Terahertz solitons

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This paper presents a theoretical study of soliton and solitonlike propagation regimes of broad-band terahertz pulses in nonlinear crystals. The standard approximation of slowly varying amplitudes for quasi-monochromatic signals is not used in this case. Solitonlike propagation regimes known in other areas of physics and those unknown earlier are analyzed. Solitons in media that contain tunnel quantum transitions are studied in detail. © *2015 Optical Society of America.*

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I. INTRODUCTION

The soliton is one of the fundamental concepts of modern physics. A solitary wave capable of being formed in nonlinear dispersive media and possessing the property of elastically interacting with waves similar to itself is customarily called a soliton. This means that, at some time after two or more solitons collide, they completely recover their original profiles. It should be once more emphasized here that all this occurs in a nonlinear medium. Therefore, the superposition principle, as understood in linear media, is invalid. Solitons interact with each other, being initially deformed and then recovering their original parameters. This property reflects the deep mathematical structure of hyperbolic equations, which possess soliton solutions. Unlike linear equations, no general methods or recipes exist here. Each equation or system has to be considered separately.

Fairly many equations and systems are currently known that generate soliton solutions.[1] In an applied sense, a good "supplier" of such equations is nonlinear optics.[2] Various equations have been obtained that describe the propagation of optical solitons with a width from nano–to femtoseconds.[3,4] Both quasi-monochromatic solitons that contain from ten to a million light oscillations and broad-band solitons with a width of up to one optical period[5] (ultrashort pulses, or USPs) are considered. Good results have been obtained in the former case from the approximation of a slowly varying envelope (SVE).[4] However, the given approximation is invalid in the latter case, and other approaches have to be used that satisfy the initial equations. Here the most satisfactory approximation is that of a slowly varying profile (SVP), which makes it possible, as in the case of an SVE, to reduce the wave equation from second to first order.[1] However, the equation is now written not for the envelope of the electric field of the pulse, but for the field itself. In this case, we neglect the wave reflected from the deformation of the medium induced by it because of nonlinearity, and we take into account only the wave that propagates forward. It is for this reason that the wave equation is reduced from second to first degree.

Studies devoted to methods of generating terahertz radiation are attracting more and more interest. It is customary to include electromagnetic frequencies from 0.1 to 10 THz in this range, and this approximately corresponds to wavelengths in the 1–0.01-mm interval. The sensitivity of the vibrational, rotational, vibrational–rotational, tunneling, and other quantum transitions to the terahertz (THz) range creates important prospects in the development of THz spectroscopy. Signals in the THz range today are finding many applications in image processing, security systems, astronomy, biology, medicine, and other areas.[6,7]

One of the most efficient generation methods is that based on the mechanism of optical rectification.[8] In this case, the spectrum of the THz signal is broad-band—i.e., the spectral pulse width is commensurable with the central frequency of its spectrum. The width of the generated pulse is such that it encompasses about one period of the oscillations of the THz range. Thus, with the optical method of generation, a THz signal possesses the properties of a USP. This means that the SVE approximation is inapplicable in the theoretical treatment of the interaction of such a pulse with matter.

It should be pointed out that the interaction of THz radiation with matter is less studied by comparison with the radiation of other frequency ranges. It therefore becomes necessary to construct theoretical models that describe such interaction. Here it is extremely difficult to construct any universal theory. One can more likely hope to describe interaction processes in which the dynamics mainly involve some kind of isolated degrees of freedom or quantum transitions. As mentioned above, tunneling states of the medium, in particular, can be susceptible to intense interaction with THz radiation. Such states are encountered, for example, in ferroelectrics of the order–disorder type,[9] in polymers that contain organic molecules,[10]

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metamaterials consisting of quantum dots,[11,12] wells, filaments, etc.

The soliton topic in the scientific press has mainly related to the optical and near-IR frequency ranges and has virtually avoided the THz range. Meanwhile, because the number of applications of the THz electromagnetic range is increasing, it has become necessary to carry out the corresponding studies.

This paper is devoted to a brief discussion of situations in which it is possible to form broad-band solitons whose spectrum lies in the THz range.

II. OPTICO-TERAHERTZ SOLITONS

The essence of the optical method mentioned above for generating broad-band THz radiation in quadratically nonlinear media, based on the optical rectification effect, is as follows: A femtosecond optical pulse whose spectrum contains frequencies at the difference of which THz radiation can be generated is fed to a nonlinear medium. The generation condition can be obtained from the laws of the conservation of momentum and energy for elementary scattering events. The dispersion law of light for a crystal has the form $\omega(\mathbf{k})$, where ω is the frequency of the light wave, and $\mathbf{k}$ is the wave number that corresponds to the given frequency. Let the frequency and the wave vector of the generated THz signal equal, respectively, Ω and $\mathbf{q}$. The law of conservation of energy and momentum is written in the form $\omega(\mathbf{k}) = \omega(\mathbf{k}-\mathbf{q}) + \Omega(\mathbf{q})$. Taking into account that $q \ll k$, we have $\omega(\mathbf{k}-\mathbf{q}) \approx \omega(\mathbf{k}) - \mathbf{q}\cdot\partial\omega/\partial\mathbf{k} = \omega(\mathbf{k}) - \mathbf{q}\cdot\mathbf{v}_g$, where $\mathbf{v}_g$ is the group velocity vector of an optical pulse with wave vector $\mathbf{k}$. Introducing angle φ between $\mathbf{v}$ and $\mathbf{q}$, we get

$$\cos\varphi = v_{\mathrm{ph}}/v_g, \tag{1}$$

where $v_{\mathrm{ph}} = \Omega/q$ is the phase velocity corresponding to the frequency Ω of the THz signal.

We have from Eq. (1) that the optical group velocity must exceed the phase velocity in the THz range—i.e., the generation has a Cherenkov nature.

Let the width of an input femtosecond pulse with carrier frequency ω equal τ_p. Its spectral width is then $\delta\omega \sim 1/\tau_p$. This quantity has the meaning of the distance in frequency between the "edge" Fourier components of the spectrum of the pulse or the characteristic frequency Ω of the generated signal. We thus have $\Omega\tau_p \sim 1$. It is assumed that the width of the input optical and generated THz pulses are of the same order of magnitude. Consequently, the THz signal contains approximately one vibrational period. Its spectral width in this case is of the same order as the central frequency of the spectrum. That is, the given signal possesses the properties of a supercontinuum. Taking $\tau_p \sim 100$ fs, we find for the central frequency of the spectrum of the generated signal $\nu \approx \delta\omega/(2\pi) \sim 1$ THz.

The idea of generating a THz pulse by means of a femtosecond optical pulse was first expressed in a theoretical paper.[13] A Cherenkov pulse was recorded experimentally some time later.[14,15]

When the propagation is noncollinear because the femtosecond pulse and the generated THz pulse are spatially separated, the generation efficiency is not high and equals 10^{-6}–10^{-5} with respect to energy. It therefore becomes necessary to implement collinear geometry ($\varphi = 0$). In this case, the condition given in Eq. (1) takes the form

$$v_g(\omega) = v_{\mathrm{ph}}(\Omega). \tag{2}$$

Here the parentheses indicate the arguments in order to emphasize that the group velocity relates to the optical range of frequencies, while the phase velocity relates to the THz range of frequencies.

In the theory of nonlinear waves of various physical natures, the condition given in Eq. (2) is called the Zakharov–Benney resonance (ZBR) condition.[1]

Nonlinearity that is quadratic in the field is needed for the generation method described above.

Let the light pulse and the THz signal generated by it propagate along the z axis, normal to the optical axis of the crystal. We then write the wave equation and the expansion of the polarization response P in powers of electric field E, respectively, in the form

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{3}$$

$$P(z,t) = \int_0^\infty \chi(\tau)E(z,t-\tau)\mathrm{d}\tau + \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \chi_2(\tau_1,\tau_2)E(z,t-\tau_1)E(z,t-\tau_2)\mathrm{d}\tau_1, \tag{4}$$

where t is time, c is the speed of light in vacuum, and χ and χ_2 are the linear and quadratically nonlinear susceptibilities, respectively.

We represent the field in the form of a sum of the THz E_T and an optical component with a slowly varying envelope ψ,

$$E = E_T + \Psi(z,t)\exp[i(\omega t - kz)] + \Psi^*(z,t)\exp[-i(\omega t - kz)]. \tag{5}$$

Substituting Eq. (5) into Eq. (4), we take into account the linear group dispersion of the light signal by means of the expansion

$$\Psi(z,t-\tau) \approx \Psi(z,t) - \tau\frac{\partial\Psi}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2\Psi}{\partial t^2} - \frac{\tau^3}{6}\frac{\partial^3\Psi}{\partial t^3}. \tag{6}$$

However, in the terms of the nonlinear part of the response given in Eq. (4), we set $\psi(z,t-\tau_1) \approx \psi(z,t-\tau_2) \approx \psi(z,t)$, $E_T(z,t-\tau_1) \approx E_T(z,t-\tau_2) \approx E_T(z,t)$. Moreover, because $|\psi|^2 \gg E_T^2$, we neglect the intrinsic nonlinearity of the THz component. This nonlinearity is determined by the square of the electric field E_T. We also throw away the last term in the expansion given in Eq. (6). Then, applying the SVE approximation to ψ and the SVP approximation to E_T and using the condition given in Eq. (2), we arrive at the Yajima–Oikawa (YO) system[16,17]

$$i\left(\frac{\partial\Psi}{\partial z} + \frac{1}{v_g}\frac{\partial\Psi}{\partial t}\right) = -\frac{k_2}{2}\frac{\partial^2\Psi}{\partial t^2} + aE_T\Psi, \tag{7}$$

$$\frac{\partial E_T}{\partial_z}+\frac{1}{v_g}\frac{\partial E_T}{\partial t}=-\sigma\frac{\partial}{\partial t}(|\Psi|^2), \tag{8}$$

where the group velocity is defined by $v_g = c[1+2\pi(\chi_\omega + \omega\partial\chi_\omega/\partial\omega)]^{-1}$, $k_2 = 2\pi(2\partial\chi_\omega/\partial\omega + \omega\partial^2\chi_\omega/\partial\omega^2)/c$ is the second-order group-velocity dispersion (GVD) coefficient, and parameters a and σ characterize the quadratic linearity; in this case, $a = 4\pi\omega\chi^{(2)}(\omega,0)/c$, $\sigma = 4\pi\chi^{(2)}(\omega,-\omega)/c$, $\chi^{(2)}(\omega_1,\omega_2) = \int d\tau_2 \int \chi_2(\tau_1,\tau_2)\exp[-1(\omega_1\tau_1+\omega_2\tau_2)]d\tau_1$, and $\chi_\omega = \int \chi(\tau)\exp(-i\omega\tau)d\tau$.

The YO system is integrable.[1] Its soliton solution has the form

$$\Psi = \frac{|k_2|}{\tau_p}\sqrt{\frac{\Omega}{a\sigma}}\exp[-i(\Omega t - qz)]\operatorname{sech}\left(\frac{t-z/v}{\tau_p}\right), \tag{9}$$

$$E_T = -\frac{k_2}{a\tau_p^2}\operatorname{sech}^2\left(\frac{t-z/v}{\tau_p}\right), \tag{10}$$

where Ω is the nonlinear red shift of the carrier frequency of the optical pulse, τ_p is the duration of the soliton, v and q are its velocity and a nonlinear additive to the wave number, determined by the expressions

$$\frac{1}{v}=\frac{1}{v_g}-k_2\Omega, \qquad q=\frac{k_2}{2}(\tau_p^{-2}-\Omega^2)+\frac{\Omega}{v_g}. \tag{11}$$

It follows from Eqs. (11) that the red shift is proportional to the intensity of the optical component of the soliton. This conclusion has been confirmed experimentally.[18]

Numerical experiments carried out with the YO system of Eqs. (7) and (8) show that, along with the unipolar half-wave soliton of Eq. (10), a similar half-wave surge appears with polarity opposite to that of the soliton of Eq. (10). When k_2 is positive, this surge lags behind the optico-terahertz soliton of Eqs. (9) and (10); otherwise it outruns it. In both cases, the velocity of the given surge equals the linear velocity v_g. This is understandable, since there is no optical component in the region of the given surge. Then, as can be seen from Eq. (8), the E_T dynamics obey a linear unidirectional wave equation. On the other hand, as the combined THz surge considered here propagates, the intrinsic nonlinear and dispersion effects should build up. The system of Eqs. (7) and (8) should be modified accordingly in order to take these effects into account. In this case, the intrinsic nonlinearity of the THz response discarded above should be taken into account in Eq. (4), and $E_T(z,t-\tau) \approx E_T(z,t) - \tau\partial E_T/\partial t + 0.5\tau^2\partial^2 E_T/\partial t^2$ should be used in its linear parts. Besides this, we take into account the last term in the expansion given in Eq. (6) and the inertialess cubic nonlinearity of the optical component, setting $\psi(z,t-\tau_{1,2}) \approx \psi(z,t) - \tau_{1,2}\partial\psi/\partial t$ in the quadratically nonlinear part of the response. We then arrive at the following generalized system of Eqs. (7) and (8):

$$i\left(\frac{\partial\Psi}{\partial z}+\frac{1}{v_g}\frac{\partial\Psi}{\partial t}\right) = -\frac{k_2}{2}\frac{\partial^2\Psi}{\partial t^2}+i\frac{k_3}{6}\frac{\partial^3\Psi}{\partial t^3}+aE_T\Psi - ib\Psi\frac{\partial E_T}{\partial t}-i\mu E_T\frac{\partial\Psi}{\partial t}+\varepsilon|\Psi|^2\Psi, \tag{12}$$

$$\frac{\partial E_T}{\partial_z}+\frac{1}{v_g}\frac{\partial E_T}{\partial t} = \eta\frac{\partial^3 E_T}{\partial t^3}-\lambda E_T\frac{\partial E_T}{\partial t}-\sigma\frac{\partial}{\partial t}(|\Psi|^2) + ig\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial t}-\Psi\frac{\partial\Psi^*}{\partial t}\right). \tag{13}$$

Here $k_3 = 2\pi(3\partial^2\chi_\omega/\partial\omega^2 + \omega\partial^3\chi_\omega/\partial\omega^3)/c$ is the third-order optical GVD parameter, $\eta = \pi(\partial^2\chi/\partial\omega^2)|_{\omega=0}/c$ is the THz dispersion parameter, $b = 4\pi\chi^{(2)}(\omega,0)/c$, $\mu = 4\pi[\chi^{(2)}(\omega,0)+\omega\partial^2\chi^{(2)}(\omega,0)/\partial\omega^2]/c$, $\lambda = 4\pi\chi^{(2)}(0,0)/c$, coefficient g is determined exclusively by the dispersion of the quadratic nonlinearity: $g = 4\pi(\partial\chi^{(2)}/\partial\omega)/c$, $\varepsilon = 6\pi\chi^{(3)}(\omega,\omega,-\omega)/c$, and $\chi^{(3)}(\omega,\omega,-\omega)$ is the nonlinear cubic susceptibility that corresponds to the carrier frequency of the optical pulse.

The last term in Eq. (13) determines the effect of the phase of the optical pulse on the generation of the THz radiation. The idea of this effect was suggested to the author by A. P. Sukhorukov in 2013. A study in the fixed-field approximation of the optical pulse was carried out in Ref. 19.

With the definite relationships between the coefficients in Eqs. (12) and (13) established in Ref. 20, the given system turns out to be integrable and generates solitons in the strict sense, which experience elastic interaction with each other. In this case, the one-soliton solution of Eqs. (12) and (13) coincides with the solution of Eqs. (9) and (10). However, parameters v and q, unlike Eq. (11), are now determined by expressions of the form

$$\frac{1}{v}=\frac{1}{v_g}-k_2\Omega+\frac{k_3}{6}\left(3\Omega^2-\frac{1}{\tau_p^2}\right),$$
$$q=\frac{\Omega}{v_g}-\left(k_2+\frac{k_3\Omega}{3}\right)\frac{\Omega^2}{2}+\frac{k_2-k_3\Omega}{2\tau_p^2}. \tag{14}$$

It can be seen from this that the soliton's velocity now depends on its duration. Moreover, the qualitative character of this dependence is determined by the sign of the third-order group dispersion: When $k_3 > 0$, the soliton's velocity increases as its duration gets shorter, and it decreases when $k_3 < 0$.

Shortening the width τ_p of a light pulse strengthens the role of its phase. Taking typical values of the carrier frequency of a light signal that corresponds to the visible region, it is easy to conclude that what has been said occurs for widths of the order of 10 fs. For even shorter widths, the validity of the SVE approximation and, as a consequence, the validity of the system of Eqs. (12) and (13) can be in doubt for the optical signal.

Thus, to describe the self-consistent generation regime of THz radiation as the width of the optical pulse decreases to 10 fs, it is necessary to modify the YO system, in which the influence of the phase of the given pulse is neglected. This is especially important in the neighborhood of small values of the second-order GVD parameter. The system of Eqs. (12) and (13) solves the given problem.

III. EXCLUSIVELY TERAHERTZ SOLITONS

If $\Omega = 0$ is formally set in Eq. (9), we get $\psi = 0$. Then, from Eq. (13), we have for E_T the Korteweg–de Vries equation

$$\frac{\partial E_T}{\partial_z}+\frac{1}{v_T}\frac{\partial E_T}{\partial t}=\eta\frac{\partial^3 E_T}{\partial t^3}-\lambda E_T\frac{\partial E_T}{\partial t}, \tag{15}$$

which has a soliton solution of the form

$$E_T=-\frac{3\eta}{\lambda\tau_p^2}\,\mathrm{sech}^2\left(\frac{t-z/v}{2\tau_p}\right), \tag{16}$$

where the soliton velocity v is determined by

$$\frac{1}{v}=\frac{1}{v_g}-\frac{\eta}{\tau_p^2}. \tag{17}$$

It follows from what was explained in the preceding section that, after an optical pulse is supplied to a crystal, connected optico-terahertz soliton states and exclusively THz solitons can be formed virtually simultaneously in the process of the dynamics.

The THz soliton given in Eq. (16) is formed in a quadratically nonlinear medium in the spectral region lying below the frequencies of the THz absorption ($\eta>0$).

We now consider the situation in which the spectrum of the THz signal covers the resonance frequencies of the medium. It was pointed out above that the THz region includes, for example, the frequencies of the tunnel transitions. Therefore, below we study the nonlinear propagation of a broad-band THz pulse in a medium formed by tunnel transitions.

The characteristic frequency ω_{21} of a transition between tunnel states is of the order of 10^{12} sec^{-1}. Let the characteristic time scale t_p of a THz pulse be of the order of 10^{-13} sec. Then we have the small parameter

$$\mu_1\equiv\omega_{21}t_p\ll 1. \tag{18}$$

In the case of a broad-band pulse of USP type, t_p has the meaning of its width. Then the meaning of Eq. (18) is that the spectral pulse width is $\delta\omega\sim 1/t_p\gg\omega_{21}$. That is, the spectrum of the pulse significantly overlaps the $1\leftrightarrow 2$ quantum transition considered here, and this causes them to strongly interact with each other. On the other hand, such a broad signal spectrum is capable of entraining into interaction with it quantum transitions to states that lie above, both from levels 1 and 2. We designate these states, respectively, as levels 3 and 4, thus restricting ourselves to the approximation of a four-level medium (Fig. 1). We shall assume that frequencies ω_{31} and ω_{42} of the allowed transitions $1\leftrightarrow 3$ and $2\leftrightarrow 4$ satisfy the conditions of transparency, contrary to the conditions of Eq. (18),

$$\mu_{1.2}\sim(\omega_{31}t_p)^{-1}\sim(\omega_{42}t_p)^{-1}\ll 1. \tag{19}$$

In the case of hydrogen-containing ferroelectrics, $\omega_{31}\approx 10^{14}$ sec^{-1}, and this completely agrees with the approximation of Eq. (19). We neglect the transitions between states 3 and 4 as higher-order effects.

As applied to questions of the propagation of USPs in various media, the approximations given by Eqs. (18) and (19) were first used in Refs. 21 and 22.

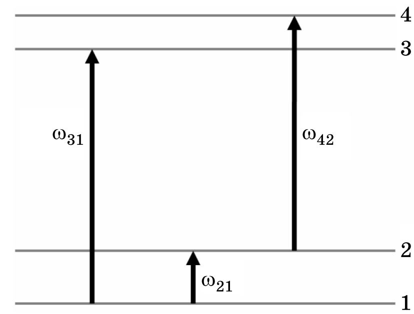

FIG. 1. Quantum model of the medium. The arrows indicate the allowed transitions, and the corresponding frequencies are designated.

In accordance with the diagram of allowed transitions in Fig. 1, we write a system of equations for the elements of the corresponding density matrix ρ

$$\begin{aligned}\frac{\partial\rho_{21}}{\partial t}&=-i\omega_{21}\rho_{21}+i\Omega_{21}(\rho_{11}-\rho_{22})+\Omega_{42}\rho_{41}-i\Omega_{31}\rho_{32}^*,\\ \frac{\partial\rho_{31}}{\partial t}&=-i\omega_{31}\rho_{31}+i\Omega_{31}(\rho_{11}-\rho_{33})-i\Omega_{21}\rho_{32},\\ \frac{\partial\rho_{42}}{\partial t}&=-i\omega_{42}\rho_{42}+i\Omega_{42}(\rho_{22}-\rho_{44})-i\Omega_{21}\rho_{41},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial\rho_{32}}{\partial t}&=-i\omega_{32}\rho_{32}+i(\Omega_{31}\rho_{21}^*-\Omega_{21}\rho_{31}-\Omega_{42}\rho_{43}^*),\\ \frac{\partial\rho_{41}}{\partial t}&=-i\omega_{41}\rho_{41}+i(\Omega_{42}\rho_{21}-\Omega_{31}\rho_{43}-\Omega_{21}\rho_{42}),\\ \frac{\partial\rho_{43}}{\partial t}&=-i\omega_{43}\rho_{43}+i(\Omega_{42}\rho_{32}^*-\Omega_{31}\rho_{41}),\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial\rho_{11}}{\partial t}&=i\Omega_{21}(\rho_{21}-\rho_{21}^*)+i\Omega_{31}(\rho_{31}-\rho_{31}^*),\\ \frac{\partial\rho_{22}}{\partial t}&=-i\Omega_{21}(\rho_{21}-\rho_{21}^*)+i\Omega_{42}(\rho_{42}-\rho_{42}^*),\\ \frac{\partial\rho_{33}}{\partial t}&=-i\Omega_{31}(\rho_{31}-\rho_{31}^*),\\ \frac{\partial\rho_{44}}{\partial t}&=-i\Omega_{42}(\rho_{42}-\rho_{42}^*).\end{aligned} \tag{22}$$

Here $\Omega_{jk}=d_{jk}E/\hbar$, d_{jk} are the dipole moments of the allowed quantum transitions, assumed to be real, and $\hbar$ is the reduced Planck's constant.

Equations (20) and (21) describe the dynamics of the off-diagonal elements of ρ for allowed and forbidden transitions, respectively. The system of Eqs. (22) corresponds to the population dynamics of the quantum levels.

The polarization response in the case under consideration has the form

$$P=n[d_{21}(\rho_{21}+\rho_{21}^*)+d_{31}(\rho_{31}+\rho_{31}^*)+d_{42}(\rho_{42}+\rho_{42}^*)], \tag{23}$$

where n is the concentration of tunnel centers.

The conditions given in Eqs. (18) and (19) make it possible to express in first approximation in parameters μ_1 and μ_2 the density-matrix elements in terms of the electric field of the pulse, as is done in Refs. 21 and 22 for two-level atoms. These calculations become more burdensome in the four-level case considered here, but do not differ in principle from those in the papers mentioned above. Then we get

$$P = P^{(0)} + P^{(1)}. \tag{24}$$

In this case,

$$P^{(0)} = \frac{2n}{\hbar}\left[\frac{d_{31}^2}{\omega_{31}}\left(\frac{1}{2} - W_\infty\right) + \frac{d_{42}^2}{\omega_{42}}\left(\frac{1}{2} + W_\infty\right)\right]E, \tag{25}$$

$$\frac{\partial P^{(1)}}{\partial t} = -2nW_\infty\left\{d_{21}\left[\omega_{21} + \frac{1}{\hbar^2}\left(\frac{d_{31}^2}{\omega_{31}} - \frac{d_{42}^2}{\omega_{42}}\right)E^2\right]\sin\theta - \frac{2}{\hbar}\frac{\partial}{\partial t}\left[\left(\frac{d_{31}^2}{\omega_{31}} - \frac{d_{42}^2}{\omega_{42}}\right)E\sin^2\frac{\theta}{2}\right]\right\}, \tag{26}$$

where $W_\infty = (\rho_{22} - \rho_{11})/2|_{t=-\infty}$ is the normalized initial population difference of the tunnel states, while

$$\theta = \frac{2d_{21}}{\hbar}\int_{-\infty}^{t} E_T \mathrm{d}t'.$$

Substituting Eqs. (24)–(26) into the right-hand side of wave Eq. (3), after using the SVP approximation, we get

$$\frac{\partial^2\theta}{\partial z\partial t} + \frac{1}{v_0}\frac{\partial^2\theta}{\partial t^2} - 4\beta\frac{\partial^2\theta}{\partial t^2}\sin^2\frac{\theta}{2}\left[\alpha - \beta\left(\frac{\partial\theta}{\partial t}\right)^2\right]\sin\theta = 0. \tag{27}$$

Here $\alpha = -8\pi d_{21}^2 n\omega_{21}v_0W_\infty/(\hbar c^2)$, $\beta = -2\pi v_0W_\infty[d_{31}^2/\omega_{31} - d_{42}^2/\omega_{42}]/(\hbar c^2)$, and v_0 is the linear velocity, determined from

$$\frac{1}{v_0} = \frac{1}{c}\sqrt{1 + \frac{8\pi n}{\hbar}\left[\frac{d_{31}^2}{\omega_{31}}\left(\frac{1}{2} - W_\infty\right) + \frac{d_{42}^2}{\omega_{42}}\left(\frac{1}{2} + W_\infty\right)\right]}. \tag{28}$$

Substituting $\beta = 0$ into Eq. (27), we arrive at the well-known sine-Gordon equation, which has soliton solutions.

It can be seen from Eq. (27) that, in the approximation used in deriving it, linear dispersion is created only by the tunnel transition $1 \leftrightarrow 2$ (coefficient α). However, transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ alter the linear, inertialess part of the refractive index of the medium, determined by the square root in Eq. (28), and create additional nonlinearity, taken into account in Eq. (27) by coefficient β.

We especially emphasize that Eq. (27), generally speaking, cannot be regarded as a weakly perturbed sine-Gordon equation. Actually, the ratio of the second term in brackets in Eq. (27) to the first is of the same order of magnitude as $(d/d_{21})^2(\mu_2/\mu_1)$, where $d \sim d_{31}, d_{42}$. It can be seen from this that the given terms can be in an arbitrary quantitative ratio with respect to each other, including being of the same order of magnitude. Consequently, transitions $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, and $2 \leftrightarrow 4$ in the approximation considered here can introduce comparable contributions into the pulse-propagation dynamics.

The solitonlike solution of Eq. (27) for the electric field of the pulse has the form

$$E_T = \frac{\hbar}{d_{21}\sqrt{\tau_p^2 + 4\beta/\alpha}}\operatorname{sech}\xi, \tag{29}$$

where τ_p is a free parameter associated with the pulse width (see below), while the dynamic variable ξ when $\beta/\alpha > 0$ is determined from the transcendental equation

$$\xi + \frac{2}{\tau_p}\sqrt{\frac{\beta}{\alpha}}\arctan\left(\frac{2}{\tau_p}\sqrt{\frac{\beta}{\alpha}}\tanh\xi\right) = \frac{t - z/v}{\tau_p}. \tag{30}$$

However, if $\beta/\alpha < 0$, we have

$$\xi - \frac{2}{\tau_p}\sqrt{-\frac{\beta}{\alpha}}\operatorname{arctanh}\left(\frac{2}{\tau_p}\sqrt{-\frac{\beta}{\alpha}}\tanh\xi\right) = \frac{t - z/v}{\tau_p}. \tag{31}$$

In this case, the propagation velocity v of the USP is associated with τ_p by

$$\frac{1}{v} = \frac{1}{v_0} + \alpha\tau_p^2. \tag{32}$$

When $\beta = 0$, as expected, the solution in both cases results in a soliton of the sine-Gordon equation.

It can be seen from Eqs. (29), (30), and (31) that the "area" of the solitonlike pulse in both cases ($\beta/\alpha > 0$ and $\beta/\alpha < 0$) is

$$A = \frac{2d_{21}}{\hbar}\int_{-\infty}^{+\infty} E\mathrm{d}t = 2\pi.$$

It follows from this that, during its propagation, the populations of the levels of the $1 \leftrightarrow 2$ tunnel transition experience complete inversion, after which they return to their initial values. Below we present an analysis of the solution of Eqs. (29)–(32) separately for $\beta/\alpha > 0$ and $\beta/\alpha < 0$.

1. *The $\beta/\alpha > 0$ case*. A graph of the dependence of the electric field on the "running time" is shown in Fig. 2 for various τ_p values. For comparison, the same figure has a dotted line that shows for the same value of τ_p the corresponding dependence when $\beta = 0$, when only the tunnel quantum transition $1 \leftrightarrow 2$ participates in the interaction with the field; this corresponds to a soliton of the sine-Gordon equation. It can be seen that the "soliton" in the case under consideration is less sharp and more spread out in time than the sine-Gordon

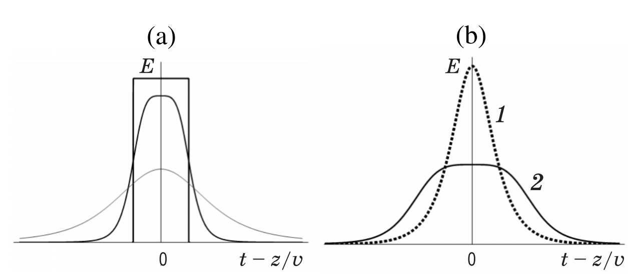

FIG. 2. Profiles of the "soliton" of Eq. (20) for $\beta/\alpha > 0$; the thickness of the curves increases as the pulse width t_p decreases (a). Profiles of a sine-Gordon soliton (1) and the soliton of Eq. (27) (2) for $\beta/\alpha > 0$ and the same value of free parameter τ_p (b).

soliton. The given difference becomes more and more prominent as the free parameter τ_p decreases, and the electric field profile takes a rectangular form when $\tau_p = 0$ (Fig. 2). The trend to a "limiting soliton" as $\tau_p \to 0$ thus becomes comprehensible in the approximation used here.

In general, we define the pulse width t_p as the interval between the times at a fixed z where the electric field is a factor of $e \approx 2.718$ less than the amplitude. Then

$$t_p^* = 3.315\tau_p^* + 2\ \arctan\left(\frac{0.930}{\tau_p^*}\right). \tag{33}$$

Here and below, $t_p^* = 0.5t_p = (|\alpha/\beta|)^{1/2}$, and $\tau_p^* = 0.5\tau_p = (|\alpha/\beta|)^{1/2}$.

Figure 3 shows the $t_p^*(\tau_p^*)$ dependence. It can be seen from the figure that the minimum width $t_p^{(\min)}$ is reached when $\tau_p = 0$ and corresponds to a "limiting soliton" of rectangular profile in Fig. 2 with maximum amplitude $E_m = 0.5(\hbar/d_{21})(\alpha/\beta)^{1/2}$ and propagation velocity $v = v_0$. The minimum width is determined from

$$t_p^{(\min)} = 2\pi\sqrt{\beta/\alpha}. \tag{34}$$

Let us present numerical estimates of the parameters of the "limiting soliton." It follows from the definitions of coefficients α and β that $|\alpha| \sim \omega_c^{(21)}\omega_{21}$ and $|\beta| \sim \omega_c/(c\omega_0)$, where $\omega_c^{(21)} = 4\pi d_{21}^2 n/\hbar$ and $\omega_c = 4\pi d_{31}^2 n/\hbar \sim 4\pi d_{42}^2 n/\hbar$ are the collective frequencies at the transitions $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$, respectively, $\omega_0 \sim \omega_{31} \sim \omega_{42}$. Taking $d_{21} \sim d_{31} \sim d_{42} \sim 10^{-18}$ esu, and $n \sim 10^{22}\ \mathrm{cm}^{-3}$, we get $\omega_c^{(21)} \sim \omega_c \approx 10^{14}\ \mathrm{sec}^{-1}$, $v \approx v_0 \sim c$. Moreover, $\Omega_{31} \sim d_{31}E_m/\hbar \sim (\omega_{21}\omega_0)^{1/2}$, and $t_p \sim (\omega_{21}\omega_0)^{-1/2}$. Let $d_{31} \sim d_{21}$, $\omega_{21} \sim 10^{12}\ \mathrm{sec}^{-1}$, and $\omega_0 \sim 10^{14}\ \mathrm{sec}^{-1}$. Then $\Omega_{31} \sim 10^{13}\ \mathrm{sec}^{-1}$ and $t_p \sim 10^{-13}$ sec, which agrees well with the conditions given by Eqs. (18) and (19), if as the time scale of the pulse is used its width t_p. Using the values given above, we find for the pulse intensity $I \approx cE^2/(4\pi) = c(\hbar\Omega_{31}/d_{31})^2 \sim 10^{10}\ \mathrm{W/cm}^2$. It is expected that such intensities of the THz signals can be obtained in the next few years by using various methods of focusing them.[23–25]

2. *The $\beta/\alpha < 0$ case*. After determining the width of the soliton, as in the preceding case, we write

$$t_p^* = 3.315\tau_p^* - 2\,\mathrm{arctanh}\left(\frac{0.930}{\tau_p^*}\right). \tag{35}$$

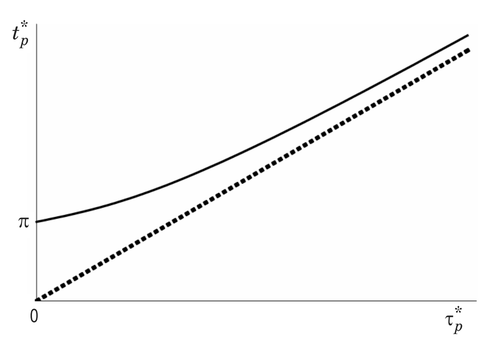

FIG. 3. Pulse width t_p of a "soliton" versus parameter τ_p for $\beta/\alpha > 0$. For convenience, both parameters are made dimensionless by normalizing them by the quantity $2(|\beta/\alpha|)^{1/2}$.

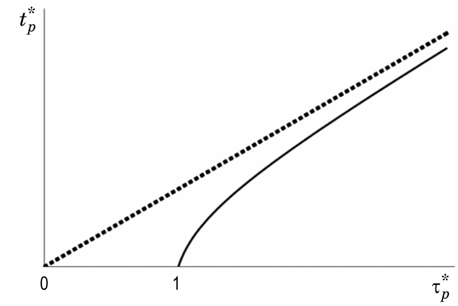

FIG. 4. Pulse width t_p of a "soliton" versus parameter τ_p for $\beta/\alpha < 0$. The parameters are made dimensionless by the same rule as in Fig. 3.

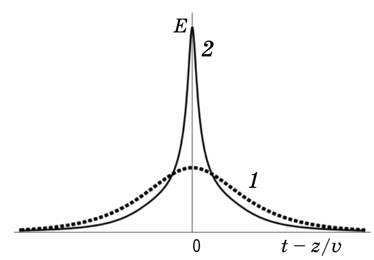

FIG. 5. Profiles of a sine-Gordon soliton (1) and a soliton of Eq. (27) (2) for $\beta/\alpha < 0$ and the same value of free parameter τ_p.

The corresponding dependence is shown in Fig. 4. As follows from this, as well as from Eqs. (29) and (31), $t_p^* > 2(|\beta/\alpha|)^{1/2}$ (or $\tau_p^* > 1$). As τ_p decreases (and with it also t_p), the soliton becomes even sharper, and, when $\tau_p \to 2|\beta/\alpha|^{1/2}$, its amplitude sharply increases, the width t_p sharply decreases in the same way. With the equilibrium initial population of tunnel states, $\alpha > 0$. Consequently, $\beta < 0$ and $v < v_0$. However, in the nonequilibrium case, $\beta > 0$, and therefore $v > v_0$. Note that, in the unidirectional-propagation approximation used here, the soliton velocity insignificantly differs from v_0. Therefore, the situation for which v is negative is excluded here. It can be seen from Fig. 5 that the "soliton," in contrast to the case $\beta/\alpha > 0$, is higher and sharper here than the sine-Gordon soliton for the same value of τ_p. It can therefore be called a "sharpened soliton."

In both cases, when $\tau_p \gg t_p^{(\min)}$, the solitonlike solutions considered here approximate those of a sine-Gordon soliton in their properties.

As the "soliton" propagates, the populations are totally exchanged between the tunnel levels, with a final return to their original values. However, here the influence of the remote states on the population dynamics of the tunnel states can be fundamental. When $\beta/\alpha > 0$, the inverted population difference of the tunnel states is long-lived by comparison with the sine-Gordon case. When a "sharpened soliton" ($\beta/\alpha < 0$) propagates, however, the population difference of the tunnel states, on the contrary, is short-lived.

IV. CONCLUSION

This paper has thus discussed various cases of the formation in nonlinear media of optico-terahertz and exclusively

THz solitons. A fairly complete review of the questions of the optical generation of THz radiation, including soliton regimes, is contained in Ref. 26. Besides this, a model has been proposed of a medium that contains tunnel transitions and experiences interaction with pulses of THz radiation. We especially emphasize that, to adequately describe the given interaction, all the quantum transitions considered here are important, since they can cause contributions that are commensurable with each other to interact with the pulse. Therefore, the generalized sine-Gordon Eqs. (27) should not be regarded as just a correctional modification of the sine-Gordon equation, where only the tunnel transition $1 \leftrightarrow 2$ is taken into account. That is why the solitonlike solutions of Eq. (27) found here can be fundamentally different from sine-Gordon solitons. The question of whether or not these solutions are solitons in the strict sense of the word remains open and is obviously of interest. An answer can be obtained here after a comprehensive study of the mathematical structure of Eq. (27).

It may be that the theoretical model involving the interaction of THz pulses with tunnel transitions used in this paper can be adapted to modify the transfer processes of electrons and excitation quasi-particles in the system of quantum dots considered in Ref. 12. These processes may be effective during the action of broad-band THz signals.

In the light of what was said above in this paper, the question also arises of studying the nonlinear dynamics of broad-band THz pulses in a system not only of tunnel processes but also of other degrees of freedom of the medium, including vibrational, rotational, etc. Vibrational degrees of freedom were considered, for example, in Ref. 27, but only in the linear approximation. In this connection, there is interest in constructing a very simple but universal theoretical model of nonlinear propagation of broad-band THz pulses in dielectric media, taking into account all the main interaction mechanisms. There is also interest in studying the diffraction of broad-band THz signals in nonlinear media. As shown by the theoretical studies of Ref. 28, spatiotemporal and spectral distortions that are absent in the case of quasi-monochromatic pulses already show up here in the linear approximation.

The investigation presented in this paper, stimulated by the significant growth in recent years of the efficiency with which THz radiation is generated and by the increase in its intensity, makes it urgently necessary to develop *nonlinear terahertz optics* as a separate area of working on the interaction of radiation with matter.

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