Relativistic Wave Equations* (Nauka, Novosibirsk, 1982; Kluwer, Dordrecht, 1990).
16. I. M. Ternov, V. Ch. Zhukovskiĭ, and A. V. Borisov, *Quantum Processes in Strong External Fields* (Mosk. Gos. Univ., Moscow, 1989).
17. V. I. Ritus, Tr. Fiz. Inst. Akad. Nauk SSSR **111**, 5 (1979).
18. V. O. Papanyan and V. I. Ritus, Tr. Fiz. Inst. Akad. Nauk SSSR **168**, 120 (1986).
19. G. G. Raffelt, *Stars as Laboratories for Fundamental Physics* (Univ. of Chicago Press, Chicago, 1996).
20. M. Giller and W. Michalak, in *Proceedings of the International Conference "Relativistic Jets in AGNs," Kraków, May 27–30, 1997* (Kraków, 1997), p. 189.

Translated by A. Isaakyan