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Singular value decomposition for the 2D fan-beam Radon transform of tensor fields

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Abstract — In this article we study the fan-beam Radon transform $\mathcal{D}_m$ of symmetrical solenoidal 2D tensor fields of arbitrary rank m in a unit disc $\mathbb{D}$ as the operator, acting from the object space $\mathbf{L}_2(\mathbb{D}; \mathbf{S}_m)$ to the data space $L_2([0, 2\pi) \times [0, 2\pi))$. The orthogonal polynomial basis $\mathbf{s}_{n,k}^{(\pm m)}$ of solenoidal tensor fields on the disc $\mathbb{D}$ was built with the help of Zernike polynomials and then a singular value decomposition (SVD) for the operator $\mathcal{D}_m$ was obtained. The inversion formula for the fan-beam tensor transform $\mathcal{D}_m$ follows from this decomposition. Thus obtained inversion formula can be used as a tomographic filter for splitting a known tensor field into potential and solenoidal parts. Numerical results are presented.

1. INTRODUCTION

The problem of determining vector or tensor field from the integral information arises in various applications, for instance in ultrasound probing of fluid or gas flows and deformed elastic media. In the first case it's required to determine the velocity vector field in the flow and in the second case — the stress tensor field.

One of the most complete monograph of the tensor tomography is [23]. Reversibility and stability of different kinds of transforms of tensor fields on the Riemannian manifolds are studied there. In [5, 19] the solution of the vector tomography problem is reduced to the scalar Radon problem. An approximate solution of the vector and tensor (of rank 2) tomography problem is given in [9, 10] with the help of polynomial non-orthogonal basis.

More information and references about vector and tensor tomography problems are given in [2, 22, 24, 25].

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In this article we derive an inversion formula on the basis of singular value decomposition (SVD) for the fan-beam transform of tensor fields. To this end the orthonormal polynomial basis of solenoidal tensor fields, supported in unit disk, are built from Zernike polynomials. In the scalar case thus obtained SVD corresponds to the known SVD for the Radon transform in the classical (parallel) formulation [6, 7, 12, 16, 15].

Unlike the scalar case, Radon transform of tensor fields has a non-zero kernel and it's possible to reconstruct uniquely (without additional information) only the solenoidal part of a tensor field, so the inversion formula can be used as a tomographic filter for splitting a known tensor field into potential and solenoidal parts.

This article is organized as follows: In Section 2 we formulate the problem of 2D tensor tomography. In Section 3 we review those part of the tensor fields theory that are needed in this paper. Section 4 contain a novel properties of Zernike polynomials. Sections 5 is devoted to the orthogonal polynomial basis in the space of solenoidal (divergence free) tensor fields and a singular value decomposition (SVD) for the tensor tomograph problem. A short description of the implementation issues and numerical tests are presented in Section 6.

2. FORMULATION OF THE PROBLEM

Let us consider the Cartesian coordinate system (x^1, x^2) on the plane ^{2} and let $\mathbf{T}_m$ denote for $m = 0, 1, \dots$ the space of all real-valued *m-covariant tensors*

$$\mathbf{a} := a_{i_1 \dots i_m} dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_m} \text{ or } \mathbf{a} = \{a_{i_1 \dots i_m},\ i_1, \dots, i_m = 1, 2\},$$

where $\otimes$ is the tensor product and $a_{i_1 \dots i_m}$ are the components of $\mathbf{a}$ in the Cartesian basis (x^1, x^2). Here and throughout we imply the summation convention. By $\mathbf{S}_m$ we denote the subspace of *symmetric* m-covariant tensors and there exists a canonical projection $\sigma : \mathbf{T}_m \to \mathbf{S}_m$ (called symmetrization) onto this space defined by the equation

$$(\sigma \mathbf{a})_{i_1 \dots i_m} := \frac{1}{m!} \sum_{\pi \in \Pi_m} a_{i_{\pi(1)} \dots i_{\pi(m)}}, \tag{2.1}$$

where Π_m is the group of all permutations of degree m. A symmetric m-covariant tensor $\mathbf{a} = \{a_{i_1 \dots i_m}, i_1, \dots, i_m = 1, 2\}$ has only $m+1$ independent components which we denoted by a_k, so that

$$a_k := a_{\underbrace{1 \dots 1}_{k} \underbrace{2 \dots 2}_{m-k}}, \ (k = 0, \dots, m). \tag{2.2}$$

We will always denote vector and tensor fields and any related quantities such as functional spaces by boldface characters.

Let $\mathbb{D} := \{(x^1, x^2) \in {}^2 \mid (x^1)^2 + (x^2)^2 < 1\}$ be a unit disc on the plane ^{2}. The symmetric m-covariant tensor field $\mathbf{a}(x^1, x^2)$ defined on $\mathbb{D}$ can be treated as a mapping

$$\mathbf{a} : \mathbb{D} \to \mathbf{S}_m,\ \mathbf{a}(x^1, x^2) = \{a_{i_1 \dots i_m}(x^1, x^2),\ i_1, \dots, i_m = 1, 2\}.$$

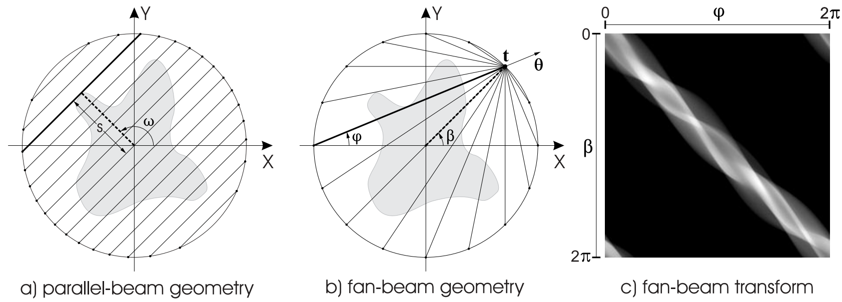

Figure 1. Left: parallel-beam scanning geometry. Middle: fan-beam scanning geometry. Right: an example of the fan-beam transform (the data function or sinogram) $f(\beta,\varphi)$, the angle β defines *the vertex point* of the fan-beam projection $f(\beta,\cdot)$ and the angle φ defines the direction of scanning

The fan-beam Radon transform $\mathcal{D}_m$ of tensor field $\mathbf{a}(x^1,x^2)$ is defined by

$$[\mathcal{D}_m\mathbf{a}](\beta,\varphi) := \int_0^{2\cos(\beta-\varphi)} \theta^{i_1}\cdot\theta^{i_2}\cdot\dots\cdot\theta^{i_m}\, a_{i_1\dots i_m}(\cos\beta - l\cos\varphi, \sin\beta - l\sin\varphi)\,\mathrm{d}l, \tag{2.3}$$

where $\beta\in[0,2\pi)$, $\boldsymbol{\theta} = \begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$, $|\beta-\varphi|\le\dfrac{\pi}{2}$.

The difference between *the parallel-beam* and *the fan-beam geometry* is shown in Figure 1.

For $|\beta-\varphi|>\pi/2$ we complete the definition of the fan-beam transform (2.3) with the condition

$$[\mathcal{D}_m\mathbf{a}](\beta,\varphi) := (-1)^{m+1}[\mathcal{D}_m\mathbf{a}](\beta,\varphi+\pi). \tag{2.4}$$

Note, that the case $m=0$ corresponds to the fan-beam Radon transform $\mathcal{D}_0\equiv\mathcal{D}$ of a scalar function $a(x^1,x^2)$.

Now, the problem is to recover the unknown tensor field $\mathbf{a}(x^1,x^2)$ in the unit disc $\mathbb{D}$ from the data function $f(\beta,\varphi)$, see Figure 1c, such that

$$[\mathcal{D}_m\mathbf{a}](\beta,\varphi) = f(\beta,\varphi),\ (\beta,\varphi)\in[0,2\pi)\times[0,2\pi).$$

This problem will be solved here by the SVD-method.

3. PRELIMINARIES

In this section, we introduce the definition of SVD method and then review some facts from vector and tensor analysis [23] and, in particular, consider real-valued tensor fields in complex coordinates (variables) [26]. We define here some functional spaces of tensor fields — $\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$, for example, and also establish the notations that will be used in the sequel.

3.1. Singular value decomposition (SVD)

Now we define the concept of a *singular value decomposition*, see [17, 18, 15, 21]. Let U and V be Hilbert spaces, and A be a compact linear operator from U to V, $A \in \mathcal{L}(U, V)$. Then there exists a sequence $\{\sigma_k\}_{k\geq 1}$ of positive numbers, monotonically tending to zero (or a finite sequence) and two orthonormal systems $\{u_k\}_{k\geq 1} \subset U$, $\{v_k\}_{k\geq 1} \subset V$, such that for all $u \in U$ we have a singular value decomposition

$$Au = \sum_{k=1}^{\infty} \sigma_k (u, u_k)_U v_k, \qquad Au_k = \sigma_k v_k, \qquad \sigma_1 \geq \sigma_2 \geq \cdots > 0.$$

The adjoint of A is given by

$$A^* v = \sum_{k=1}^{\infty} \sigma_k (v, v_k)_V u_k, \qquad A^* v_k = \sigma_k u_k$$

and the generalized inverse of A is

$$A^+ v = \sum_{k=1}^{\infty} \sigma_k^{-1} (v, v_k)_V u_k.$$

Operator A^+ can be unbounded, so one can use a truncated SVD for its regularization

$$T_\gamma v = \sum_{k\leq 1/\gamma} \sigma_k^{-1} (v, v_k)_V u_k,$$

where γ is the parameter of regularization. SVD is one of the methods for solving ill-posed problems and it allows to characterize the range of the operator, invert it and estimate an incorrectness of the corresponding inverse problem.

3.2. Tensor fields in complex coordinates

Let's identify ^{2} with the complex plane by the usual way

$$z^1 \equiv z := x^1 + \mathrm{i}x^2, \qquad z^2 \equiv \bar{z} := x^1 - \mathrm{i}x^2, \qquad \mathrm{i}^2 = -1.$$

Let $\mathbf{a} = \{a_{i_1\ldots i_m}(x^1, x^2)\}$ be an m-covariant real-valued tensor field in Cartesian coordinates (x^1, x^2), then in complex coordinates or variables $(z, \bar{z})$ it will have new components $A_{i_1\ldots i_m}(z, \bar{z})$, which are formally expressed by *the covariant tensor law*

$$A_{i_1\ldots i_m}(z, \bar{z}) = \frac{\partial x^{s_1}}{\partial z^{i_1}} \cdots \frac{\partial x^{s_m}}{\partial z^{i_m}} a_{s_1\ldots s_m}(x^1, x^2), \tag{3.1}$$

where the Jacobian matrix is

$$J \equiv (J^i_j) := \begin{pmatrix} \dfrac{\partial z^1}{\partial x^1} & \dfrac{\partial z^1}{\partial x^2} \\ \dfrac{\partial z^2}{\partial x^1} & \dfrac{\partial z^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}$$

and the inverse matrix of it is

$$J^{-1} = \begin{pmatrix} \dfrac{\partial x^1}{\partial z^1} & \dfrac{\partial x^1}{\partial z^2} \\ \dfrac{\partial x^2}{\partial z^1} & \dfrac{\partial x^2}{\partial z^2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}.$$

Here the formal partial derivatives with respect to z^1 and z^2 are defined in the usual way

$$\begin{aligned} \frac{\partial}{\partial z^1} &\equiv \frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x^1} - \mathrm{i}\,\frac{\partial}{\partial x^2}\Big), \\ \frac{\partial}{\partial z^2} &\equiv \frac{\partial}{\partial \bar z} := \frac{1}{2}\Big(\frac{\partial}{\partial x^1} + \mathrm{i}\,\frac{\partial}{\partial x^2}\Big). \end{aligned} \tag{3.2}$$

We shall write transformation (3.1) as

$$\mathbf{a} = \{a_{i_1\ldots i_m}(x^1,x^2)\} \quad \rightarrowtail \quad \mathbf{A} = \{A_{i_1\ldots i_m}(z,\bar z)\}.$$

From now on small letters will be used to denote tensor fields in the initial Cartesian coordinate system (x^1, x^2) and capital letters will be used for the same tensor fields in complex coordinates $(z, \bar z)$.

An inverse relationship also takes place

$$a_{i_1\ldots i_m}(x^1,x^2) = \frac{\partial z^{s_1}}{\partial x^{i_1}}\cdots\frac{\partial z^{s_m}}{\partial x^{i_m}} A_{s_1\ldots s_m}(z,\bar z), \tag{3.3}$$

and we shall also write this as

$$\mathbf{A} = \{A_{i_1\ldots i_m}(z,\bar z)\} \quad \rightarrowtail \quad \mathbf{a} = \{a_{i_1\ldots i_m}(x^1,x^2)\}.$$

A symmetric m-covariant tensor $\mathbf{A}$ could also be given by its components A_k

$$A_k := A_{\underbrace{1\ldots1}_{k}\underbrace{2\ldots2}_{m-k}}, \quad (k = 0,\ldots,m) \tag{3.4}$$

and subject to the conditions

$$A_k = \overline{A}_{m-k}, \quad (k = 0,\ldots,m). \tag{3.5}$$

So we may image the symmetric tensor as pseudovector, expanding the one as a column array for convenience, that the following notations will be used

$$\mathbf{a} = \begin{pmatrix} a_m \\ a_{m-1} \\ \ldots \\ a_1 \\ a_0 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} A_m \\ A_{m-1} \\ \ldots \\ A_1 \\ A_0 \end{pmatrix}. \tag{3.6}$$

Taking into account the tensor law (3.1), (3.3) we get the formulae that link independent components (2.2) and (3.4) in pseudovectors (3.6)

$$a_k = (-\mathrm{i})^{m-k} \sum_{p=0}^{m-k} \sum_{q=0}^{k} C_{m-k}^{p} C_k^q (-1)^p A_{p+q}, \tag{3.7}$$

$$A_k = \frac{\mathrm{i}^{m-k}}{2^m} \sum_{r=0}^{m-k} \sum_{s=0}^{k} C_{m-k}^{r} C_k^s (-\mathrm{i})^{k+r-s} a_{r+s}, \tag{3.8}$$

where $k = 0, 1, \ldots, m$ and C_j^i are binomial coefficients.

3.3. Metric tensor G and pointwise inner product in complex coordinates

On parity with covariant components of the tensor we shall also use its *contravariant components*. In Cartesian coordinates (x^1, x^2) covariant and contravariant components g_{ij} and g^{ij} of the metric tensor $\mathbf{g}$ are the same

$$\mathbf{g} = \{g_{ij}\} = \{g^{ij}\} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}. \tag{3.9}$$

Thus contravariant components of the tensor $\mathbf{a}$ coincide with its corresponding covariant components, $a_{i_1 \ldots i_m} = a^{i_1 \cdots i_m}$. The pointwise inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{S}_m$ induced by the Euclidian metric $\mathbf{g}$ (3.9) is defined by the formula

$$\langle \mathbf{a}, \mathbf{b} \rangle := a^{i_1 i_2 \ldots i_m} b_{i_1 i_2 \ldots i_m}.$$

In complex coordinates $(z, \bar{z})$ the metric tensor $\mathbf{G}$ has the following covariant

$$\{G_{ij}\} = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{Bmatrix}$$

and contravariant components

$$\{G^{ij}\} = \begin{Bmatrix} 0 & 2 \\ 2 & 0 \end{Bmatrix}. \tag{3.10}$$

Contravariant components of tensor $\mathbf{A}$ in complex coordinates are obtained by raising indexes with contravariant components of the metric tensor (3.10)

$$A^{i_1 i_2 \ldots i_m} = G^{i_1 j_1} G^{i_2 j_2} \ldots G^{i_m j_m} A_{j_1 j_2 \ldots j_m}$$

and the pointwise inner product of tensor fields is evaluated by formula

$$\langle \mathbf{A}, \mathbf{B} \rangle = A^{i_1 i_2 \ldots i_m} B_{i_1 i_2 \ldots i_m} = A_{i_1 i_2 \ldots i_m} B^{i_1 i_2 \ldots i_m}.$$

If tensors $\mathbf{A} = \{A_k\}$ and $\mathbf{B} = \{B_k\}$ are considered as pseudovectors in complex coordinates then their pointwise inner product will be equal to

$$\langle \mathbf{A}, \mathbf{B} \rangle = 2^m \sum_{k=0}^{m} C_m^k A_k B_{m-k}. \tag{3.11}$$

The pointwise norm of tensor $\mathbf{A}$ then will be

$$|\mathbf{A}|^2 = 2^m \sum_{k=0}^{m} C_m^k |A_k|^2. \tag{3.12}$$

It is clear that the pointwise inner product is invariant, i.e. if $\mathbf{a} \rightarrowtail \mathbf{A}$ and $\mathbf{b} \rightarrowtail \mathbf{B}$, than

$$\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{A}, \mathbf{B} \rangle. \tag{3.13}$$

3.4. The space of integrable tensor fields $\mathbf{L}_2(\mathbb{D}\,; \mathbf{S}_m)$

Let $\mathbf{L}_2(\mathbb{D}\,; \mathbf{S}_m)$ denote a Hilbert space comprising real-valued symmetric m-covariant tensor fields on $\mathbb{D}$ with the inner product, denoted by $\langle\langle \cdot, \cdot \rangle\rangle$

$$\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle \equiv \langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle_{\mathbf{L}_2(\mathbb{D};\mathbf{S}_m)} := \iint_{\mathbb{D}} \langle \mathbf{a}(x^1, x^2), \mathbf{b}(x^1, x^2) \rangle \, \mathrm{d}V^2,$$
$$\mathrm{d}V^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and the finite norm $\|\cdot\|$

$$\|\mathbf{a}\|^2 \equiv \|\mathbf{a}\|^2_{\mathbf{L}_2(\mathbb{D};\mathbf{S}_m)} := \iint_{\mathbb{D}} \langle \mathbf{a}(x^1, x^2), \mathbf{a}(x^1, x^2) \rangle \, \mathrm{d}V^2.$$

In complex coordinates for $\mathbf{a} \rightarrowtail \mathbf{A}$ and $\mathbf{b} \rightarrowtail \mathbf{B}$ we have

$$\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle = \iint_{\mathbb{D}} \langle \mathbf{A}(z, \bar{z}), \mathbf{B}(z, \bar{z}) \rangle \, \mathrm{d}V^2, \qquad \mathrm{d}V^2 = \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{-2\mathrm{i}}.$$

By virtue of invariance of inner product (3.13) the following equalities take place

$$\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle = \langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle, \qquad \|\mathbf{a}\| = \|\mathbf{A}\|.$$

3.5. Differential operations on symmetric tensor fields

We shall denote the class of real-valued m-covariant symmetric tensor fields $\mathbf{a} = \{a_{i_1 \ldots i_m}(x^1, x^2)\}$, whose all components are functions from $C^k(\mathbb{D}\,)$, $1 \le k \le \infty$ by $\mathbf{C}^k(\mathbb{D}\,; \mathbf{S}_m)$. A subset of $\mathbf{C}^k(\mathbb{D}\,; \mathbf{S}_m)$ whose finite support is contained in $\mathbb{D}$ will be denoted by $\mathbf{C}^k_0(\mathbb{D}\,; \mathbf{S}_m)$.

The operator of *covariant differentiation* ∇ (in the vectorial case $\equiv$ grad)

$$\nabla\colon \quad \mathbf{C}^\infty(\mathbb{D}\,; \mathbf{S}_m) \to \mathbf{C}^\infty(\mathbb{D}\,; \mathbf{S}_{m+1})$$

in Cartesian coordinate system (x^1, x^2) is defined by equation

$$\nabla \mathbf{a} := \{a_{i_1 \ldots i_m ; j}\} = \Big\{ \frac{\partial a_{i_1 \ldots i_m}}{\partial x^j}, \ j = 1, 2 \Big\}.$$

The covariant differentiation ∇ operates on any tensor field of rank $m \ge 0$ and produces a tensor field that is one rank higher. For example, the gradient of a (co)vector field is a second rank tensor field.

In complex coordinates we have

$$\nabla \mathbf{A} = \{A_{i_1\dots i_m;j}\} = \Big\{\frac{\partial A_{i_1\dots i_m}}{\partial z^j},\ j=1,2\Big\},$$

where formal partial derivatives with respect to z^1 and z^2 are defined by (3.2).

The operator of *divergence* δ (in the vectorial case $\equiv$ div)

$$\delta\colon\quad \mathbf{C}^\infty(\mathbb{D}\,;\mathbf{S}_m) \to \mathbf{C}^\infty(\mathbb{D}\,;\mathbf{S}_{m-1})$$

in Cartesian coordinate system (x^1, x^2) is defined by

$$\delta\mathbf{a} := \{a^{i_1 i_2\dots i_{m-1}s}_{;x^j}\} = \Big\{\frac{\partial a^{i_1 i_2\dots i_{m-1}j}}{\partial x^j},\ j=1,2\Big\}.$$

The divergence δ can operate on any tensor field of rank $m \ge 1$ and above produces a tensor that is one rank lower. For example, the divergence of a second rank tensor field is a (co)vector field.

In complex variables the divergence is calculated with the help of contravariant components G^{ij} of the metric tensor (3.10)

$$\begin{aligned}\delta\mathbf{A} &= \Big\{A_{i_1 i_2\dots i_m;z^s} G^{i_m s}\Big\}\\ &= \Big\{2\,\frac{\partial A_{i_1 i_2\dots i_{m-1}2}}{\partial z} + 2\,\frac{\partial A_{i_1 i_2\dots i_{m-1}1}}{\partial \bar z},\ i_1,\dots,i_{m-1}=1,2\Big\}.\end{aligned} \tag{3.14}$$

A smooth tensor field $\mathbf{a} \in \mathbf{C}^k(\mathbb{D}\,;\mathbf{S}_m)$ is called *solenoidal* if its divergence equals to zero. The condition for the tensor field $\mathbf{A}$ to be solenoidal can be expressed in complex coordinates in terms of its independent components $A_0, \dots, A_m$

$$\begin{cases} (A_0)_z + (A_1)_{\bar z} = 0\\ \dots\\ (A_k)_z + (A_{k+1})_{\bar z} = 0\\ \dots\\ (A_{m-1})_z + (A_m)_{\bar z} = 0\end{cases} \quad\text{or}\quad \frac{\partial}{\partial \bar z}\begin{pmatrix} A_m\\ A_{m-1}\\ \dots\\ A_2\\ A_1\end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} A_{m-1}\\ A_{m-2}\\ \dots\\ A_1\\ A_0\end{pmatrix} = 0. \tag{3.15}$$

The next differential operation on the symmetric tensor fields is *the symmetric inner differentiation* d

$$d\colon\quad \mathbf{C}^\infty(\mathbb{D}\,;\mathbf{S}_{m-1}) \to \mathbf{C}^\infty(\mathbb{D}\,;\mathbf{S}_m),$$

defined in the following way

$$d := \sigma\nabla,$$

where σ is the symmetrization operator (2.1).

A tensor field $\mathbf{a} \in \mathbf{C}^\infty(\mathbb{D}\,;\mathbf{S}_m)$ is called a smooth *potential field*, if for some tensor field $\mathbf{v} \in \mathbf{C}^\infty_0(\mathbb{D}\,;\mathbf{S}_{m-1})$ with boundary condition $\mathbf{v}|_{\partial\mathbb{D}} = 0$ we have $\mathbf{a} = d\mathbf{v}$ and $\mathbf{v}$ is the potential.

The symmetric inner differentiation d in complex variables is calculated in the following manner. If $\mathbf{a} = d\mathbf{v}$ and $\mathbf{a} \rightarrowtail \mathbf{A}$, $\mathbf{v} \rightarrowtail \mathbf{V}$ then $\mathbf{A} = d\mathbf{V}$ and

$$A_k = \frac{m-k}{m}\,\frac{\partial V_k}{\partial \bar z} + \frac{k}{m}\,\frac{\partial V_{k-1}}{\partial z} \quad (m \ge 1,\ k = 0, 1, \dots, m). \tag{3.16}$$

3.6. Orthogonal decomposition of the space $\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$ into the sum of solenoidal and potential parts

Operators d and $-\delta$ are formally conjugate and for a bounded region with a piecewise-smooth boundary ∂ the Gauss–Ostrogradsky formula takes place

$$\iint_{\mathbb{G}} \left(\langle d\mathbf{v},\ \mathbf{a}\rangle + \langle \mathbf{v}, \delta\mathbf{a}\rangle\right) \mathrm{d}V^2 = \int_{\partial\mathbb{G}} \langle i_{\mathbf{n}}\mathbf{v}, \mathbf{a}\rangle \,\mathrm{d}V^1, \tag{3.17}$$

where $\mathbf{a} \in \mathbf{S}_m,\ \mathbf{v} \in \mathbf{S}_{m-1}$ are smooth tensor fields, and $\mathbf{n} = \{n_1, n_2\} \in \mathbf{S}_1$ is a unit covector of outward normal to the boundary ∂ , and $i_{\mathbf{n}}$ is the operator of *symmetric multiplication* with the covector $\mathbf{n}$

$$i_{\mathbf{n}} : \mathbf{S}_m \to \mathbf{S}_{m+1},$$

which is defined by the equation

$$(i_{\mathbf{n}}\mathbf{v})_{i_1\ldots i_m i_{m+1}} := \sigma(n_{i_1} v_{i_2\ldots i_{m+1}}).$$

In terms of the Gauss–Ostrogradsky formula (3.17) we can define that a tensor field $\mathbf{a} \in \mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$ is solenoidal if the following equation takes place

$$\langle\langle d\mathbf{v},\ \mathbf{a}\rangle\rangle = \iint_{\mathbb{D}} \langle d\mathbf{v}, \mathbf{a}\rangle \,\mathrm{d}V^2 = 0 \tag{3.18}$$

for all smooth tensor fields $\mathbf{v}(x^1, x^2) \in \mathbf{C}_0^\infty(\mathbb{D}\,;\mathbf{S}_{m-1})$.

We denote by $\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta)$ the graph space of δ over $\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$, i.e.

$$\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta) := \{\mathbf{u} \in \mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m) \mid \delta\mathbf{u} \in \mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)\}.$$

It is a Hilbert space under the graph norm

$$\langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle_{\mathbf{H}(\mathbb{D};\mathbf{S}_m, \delta)} := \langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle + \langle\langle \delta\mathbf{u}, \delta\mathbf{v}\rangle\rangle, \quad \|\mathbf{u}\|^2_{\mathbf{H}(\mathbb{D};\mathbf{S}_m, \delta)} := \|\mathbf{u}\|^2 + \|\delta\mathbf{u}\|^2.$$

Finally, we define subspace of solenoidal tensor fields (i.e. which satisfies the equation (3.18))

$$\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta = 0) := \{\mathbf{a} \in \mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta) \mid \delta\mathbf{a} = 0\}$$

and it is clear that this subspace is a completion of the set of smooth solenoidal tensor fields with respect to the norm $\|\cdot\|$ of $\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$.

By $\mathbf{H}_N(\mathbb{D}\,;\mathbf{S}_m, \delta = 0)$ we denote the finite-dimensional subspace of polynomial (of degree at most N) solenoidal m-covariant tensors fields. Then we have

$$\mathbf{H}_0(\mathbb{D}\,;\mathbf{S}_m, \delta = 0) \subset \mathbf{H}_1(\mathbb{D}\,;_m, \delta = 0) \subset \ldots \\ \subset \mathbf{H}_N(\mathbb{D}\,;\mathbf{S}_m, \delta = 0) \subset \cdots \subset L_2(\mathbb{D}\,;\mathbf{S}_m)$$

and

$$\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta = 0) = \mathbf{clos}\Big(\bigcup_{N=0}^{\infty} \mathbf{H}_N(\mathbb{D}\,;\mathbf{S}_m, \delta = 0)\Big),$$

where **clos** means the closure in $\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$.

It is well known, see [8, 27], that a vector field can be represented as a sum of solenoidal and potential vector fields. The classical result in this direction belongs to H. Weyl and is connected with the decomposition of the L_2 space of vector fields into the orthogonal sum of solenoidal and potential fields. The analogous result is true for tensor fields, see [11, 14, 23]. Namely, for $\mathbf{u} \in \mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$ we have

$$\mathbf{u} = \mathbf{a} + d\mathbf{v}, \qquad \langle\langle \mathbf{a}, d\mathbf{v}\rangle\rangle = 0,$$

where $\mathbf{a}$ is a solenoidal tensor field and $\mathbf{v} \in \mathbf{H}_0^1(\mathbb{D}\,;\mathbf{S}_{m-1})$. Or, in another words the orthogonal decomposition

$$\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m) = \mathbf{H}(\mathbb{D}\,;\mathbf{S}_m, \delta = 0) \oplus d\mathbf{H}_0^1(\mathbb{D}\,;\mathbf{S}_{m-1})$$

takes place, where the Sobolev space $\mathbf{H}_0^1(\mathbb{D}\,;\mathbf{S}_{m-1})$ is a completion of the space of smooth tensor fields $\mathbf{C}_0^1(\mathbb{D}\,;\mathbf{S}_{m-1})$ with respect to the Sobolev norm $\|\cdot\|_1$, corresponding to the scalar product $\langle\langle\cdot\,,\cdot\rangle\rangle_1$ that is defined by the formula

$$\langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle_1 = \langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle + \langle\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle\rangle.$$

3.7. Fan-beam Radon transform $\mathcal{D}_m$ of tensor fields in complex variables

Let's assume that some (constant) vector field is given in Cartesian coordinates

$$\boldsymbol{\theta} = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix},$$

then according to the tensor law for contravariant components its representation in complex coordinates will look like

$$\boldsymbol{\theta} \rightarrowtail \boldsymbol{\Theta}, \qquad \Theta^j = \frac{\partial z^j}{\partial x^s}\,\theta^s, \qquad \boldsymbol{\Theta} = \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\varphi} \\ e^{-\mathrm{i}\varphi} \end{pmatrix}.$$

Then we denote by $\boldsymbol{\theta}^m$ the tensor product

$$\boldsymbol{\theta}^m := \underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m} = \{\theta^{j_1}\cdot\theta^{j_2}\cdot\dots\cdot\theta^{j_m}\},$$

and $\boldsymbol{\theta}^m$ will be an m-contravariant tensor in Cartesian coordinates and in complex coordinates we have the tensor product

$$\boldsymbol{\Theta}^m := \underbrace{\boldsymbol{\Theta} \otimes \boldsymbol{\Theta} \otimes \cdots \otimes \boldsymbol{\Theta}}_{m} = \{\Theta^{j_1}\cdot\Theta^{j_2}\cdot\dots\cdot\Theta^{j_m}\}.$$

It is clear that $\boldsymbol{\theta}^m \rightarrowtail \boldsymbol{\Theta}^m$. As soon as the inner product of tensors is invariant (3.13), we get

$$\langle \mathbf{a}, \boldsymbol{\theta}^m\rangle = a_{j_1\cdots j_m}\theta^{j_1}\cdot\theta^{j_2}\cdot\dots\cdot\theta^{j_m} = \langle \mathbf{A}, \boldsymbol{\Theta}^m\rangle = A_{j_1\cdots j_m}\Theta^{j_1}\cdot\Theta^{j_1}\cdot\dots\cdot\Theta^{j_m}.$$

Thus we can evaluate the fan-beam transform (2.3) through the components of the tensor $\mathbf{A}(z, \bar{z})$

$$\begin{aligned}
[\mathcal{D}_m \mathbf{a}](\beta, \varphi) &= \int_0^{2\cos(\beta-\varphi)} \langle \boldsymbol{\theta}^m, \mathbf{a}(\cos\beta - l\cos\varphi, \sin\beta - l\sin\varphi)\rangle \,\mathrm{d}l \\
&= \int_{\tau(t,\varphi)}^{t} \langle \boldsymbol{\Theta}^m, \mathbf{A}(\zeta,\bar{\zeta})\rangle |\,\mathrm{d}\zeta| = \int_{\tau(t,\varphi)}^{t} \Theta^{j_1}\dots\Theta^{j_m} A_{j_1\dots j_m}(\zeta,\bar{\zeta})\,|\mathrm{d}\zeta|, \quad (3.19)
\end{aligned}$$

where $t = e^{\mathrm{i}\beta}$, $\tau(t,\varphi) = -\bar{t}e^{2\mathrm{i}\varphi} = -e^{\mathrm{i}(2\varphi-\beta)}$, $\beta \in [0, 2\pi)$, $\varphi \in [\beta-\pi/2, \beta+\pi/2]$.

Here and in the sequel, we use notation

$$\int_{z_1}^{z_2} \dots |\mathrm{d}\zeta|$$

for a line integral along the line segment with end points $z_1, z_2 \in \overline{\mathbb{D}}$.

At last we can get the fan-beam transform (3.19) in terms of components A_k and for $\varphi \in [\beta-\pi/2, \beta+\pi/2]$ we have

$$[\mathcal{D}_m \mathbf{a}](\beta, \varphi) = \int_{\tau(t,\varphi)}^{t} \sum_{k=0}^{m} C_m^k e^{\mathrm{i}k\varphi} e^{-\mathrm{i}(m-k)\varphi} A_k(\zeta,\bar{\zeta})\,|\mathrm{d}\zeta| \quad (3.20)$$

$$= \sum_{k=0}^{m} C_m^k e^{\mathrm{i}(2k-m)\varphi} \int_{\tau(t,\varphi)}^{t} A_k(\zeta,\bar{\zeta})\,|\mathrm{d}\zeta| = \sum_{k=0}^{m} C_m^k e^{\mathrm{i}(2k-m)\varphi}[\mathcal{D}A_k] \quad (3.21)$$

$$= \sum_{k=0}^{m} C_m^k e^{\mathrm{i}(m-2k)\varphi}[\mathcal{D}A_{m-k}]. \quad (3.22)$$

Recall that for $|\beta - \varphi| \geq \pi/2$ the fan-beam transform $\mathcal{D}_m\mathbf{a}$ is defined by condition (2.4).

Now we verify that the potential part of a tensor field is “invisible” for tensor transform $\mathcal{D}_m$. Let $\mathbf{a} = d\mathbf{v}$ and $\mathbf{a} \rightarrowtail \mathbf{A} = d\mathbf{V}$, $\mathbf{v} \rightarrowtail \mathbf{V}$. Substituting the potential tensor (3.16) in the expansion (3.20) and making evident evaluations, we get

$$\begin{aligned}
[\mathcal{D}_m \mathbf{a}](\beta, \varphi) &= \int_{\tau(t,\varphi)}^{t} \sum_{k=0}^{m} C_m^k e^{\mathrm{i}k\varphi} e^{-\mathrm{i}(m-k)\varphi} \Big(\frac{m-k}{m}\frac{\partial V_k}{\partial \bar{z}} + \frac{k}{m}\frac{\partial V_{k-1}}{\partial z}\Big)\,\mathrm{d}|z| \\
&= \int_{\tau(t,\varphi)}^{t} \sum_{k=0}^{m} e^{\mathrm{i}(2k-m)\varphi}\Big(C_{m-1}^k \frac{\partial V_k}{\partial \bar{z}} + C_{m-1}^{k-1}\frac{\partial V_{k-1}}{\partial z}\Big)\,\mathrm{d}|z| \\
&= \int_{\tau(t,\varphi)}^{t} \Big(\sum_{k=0}^{m-1} e^{\mathrm{i}(2k-m)\varphi} C_{m-1}^k \frac{\partial V_k}{\partial \bar{z}} + \sum_{k=1}^{m} e^{\mathrm{i}(2k-m)\varphi} C_{m-1}^{k-1}\frac{\partial V_{k-1}}{\partial z}\Big)\,\mathrm{d}|z| \\
&= \int_{\tau(t,\varphi)}^{t} \Big(\sum_{k=0}^{m-1} e^{\mathrm{i}(2k-m)\varphi} C_{m-1}^k \frac{\partial V_k}{\partial \bar{z}} + \sum_{k=0}^{m-1} e^{\mathrm{i}(2k-m+2)\varphi} C_{m-1}^{k}\frac{\partial V_{k}}{\partial z}\Big)\,\mathrm{d}|z| =
\end{aligned}$$

$$= \sum_{k=0}^{m-1} e^{\mathrm{i}(2k-m+1)\varphi} C_{m-1}^{k} \int_{\tau(t,\varphi)}^{t} \frac{\partial V_k}{\partial \mathbf{\Theta}} \, \mathrm{d}|z|$$

$$= \sum_{k=0}^{m-1} e^{\mathrm{i}(2k-m+1)\varphi} C_{m-1}^{k} (V_k(t,\bar{t}) - V_k(\tau,\bar{\tau})) = 0.$$

Here

$$\frac{\partial}{\partial \mathbf{\Theta}} := e^{\mathrm{i}\varphi} \frac{\partial}{\partial z} + e^{-\mathrm{i}\varphi} \frac{\partial}{\partial \bar{z}} \tag{3.23}$$

is the derivative in the direction $\mathbf{\Theta} = \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\varphi} \\ e^{-\mathrm{i}\varphi} \end{pmatrix}$, written in the complex form and after integration we take into account that the potential $\mathbf{V}$ vanishes on the boundary of the disc $\mathbb{D}$.

At the end of this section we resume, that we consider the operator $\mathcal{D}_m$ as follows

$$\mathcal{D}_m : \mathbf{L}_2(\mathbb{D}\,; \mathbf{S}_m)) \to L_2([0, 2\pi) \times [0, 2\pi),$$

and $\ker \mathcal{D}_m$ coincides with the space of potential fields $d\mathbf{H}_0^1(\mathbb{D}\,; \mathbf{S}_{m-1})$, so one can say that potential fields are "invisible" for the tensorial Radon transform $\mathcal{D}_m$.

4. ZERNIKE POLYNOMIALS

We will identify complex plane with ^{2} as above. Let $\mathbb{D} = \{z \big| \, |z| < 1\}$ be the open unit disc in and $L_2(\mathbb{D})$ denote a Hilbert space comprising square integrable (complex-valued) functions on $\mathbb{D}$ with the inner product denoted by $\langle\langle \cdot, \cdot \rangle\rangle$

$$\langle\langle a, b \rangle\rangle := \iint_{\mathbb{D}} a(x^1, x^2) \overline{b(x^1, x^2)} \, \mathrm{d}V^2, \qquad \mathrm{d}V^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and the finite norm $\|\cdot\|$

$$\|a\|^2 := \iint_{\mathbb{D}} |a(x^1, x^2)|^2 \, \mathrm{d}V^2.$$

It is known that Zernike polynomials [4] form a complete orthogonal system (basis) over the Hilbert space $L_2(\mathbb{D})$. Let $z = re^{\mathrm{i}\psi} \in \mathbb{D}$, $r = |z|$, $\psi = \arg(z)$. Traditionally, Zernike polynomials, see [4, 16, 20], are defined by

$$V_{n,l}(r, \psi) := e^{\mathrm{i}l\psi} R_{n,|l|}(r) \qquad (-n \le l \le n), \tag{4.1}$$

where

$$R_{n,m}(r) := \sum_{p=0}^{(n-m)/2} (-1)^p \frac{(n-p)!}{p! \left(\frac{n+m}{2} - p\right) \left(\frac{n-m}{2} + p\right)} r^{n-2p}$$

are the so-called real-valued Zernike radial polynomials [20], defined for integers n and m so that $0 \le m \le n$ and $n - m$ is even. The family $R_{n,m}(r)$ is related to the Jacobi polynomials

$$R_{n,m}(r) = r^m P_{(n-m)/2}^{(0,m)}(2r^2 - 1),$$

where the Jacobi polynomials [13] are given through the Rodriguez formula

$$P_k^{(a,b)}(s) := \frac{(-1)^k}{k!2^k}(1-s)^{-a}(1+s)^{-b}\frac{d^k}{ds^k}\Big[(1-s)^{(k+a)}(1+s)^{(k+b)}\Big]$$

for $a, b > -1$; $k = 0, 1, 2, \ldots$, and $s \in$ form a complete orthogonal system in the Hilbert space $L_2[-1,1]$ of square integrable functions on $[-1,1]$.

In this paper we will use another numbering of Zernike polynomials (4.1) and treat them as polynomials in $z, \bar{z}$. For this reason the new notation $Z^{n,k}$ is introduced according to the transformation of indexes $l = n-2k$ in (4.1). Thus we get

$$V_{n,n-2k}(r,\psi) = z^{n-2k}P_k^{(0,|n-2k|)}(2|z|^2-1) \qquad (k = 0,1,\ldots,n).$$

So, we define

$$Z^{n,k}(z,\bar{z}) := (-1)^k V_{n,n-2k}(r,\psi) = (-1)^k z^{n-2k}P_k^{(0,|n-2k|)}(2|z|^2-1), \quad (4.2)$$

where $n = 0, 1, 2, \ldots$ and $k = 0, 1, \ldots, n$. The first index (superscript) n indicates the degree of a polynomial $Z^{n,k}$ and the second superscript k denotes its order in a bunch $Z^{n,0}, Z^{n,1}, \ldots, Z^{n,n}$. The multiplier $(-1)^k$ in (4.2) was introduced for convenience of further computations.

This definition can also be rewritten as

$$Z^{n,k}(z,\bar{z}) = \begin{cases} \sum_{s=0}^k C_k^s C_{n-k}^s z^{n-k-s}(1-z\bar{z})^s(-\bar{z})^{k-s} & k = 0,1,\ldots,[n/2] \\ (-1)^n \bar{Z}^{n,n-k}(z,\bar{z}) & k = [n/2]+1,\ldots,n, \end{cases} \quad (4.3)$$

where $[\cdot]$ denotes the integer part of a number.

After the evaluation of (4.3) we get

$$Z^{n,k}(z,\bar{z}) = \begin{cases} \sum_{s=0}^k (-1)^{k-s} C_{n-k}^s C_{n-s}^{k-s} \bar{z}^{k-s} z^{n-k-s} & k = 0,1,\ldots,[n/2] \\ (-1)^n \bar{Z}^{n,n-k}(z,\bar{z}) & k = [n/2]+1,\ldots,n. \end{cases} \quad (4.4)$$

For example, the first Zernike polynomials up to the degree (order) $n = 4$ are

$Z^{0,0} = 1$
$Z^{1,0} = z \quad Z^{1,1} = -\bar{z}$
$Z^{2,0} = z^2 \quad Z^{2,1} = 1 - 2z\bar{z} \quad Z^{2,2} = \bar{z}^2$
$Z^{3,0} = z^3 \quad Z^{3,1} = 2z - 3z^2\bar{z} \quad Z^{3,2} = 3z\bar{z}^2 - 2\bar{z} \quad Z^{3,3} = -\bar{z}^3$
$Z^{4,0} = z^4 \quad Z^{4,1} = 3z^2 - 4z^3\bar{z} \quad Z^{4,2} = 1 - 6z\bar{z} + 6z^2\bar{z}^2 \quad Z^{4,3} = 3\bar{z}^2 - 4z\bar{z}^3 \quad Z^{4,4} = \bar{z}^4.$

The Zernike polynomials are orthogonal in the unit disc $\mathbb{D}$, obey the following orthogonality relation

$$\langle\langle Z^{n,k}, Z^{m,s}\rangle\rangle = \frac{\pi}{n+1}\delta_{n,m}\delta_{k,s} \quad (4.5)$$

and their L_2-norms are equal to

$$\|Z^{n,k}\| = \sqrt{\frac{\pi}{n+1}} \qquad (k = 0,1,\ldots,n).$$

It allows the expansion of an arbitrary function $a(z,\bar z) \in L_2(\mathbb{D})$ in terms of a unique combination of Zernike polynomials.

$$a(z,\bar z) = \sum_{n=0}^{\infty} \frac{n+1}{\pi} \sum_{k=0}^{n} \langle\langle a,\ Z^{n,k}\rangle\rangle Z^{n,k}(z,\bar z). \tag{4.6}$$

Since we use the complex variables z and $\bar z$ and treat them as independent variables here, we'll sometimes write $a(z)$ instead of $a(z,\bar z)$. The formal partial derivatives with respect to z and $\bar z$ are defined in the usual way by (3.2).

In the following theorem we formulate in complex variables some novel properties of Zernike polynomials.

Theorem 1. *The following properties take place: (a) Zernike polynomials* (4.3) *have the differential representation*

$$Z^{n,k}(z,\bar z) = \frac{1}{k!}\frac{\partial^k}{\partial z^k}\Big[z^n\Big(\frac{1}{z} - \bar z\Big)^k\Big] \qquad (n \ge 0, \quad k = 0,1,\dots,n). \tag{4.7}$$

(b) Zernike polynomials (4.3) are the solution of the elliptic system

$$\begin{cases} \qquad\qquad (Z^{n,n})_z & = 0 \\ (Z^{n,n})_{\bar z} + (Z^{n,n-1})_z & = 0 \\ \qquad\qquad \dots & \\ (Z^{n,k})_{\bar z} + (Z^{n,k-1})_z & = 0 \\ \qquad\qquad \dots & \\ (Z^{n,1})_{\bar z} + (Z^{n,0})_z & = 0 \\ (Z^{n,0})_{\bar z} & = 0 \end{cases} \tag{4.8}$$

and satisfy boundary conditions

$$Z^{n,k}(t,\bar t) = (-1)^k t^{n-2k}, \qquad |t| = 1 \qquad (n \ge 0, \quad k = 0,1,\dots,n). \tag{4.9}$$

(c) Zernike polynomials (4.3) *can be represented in the form of Cauchy-type integral*

$$\frac{1}{2\pi \mathrm{i}} \int_{|t|=1} \frac{t^n(\bar t - \bar z)^k}{(t-z)^{k+1}}\,\mathrm{d}t = \begin{cases} Z^{n,k}(z,\bar z), & n \ge 0,\ k = 0,1,\dots,n \\ 0, & n \ge 0,\ k > n \text{ or } k < 0. \end{cases} \tag{4.10}$$

Proof. (a) Let's first prove (4.7) for $k = 0,1,\dots,[n/2]$.

By Leibnitz formula $[uv]^{(k)} = \sum_{s=0}^{k} C_k^s u^{(s)} v^{(k-s)}$ we get

$$\Big[z^n\Big(\frac{1}{z} - \bar z\Big)^k\Big]_z^{(k)} = \big[z^{n-k}(1 - z\bar z)^k\big]_z^{(k)} = \sum_{s=0}^{k} C_k^s \big[z^{n-k}\big]_z^{(s)} \big[(1 - z\bar z)^k\big]_z^{(k-s)} =$$

$$= \sum_{s=0}^{k} C_k^s (n-k)(n-k-1)\dots(n-k-s+1) z^{n-k-s}$$
$$\times k(k-1)\dots(s+1)(1-z\bar z)^s(-\bar z)^{k-s}$$
$$= k! \sum_{s=0}^{k} C_k^s C_{n-k}^s z^{n-k-s}(1-z\bar z)^s(-\bar z)^{k-s} = k! Z^{n,k}(z,\bar z). \tag{4.11}$$

Now let's substitute $k \to n-k$ in (4.11). Taking into account that $k \le n-k$, we get

$$\Big[z^n\Big(\frac{1}{z}-\bar z\Big)^{n-k}\Big]_z^{(n-k)} = \big[z^k(1-z\bar z)^{n-k}\big]_z^{(n-k)}$$
$$= \sum_{s=0}^{k} C_{n-k}^s \big[z^k\big]_z^{(s)} \big[(1-z\bar z)^{n-k}\big]_z^{(n-k-s)}$$
$$= \sum_{s=0}^{k} C_{n-k}^s k(k-1)\dots(k-s+1) z^{k-s}$$
$$\times (n-k)(n-k-1)\dots(s+1)(1-z\bar z)^s(-\bar z)^{n-k-s}$$
$$= (n-k)! \sum_{s=0}^{k} C_k^s C_{n-k}^s (-1)^n (\bar z)^{n-k-s}(1-z\bar z)^s(-z)^{k-s}$$
$$= (-1)^n (n-k)!\, \bar Z^{n,k}(z,\bar z) = (n-k)!\, Z^{n,n-k}(z,\bar z).$$

And the assertion (a) follows.

(b) It is easy to verify equations (4.8) by direct computation using formula (4.4). On the boundary of the unit disc $\mathbb{D}$, due to the normalization $P_k^{(0,|n-2k|)}(1) = 1$ of Jacobi polynomials, we get

$$Z^{n,k}(t,\bar t) = (-1)^k t^{n-2k}, \qquad |t| = 1, \qquad (n \ge 0, \quad k = 0,1,\dots,n).$$

(c) In (4.10) we take advantage of Newtonian binomial formula

$$\frac{1}{2\pi \mathrm{i}} = \int_{|t|=1} \frac{t^n(\bar t - \bar z)^k}{(t-z)^{k+1}}\,\mathrm{d}t = \sum_{s=0}^{k} C_k^s (-\bar z)^{k-s} \frac{1}{2\pi \mathrm{i}} \int_{|t|=1} \frac{t^n \bar t^s}{(t-z)^{k+1}}\,\mathrm{d}t$$
$$= \sum_{s=0}^{k} C_k^s (-\bar z)^{k-s} \frac{1}{2\pi \mathrm{i}} \int_{|t|=1} \frac{t^{n-s}}{(t-z)^{k+1}}\,\mathrm{d}t$$
$$= \sum_{s=0}^{k} C_k^s (-\bar z)^{k-s} \frac{1}{k!}\,\frac{k!}{2\pi \mathrm{i}} \int_{|t|=1} \frac{t^{n-s}}{(t-z)^{k+1}}\,\mathrm{d}t$$
$$= \sum_{s=0}^{k} C_k^s (-\bar z)^{k-s} \frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}z^k}\big[z^{n-s}\big] = \frac{1}{k!}\,\frac{\partial^k}{\partial z^k}\Big[\sum_{s=0}^{k} C_k^s (-\bar z)^{k-s} z^{n-s}\Big]$$
$$= \frac{1}{k!}\,\frac{\partial^k}{\partial z^k}\Big[z^n\Big(\frac{1}{z}-\bar z\Big)^k\Big].$$

Theorem 1 is proved. □

4.1. Fan-beam Radon transform of Zernike polynomials

The fan-beam Radon transform $\mathcal{D}$ of a scalar function $a(x^1, x^2)$ is defined by (3.19) for $m = 0$. We have

$$[\mathcal{D}a](\beta, \varphi) = \int_{\tau(t,\varphi)}^{t} a(\zeta, \bar{\zeta})\, |\mathrm{d}\zeta|, \tag{4.12}$$

where $t = e^{\mathrm{i}\beta}$, $\beta \in [0, 2\pi)$, $\tau(t, \varphi) = -\bar{t}e^{2\mathrm{i}\varphi}$, $|\beta - \varphi| \le \pi/2$, $\varphi = \arg(t - \tau)$. For $|\beta - \varphi| > \pi/2$ we complete the definition of the fan-beam transform (4.12) with the condition

$$[\mathcal{D}a](\beta, \varphi) = -[\mathcal{D}a](\beta, \varphi + \pi).$$

Theorem 2. *The fan-beam Radon transform $\mathcal{D}Z^{n,k}$ of Zernike polynomials $Z^{n,k}$ equals to*

$$[\mathcal{D}Z^{n,k}](\beta, \varphi) = \frac{2e^{\mathrm{i}(n-2k)\varphi}}{n+1} \times \begin{cases} \cos\left[(n+1)(\beta - \varphi)\right], & n = \mathit{even} \\ \mathrm{i}\sin\left[(n+1)(\beta - \varphi)\right], & n = \mathit{odd}, \end{cases} \tag{4.13}$$

where $\beta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$.

Proof. At first we introduce the auxiliary polynomials $X^{n,k}$ defined by

$$X^{n,k}(z, \bar{z}) := \frac{1}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} \left[z^n \left(\frac{1}{z} - \bar{z} \right)^k \right] \qquad (n \ge 1, \quad k = 1, \dots, n).$$

Then the next equations follows directly from (4.7)

$$\frac{\partial X^{n,k}}{\partial z} = Z^{n,k}(z, \bar{z}), \qquad \frac{\partial X^{n,k}}{\partial \bar{z}} = -Z^{n,k-1}(z, \bar{z}). \tag{4.14}$$

Using (4.7) we can verify, that

$$X^{n,k} = \frac{1}{k} \left(Z^{n-1,k-1} - \bar{z} Z^{n,k-1} \right).$$

Then combining above and (4.9) we obtain the boundary conditions

$$X^{n,k}(t, \bar{t}) = 0, \qquad |t| = 1. \tag{4.15}$$

Let's compute the fan-beam transformation of Zernike polynomials. To this end we use previously obtained derivatives (4.14) for computing by (3.23) the derivative of $X^{n,k}$ in the direction $\mathbf{\Theta} = \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\varphi} \\ e^{-\mathrm{i}\varphi} \end{pmatrix}$. So we get

$$\frac{\partial X^{n,k}}{\partial \mathbf{\Theta}} = e^{\mathrm{i}\varphi} \frac{\partial X^{n,k}}{\partial z} + e^{-\mathrm{i}\varphi} \frac{\partial X^{n,k}}{\partial \bar{z}} = e^{\mathrm{i}\varphi} Z^{n,k} - e^{-\mathrm{i}\varphi} Z^{n,k-1},$$

or the same in another form

$$Z^{n,k}(z, \bar{z}) = e^{-2\mathrm{i}\varphi} Z^{n,k-1} + e^{-\mathrm{i}\varphi} \frac{\partial X^{n,k}}{\partial \mathbf{\Theta}}.$$

This equation combined with (4.15) is used for evaluation of the next integral

$$\int_\tau^t Z^{n,k}\,|\mathrm{d}\zeta| = e^{-2\mathrm{i}\varphi}\int_\tau^t Z^{n,k-1}\,|\mathrm{d}\zeta| + e^{-\mathrm{i}\varphi}(X^{n,k}(t,\bar{t}) - X^{n,k}(\tau,\bar{\tau}))$$
$$= e^{-2\mathrm{i}\varphi}\int_\tau^t Z^{n,k-1}\,|\mathrm{d}\zeta|,$$

where $t = e^{\mathrm{i}\beta}$, $\tau(t,\varphi) = -\bar{t}e^{2\mathrm{i}\varphi}$. Unwrapping the recurrence relation gives

$$\int_\tau^t Z^{n,k}\,|\mathrm{d}\zeta| = e^{-2k\mathrm{i}\varphi}\int_\tau^t Z^{n,0}\,|\mathrm{d}\zeta|.$$

The last integral is computed directly, taking into account that $Z^{n,0}(z,\bar{z}) = z^n$

$$\int_\tau^t Z^{n,0}\,|\mathrm{d}\zeta| = \int_\tau^t \zeta^n\,|\mathrm{d}\zeta| = \int_0^{|t-\tau|}(\tau + se^{\mathrm{i}\varphi})^n\,\mathrm{d}s$$
$$= \frac{1}{n+1}\,(\tau + se^{\mathrm{i}\varphi})^{n+1}e^{-\mathrm{i}\varphi}\Big|_0^{|t-\tau|} = \frac{1}{n+1}\,e^{-\mathrm{i}\varphi}(t^{n+1} - \tau^{n+1}).$$

Finally, we get

$$[\mathcal{D}Z^{n,k}](\beta,\varphi) = \int_\tau^t Z^{n,k}\,|\mathrm{d}\zeta| = \frac{1}{n+1}\,(e^{-\mathrm{i}\varphi}t^{n+1} + (-1)^n e^{\mathrm{i}(2n+1)\varphi}\bar{t}^{n+1})$$
$$= \frac{2e^{\mathrm{i}(2n-k)\varphi}}{n+1} \times \begin{cases} \cos\,[(n+1)(\beta-\varphi)], & n = \text{even} \\ \mathrm{i}\sin\,[(n+1)(\beta-\varphi)], & n = \text{odd}. \end{cases}$$

Theorem 2 is proof. $\square$

5. CONSTRUCTION OF THE ORTHOGONAL POLYNOMIAL BASIS AND SVD

In this section we describe the construction of orthogonal polynomial basis in the space of solenoidal (divergence free) tensor fields $\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m,\delta = 0)$.

We are given a polynomial of degree N solenoidal m-covariant tensor field $\mathbf{a} \in \mathbf{H}_N(\mathbb{D}\,;\mathbf{S}_m,\delta = 0)$ and in complex coordinates we have $\mathbf{a} \rightarrowtail \mathbf{A} = \{A_k\}$. As was mentioned earlier, we use a pseudovector notation for the tensor field $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} A_m \\ A_{m-1} \\ \dots \\ A_1 \\ A_0 \end{pmatrix}. \tag{5.1}$$

The condition (3.5) now looks like

$$\begin{pmatrix} A_m \\ A_{m-1} \\ \dots \\ A_1 \\ A_0 \end{pmatrix} = \begin{pmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \dots \\ \bar{A}_{m-1} \\ \bar{A}_m \end{pmatrix}. \tag{5.2}$$

For $m, n \geq 0$, $k = 0, \dots, n+m$ and $2k \neq m+n$ we define polynomial of degree n symmetric tensor fields (in complex variables)

$$\mathbf{S}^{(+m)}_{n,k} := (-1)^n \begin{pmatrix} Z^{n,k} + \bar{Z}^{n,k-m} \\ Z^{n,k-1} + \bar{Z}^{n,k-m+1} \\ \dots \\ Z^{n,k-m+1} + \bar{Z}^{n,k-1} \\ Z^{n,k-m} + \bar{Z}^{n,k} \end{pmatrix},$$

$$\mathbf{S}^{(-m)}_{n,k} := \frac{1}{\mathrm{i}} \begin{pmatrix} Z^{n,k} - \bar{Z}^{n,k-m} \\ Z^{n,k-1} - \bar{Z}^{n,k-m+1} \\ \dots \\ Z^{n,k-m+1} - \bar{Z}^{n,k-1} \\ Z^{n,k-m} - \bar{Z}^{n,k} \end{pmatrix}, \tag{5.3}$$

where for convenience we set $Z^{n,k} \equiv 0$ for $k < 0$ or $k > n$.

For $2k = m+n$ we have two cases: the first one is when m and $n =$ even

$$\mathbf{S}^{(+m)}_{n,(m+n)/2} := \begin{pmatrix} Z^{n,(m+n)/2} \\ Z^{n,(m+n)/2-1} \\ \dots \\ Z^{n,(m+n)/2-m+1} \\ Z^{n,(m+n)/2-m} \end{pmatrix}, \qquad \mathbf{S}^{(-m)}_{n,(m+n)/2} := 0; \tag{5.4}$$

the second one is when m and $n =$ odd

$$\mathbf{S}^{(+m)}_{n,(m+n)/2} := 0, \qquad \mathbf{S}^{(-m)}_{n,(m+n)/2} := \frac{1}{\mathrm{i}} \begin{pmatrix} Z^{n,(m+n)/2} \\ Z^{n,(m+n)/2-1} \\ \dots \\ Z^{n,(m+n)/2-m+1} \\ Z^{n,(m+n)/2-m} \end{pmatrix}. \tag{5.5}$$

Remark 1. Polynomial tensor fields (5.4) and (5.5) can be evaluated by general formulae (5.3), but then the result should be divided by 2.

Remark 2. Note, that for $k+s=m+n$ the equation

$$\mathbf{S}_{n,k}^{(\pm m)}=(-1)^n\mathbf{S}_{n,s}^{(\pm m)}$$

takes place, therefore in (5.3) we set only $k=0,1,\dots,[(n+m)/2]$, where $[\cdot]$ defines the integer part of a number.

So, if we now make transformations (3.3) or (3.7) from complex variables to real variables

$$\mathbf{S}_{n,k}^{(\pm m)}(z,\bar z)\to\mathbf{s}_{n,k}^{(\pm m)}(x^1,x^2)\quad(n\ge 0,\ k=0,1,\dots,[(n+m)/2]),\tag{5.6}$$

then we get polynomial real-valued tensors $\mathbf{s}_{n,k}^{(\pm m)}$ in Cartesian variables (x^1,x^2).

Lemma 1. *The tensor fields* $\mathbf{s}_{n,k}^{(\pm m)}$ *(*$n=0,\dots,N,\ k=0,1,\dots,[(n+m)/2]$*) defined by* (5.6) *form an orthogonal basis of finite-dimensional subspace* $\mathbf{H}_N(\mathbb{D};\mathbf{S}_m,\delta=0)$, *thus*

$$\dim\mathbf{H}_N(\mathbb{D};\mathbf{S}_m,\delta=0)=(N+1)(N+2+2m)/2.$$

Proof. Consider a tensor field $\mathbf{a}\in\mathbf{H}_N(\mathbb{D};\mathbf{S}_m,\delta=0)$ and let $\mathbf{a}\rightarrowtail\mathbf{A}$. Expand each component A_k of pseudovector (5.1) in the sum of Zernike polynomials and use the solenoidality condition (3.15). Taking into account the property (b) from Theorem 1, we get an expansion of tensor $\mathbf{A}$ into the sum

$$\mathbf{A}=\begin{pmatrix}A_m\\A_{m-1}\\\dots\\A_1\\A_0\end{pmatrix}=\sum_{n=0}^{N}\sum_{k=0}^{n+m}c_{n,k}\begin{pmatrix}Z^{n,k}\\Z^{n,k-1}\\\dots\\Z^{n,k-m+1}\\Z^{n,k-m}\end{pmatrix}.\tag{5.7}$$

From another hand, (5.2) yields

$$\mathbf{A}=\begin{pmatrix}\bar A_0\\\bar A_1\\\dots\\\bar A_{m-1}\\\bar A_m\end{pmatrix}=\sum_{n=0}^{N}\sum_{k=0}^{n+m}\bar c_{n,k}\begin{pmatrix}\bar Z^{n,k-m}\\\bar Z^{n,k-m+1}\\\dots\\\bar Z^{n,k-1}\\\bar Z^{n,k}\end{pmatrix}=$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{n+m} \bar{c}_{n,n+m-k} \begin{pmatrix} \bar{Z}^{n,n-k} \\ \bar{Z}^{n,n-k+1} \\ \dots \\ \bar{Z}^{n,n-k+m-1} \\ \bar{Z}^{n,n-k+m} \end{pmatrix}$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{n+m} (-1)^n \bar{c}_{n,n+m-k} \begin{pmatrix} Z^{n,k} \\ Z^{n,k-1} \\ \dots \\ Z^{n,k-m+1} \\ Z^{n,k-m} \end{pmatrix}. \qquad (5.8)$$

Comparing the last expression with (5.7), we obtain

$$c_{n,k} = (-1)^n \bar{c}_{n,n+m-k} \qquad (k = 0, \dots, n+m). \qquad (5.9)$$

Splitting the coefficients $c_{n,k}$ in (5.7) into the real and imaginary parts

$$c_{n,k} = a_{n,k} + \mathrm{i} b_{n,k} \qquad (k = 0, \dots, [(n+m)/2]),$$

taking into account (5.9) and definition of $\mathbf{S}^{\pm m)}_{n,k}$, we finally get

$$\begin{pmatrix} A_m \\ A_{m-1} \\ \dots \\ A_1 \\ A_0 \end{pmatrix} = \sum_{n=0}^{N} \sum_{k=0}^{[\frac{n+m}{2}]*} a_{n,k} \begin{pmatrix} Z^{n,k} + \bar{Z}^{n,k-m} \\ Z^{n,k-1} + \bar{Z}^{n,k-m+1} \\ \dots \\ Z^{n,k-m+1} + \bar{Z}^{n,k-1} \\ Z^{n,k-m} + \bar{Z}^{n,k} \end{pmatrix} + \mathrm{i} b_{n,k} \begin{pmatrix} Z^{n,k} - \bar{Z}^{n,k-m} \\ Z^{n,k-1} - \bar{Z}^{n,k-m+1} \\ \dots \\ Z^{n,k-m+1} - \bar{Z}^{n,k-1} \\ Z^{n,k-m} - \bar{Z}^{n,k} \end{pmatrix}$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{[\frac{n+m}{2}]*} (-1)^n a_{n,k} \mathbf{S}^{(+m)}_{n,k}(z, \bar{z}) - b_{n,k} \mathbf{S}^{(-m)}_{n,k}(z, \bar{z}).$$

The sign $*$ here means that in the case of even n and even m the coefficient $b_{n,(n+m)/2}$ should be set to 0, and in the case of odd n and odd m the coefficient $a_{n,(n+m)/2}$ should be set to 0.

So, we have that the tensor field $\mathbf{a}$ is a linear combination of polynomial tensor fields (5.6).

Now we show that polynomial tensor fields (5.6) are orthogonal. Let $k \neq s$, $k, s = 0, 1, \dots, [(n+m)/2]$ and remark, that then $k + s \neq m + n$ take place.

Using formula (3.11) we have

$$\begin{aligned}\langle\langle \mathbf{s}_{n,k}^{(\pm m)}, \mathbf{s}_{n,s}^{(\pm m)}\rangle\rangle &= \langle\langle \mathbf{S}_{n,k}^{(\pm m)}, \mathbf{S}_{n,s}^{(\pm m)}\rangle\rangle \\ &= \pm 2^m \iint_{\mathbb{D}} \sum_{p=0}^{m} C_m^p (Z^{n,k-m+p} \pm \bar{Z}^{n,k-p})(Z^{n,s-p} \pm \bar{Z}^{n,s-m+p})\,\mathrm{d}V^2.\end{aligned}$$

Taking into account the orthogonality of Zernike polynomials we obtain

$$\langle\langle \mathbf{s}_{n,k}^{(\pm m)}, \mathbf{s}_{n,s}^{(\pm m)}\rangle\rangle = 0, \qquad k \neq s.$$

Let's now evaluate the norms of polynomial tensor $\mathbf{s}_{n,k}^{(\pm m)}$ by the formula (3.12). At first we consider the case $n + m \neq 2k$, then

$$\begin{aligned}\|\mathbf{s}_{n,k}^{(\pm m)}\|^2 = \|\mathbf{S}_{n,k}^{(\pm m)}\|^2 &= 2^m \iint_{\mathbb{D}} \sum_{p=0}^{m} C_m^p |Z^{n,k-m+p} \pm \bar{Z}^{n,k-p}|^2\,\mathrm{d}V^2 \\ &= 2^m \sum_{p=0}^{m} C_m^p (\|Z^{n,k-m+p} \pm \bar{Z}^{n,k-p}\|^2) \\ &= 2^m \sum_{p=0}^{m} C_m^p \|Z^{n,k-m+p}\|^2 + \|\bar{Z}^{n,k-p}\|^2 \\ &= 2^{m+1} \sum_{p=k-n}^{k} C_m^p \|Z^{n,k-p}\|^2 = \frac{2^{m+1}\pi}{n+1}\,\alpha_{n,k}^{(m)}, \end{aligned} \tag{5.10}$$

where coefficients $\alpha_{n,k}^{(m)}$ are defined by

$$\alpha_{n,k}^{(m)} = \sum_{p=k-n}^{k} C_m^p \qquad \Big(n \geq 0, \quad k = 0, 1, \ldots, \Big[\frac{m+n}{2}\Big]\Big). \tag{5.11}$$

Remark 3. In this formula for convenience we set $C_m^p = 0$ if $p < 0$ or $p > m$.

If $n + m = 2k$, i. e. $k = (n + m)/2$, then taking into account the above calculations, we get

$$\|\mathbf{s}_{n,(m+n)/2}^{(+m)}\|^2 = \begin{cases} 0, & n = \text{odd} \\ \dfrac{2^m\pi}{n+1}\,\alpha_{n,(m+n)/2}^{(m)}, & n = \text{even}, \end{cases}$$

$$\|\mathbf{s}_{n,(m+n)/2}^{(-m)}\|^2 = \begin{cases} \dfrac{2^m\pi}{n+1}\,\alpha_{n,(m+n)/2}^{(m)}, & n = \text{odd} \\ 0, & n = \text{even}. \end{cases}$$

Obviously,

$$\dim \mathbf{H}_N(\mathbb{D}\,; \mathbf{S}_m, \delta = 0) = \sum_{n=m}^{m+N} n = \frac{(N+1)(N+2+2m)}{2}.$$

Lemma 1 is proved. □

Lemma 1 and the definition of the subspace of solenoidal tensor fields $\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m\,,\delta=0)$ yield

Corollary 1. *Polynomial tensor fields* $\mathbf{s}^{(\pm m)}_{n,k}$ $(n\ \geq\ 0,\ k\ =\ 0,1,\ldots,$ $[(n+m)/2])$ *form an orthogonal basis in the subspace of solenoidal tensor fields* $\mathbf{H}(\mathbb{D}\,;\mathbf{S}_m\,,\delta=0)\subset\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$.

Now we are in a position to define the SVD for the fan-beam Radon transform $\mathcal{D}_m$.

Theorem 3. *The singular values of the operator* (2.3)

$$\mathcal{D}_m:\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)\to L_2([0,2\pi)\times[0,2\pi))$$

are given by

$$\sigma^{(m)}_{n,k}\equiv\sigma^{(\pm m)}_{n,k}:=\sqrt{\frac{8\pi}{(n+1)2^m}\alpha^{(m)}_{n,k}}\quad\Big(n\geq 0,\ k=0,1,\ldots,\Big[\frac{n+m}{2}\Big]\Big),$$

where coefficients $\alpha^{(m)}_{n,k}$ *are defined by the formula (5.11). If a solenoidal real-valued symmetrical tensor field* $\mathbf{a}(x^1,x^2)\in\mathbf{L}_2(\mathbb{D}\,;\mathbf{S}_m)$ *has an expansion*

$$\mathbf{a}(x^1,x^2)=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n+m}{2}]}{}^{*}\frac{1}{\|\mathbf{s}^{(\pm m)}_{n,k}\|}\big(a_{n,k}\mathbf{s}^{(+m)}_{n,k}(x^1,x^2)+b_{n,k}\mathbf{s}^{(-m)}_{n,k}(x^1,x^2)\big),\quad(5.12)$$

then the fan-beam Radon transform $\mathcal{D}_m\mathbf{a}$ *has the following singular value decomposition*

$$[\mathcal{D}_m\mathbf{a}](\beta,\varphi)=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n+m}{2}]}{}^{*}\sigma^{(m)}_{n,k}\big(a_{n,k}f^{(+m)}_{n,k}(\beta,\varphi)+b_{n,k}f^{(-m)}_{n,k}(\beta,\varphi)\big),\quad(5.13)$$

where singular functions are

$$f^{(+m)}_{n,k}(\beta,\varphi):=\frac{1}{\pi}\begin{cases}\cos[(n+1)(\beta-\varphi)]\cos[(n-2k+m)\varphi]\\ \sin[(n+1)(\beta-\varphi)]\sin[(n-2k+m)\varphi],\end{cases}$$

$$f^{(-m)}_{n,k}(\beta,\varphi):=\frac{1}{\pi}\begin{cases}\cos[(n+1)(\beta-\varphi)]\sin[(n-2k+m)\varphi]\\ \sin[(n+1)(\beta-\varphi)]\cos[(n-2k+m)\varphi],\end{cases}$$

when $n\geq 0$, $k=0,\ldots,[(n+m)/2]$ *and* $2k\neq m+n$ *and*

$$f^{(+m)}_{n,(m+n)/2}(\beta,\varphi):=\frac{1}{\sqrt{2}\pi}\begin{cases}\cos[(n+1)(\beta-\varphi)]\\ 0,\end{cases}$$

$$f^{(-m)}_{n,(m+n)/2}(\beta,\varphi):=\frac{1}{\sqrt{2}\pi}\begin{cases}0\\ \sin[(n+1)(\beta-\varphi)],\end{cases}$$

when $2k = m + n$. In all expressions above top line corresponds to the even values of n, and bottom line — to the odd n. The sign $*$ in (5.12) and (5.13) near by the inner sum denotes that in the case of even n and m the coefficient $b_{n,(n+m)/2}$ should be set to 0, and in the case of odd n and m — the coefficient $a_{n,(n+m)/2}$ respectively.

Proof. Note, that the system of functions $f^{(\pm)}_{n,k}$ for $n \geq 0$, $k = 0, 1, \ldots,$ $[(n+m)/2]$ is the subsystem of the standard orthonormal basis of $L_2([0, 2\pi) \times [0, 2\pi))$ and there is the basis of the image of the tensor fan beam transform $\mathcal{D}_m$.

Let's evaluate now the fan-beam transform $\mathcal{D}_m$ for basis polynomial tensor (5.6). For this we introduce

$$\mathbf{A} := \begin{pmatrix} Z^{n,k} \\ Z^{n,k-1} \\ \ldots \\ Z^{n,k-m+1} \\ Z^{n,k-m} \end{pmatrix}, \; A_{m-s} = Z^{n,k-s}, \quad \mathbf{B} := \begin{pmatrix} \bar{Z}^{n,k-m} \\ \bar{Z}^{n,k-m+1} \\ \ldots \\ \bar{Z}^{n,k-1} \\ \bar{Z}^{n,k} \end{pmatrix}, \; B_s = \bar{Z}^{n,k-s}.$$

Hence in case of $n + m \neq 2k$ we get

$$\mathbf{S}^{(+m)}_{n,k} = (-1)^n(\mathbf{A} + \mathbf{B}), \qquad \mathbf{S}^{(-m)}_{n,k} = \frac{1}{\mathrm{i}}\,(\mathbf{A} - \mathbf{B}).$$

Then by using formulae (3.21) and Theorem 2 we have

$$\begin{aligned} [\mathcal{D}_m \mathbf{A}] &= \mathcal{D}_m \begin{pmatrix} Z^{n,k} \\ Z^{n,k-1} \\ \ldots \\ Z^{n,k-m+1} \\ Z^{n,k-m} \end{pmatrix} = \sum_{s=0}^{m} C^s_m e^{\mathrm{i}(m-2s)\varphi} [\mathcal{D} Z^{n,k-s}] \\ &= \frac{2e^{\mathrm{i}(m+n-2k)\varphi}}{n+1} \sum_{s=k-n}^{k} C^s_m \times \begin{cases} \cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ \mathrm{i}\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd} \end{cases} \\ &= \frac{2e^{\mathrm{i}(m+n-2k)\varphi}}{n+1}\, \alpha^{(m)}_{n,k} \times \begin{cases} \cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ \mathrm{i}\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd}, \end{cases} \end{aligned}$$

where $\alpha^{(m)}_{n,k}$ are defined by the formula (5.11). Analogically, using the formula

(3.22) and Theorem 2 we get

$$[\mathcal{D}_m\mathbf{B}] = \mathcal{D}_m \begin{pmatrix} \bar{Z}^{n,k-m} \\ \bar{Z}^{n,k-m+1} \\ \dots \\ \bar{Z}^{n,k-1} \\ \bar{Z}^{n,k} \end{pmatrix} = \sum_{s=0}^{m} C_m^s e^{\mathrm{i}(2s-m)\varphi}[\mathcal{D}\bar{Z}^{n,k-s}]$$

$$= \frac{2e^{-\mathrm{i}(m+n-2k)\varphi}}{n+1} \sum_{s=k-n}^{k} C_m^s \times \begin{cases} \cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ -\mathrm{i}\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd} \end{cases}$$

$$= \frac{2e^{-\mathrm{i}(m+n-2k)\varphi}}{n+1} \alpha_{n,k}^{(m)} \times \begin{cases} \cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ -\mathrm{i}\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd.} \end{cases}$$

From two formulas, derived above, it follows that

$$\left[\mathcal{D}_m \mathbf{s}_{n,k}^{(+m)}\right](\beta,\varphi) = \left[\mathcal{D}_m \mathbf{S}_{n,k}^{(+m)}\right](\beta,\varphi) = (-1)^n[\mathcal{D}_m(\mathbf{A}+\mathbf{B})]$$

$$= \frac{2(-1)^n}{n+1}\alpha_{n,k}^{(m)}(e^{\mathrm{i}(n-2k+m)\varphi} \pm e^{-\mathrm{i}(n-2k+m)\varphi}) \times \begin{cases} \cos\left[(n+1)(\beta-\varphi)\right] \\ \mathrm{i}\sin\left[(n+1)(\beta-\varphi)\right] \end{cases}$$

$$= \frac{4\alpha_{n,k}^{(m)}}{n+1} \times \begin{cases} \cos\left[(n-2k+m)\varphi\right]\cos\left[(n+1)(\beta-\varphi)\right] \\ \sin\left[(n-2k+m)\varphi\right]\sin\left[(n+1)(\beta-\varphi)\right] \end{cases}$$

$$= \frac{4\|\mathbf{s}_{n,k}^{(+m)}\|^2}{\pi 2^{m+1}} \times \begin{cases} \cos\left[(n-2k+m)\varphi\right]\cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ \sin\left[(n-2k+m)\varphi\right]\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd.} \end{cases}$$

By the same way we evaluate the fan-beam transform of the other part of basis for $n+m \neq 2k$

$$\left[\mathcal{D}_m \mathbf{s}_{n,k}^{(-m)}\right](\beta,\varphi) = \left[\mathcal{D}_m \mathbf{S}_{n,k}^{(-m)}\right](\beta,\varphi) = \frac{1}{\mathrm{i}}\left[\mathcal{D}_m(\mathbf{A}-\mathbf{B})\right]$$

$$= \frac{2\alpha_{n,k}^{(m)}}{n+1}(e^{\mathrm{i}(n-2k+m)\varphi} \mp e^{-\mathrm{i}(n-2k+m)\varphi}) \times \begin{cases} (1/\mathrm{i})\cos\left[(n+1)(\beta-\varphi)\right] \\ \sin\left[(n+1)(\beta-\varphi)\right] \end{cases}$$

$$= \frac{4\alpha_{n,k}^{(m)}}{n+1} \times \begin{cases} \sin\left[(n-2k+m)\varphi\right]\cos\left[(n+1)(\beta-\varphi)\right] \\ \cos\left[(n-2k+m)\varphi\right]\sin\left[(n+1)(\beta-\varphi)\right] \end{cases}$$

$$= \frac{4\|\mathbf{s}_{n,k}^{(-m)}\|^2}{\pi 2^{m+1}} \times \begin{cases} \sin\left[(n-2k+m)\varphi\right]\cos\left[(n+1)(\beta-\varphi)\right], & n = \text{even} \\ \cos\left[(n-2k+m)\varphi\right]\sin\left[(n+1)(\beta-\varphi)\right], & n = \text{odd.} \end{cases}$$

Consider the case $n+m=2k$ and $n,m=$ even, then

$$\begin{aligned}&\big[\mathcal{D}_m\mathbf{s}^{(+m)}_{n,(m+n)/2}\big](\beta,\varphi)=\big[\mathcal{D}_m\mathbf{S}^{(+m)}_{n,(m+n)/2}\big](\beta,\varphi)\\&=\frac{4\alpha^{(m)}_{n,k}}{2(n+1)}\times\begin{cases}\cos\left[(n+1)(\beta-\varphi)\right]\\0\end{cases}\\&=\frac{4\|\mathbf{s}^{(+m)}_{n,(m+n)/2}\|^2}{\pi 2^{m+1}}\times\begin{cases}\cos\left[(n+1)(\beta-\varphi)\right], & n=\text{even}\\0, & n=\text{odd}.\end{cases}\end{aligned}$$

If $n+m=2k$ and $n,m=$ odd, then

$$\begin{aligned}&\big[\mathcal{D}_m\mathbf{s}^{(-m)}_{n,(m+n)/2}\big](\beta,\varphi)=\big[\mathcal{D}_m\mathbf{S}^{(-m)}_{n,(m+n)/2}\big](\beta,\varphi)\\&=\frac{4\alpha^{(m)}_{n,k}}{2(n+1)}\times\begin{cases}0\\\sin\left[(n+1)(\beta-\varphi)\right]\end{cases}\\&=\frac{4\|\mathbf{s}^{(-m)}_{n,(m+n)/2}\|^2}{\pi 2^{m+1}}\times\begin{cases}0, & n=\text{even}\\\sin\left[(n+1)(\beta-\varphi)\right], & n=\text{odd}.\end{cases}\end{aligned}$$

Using equations for norms (5.10), we get (5.13). Theorem 3 is proved. □

At the end of this section we present some examples.

Example 1. Let's take for instance $m=0$, that corresponds to the scalar field, hence we have $a(x^1,x^2)=A(z,\bar z)$ and the orthogonal basis $s^{(\pm 0)}_{n,k}=S^{(\pm 0)}_{n,k}$ in $L_2(\mathbb{D})$ is

$$s^{(+0)}_{n,k}=\begin{cases}(-1)^n 2\,\mathrm{Re}\,Z^{n,k}, & 2k\neq n\\ s^{(+0)}_{n,n/2}=Z^{n,n/2}, & 2k=n,\end{cases}$$

$$s^{(-0)}_{n,k}=\begin{cases}2\,\mathrm{Im}\,Z^{n,k}, & 2k\neq n\\ s^{(-0)}_{n,n/2}=0, & 2k=n,\end{cases}$$

where $n\geq 0$, $k=0,1,\dots,[n/2]$. We have $\dim\mathbf{H}_N(\mathbb{D};\mathbf{S}_0,\delta=0)=(N+1)(N+2)/2$ and singular values are

$$\sigma_{n,k}\equiv\sigma^{(\pm 0)}_{n,k}=\sqrt{8\pi/(n+1)}\qquad(n\geq 0,\quad k=0,\dots,[n/2]).$$

Example 2. For $m=1$ one gets covector field $\mathbf{a}(x^1,x^2)=\{a_1,a_2\}$, which in the complex variables according to the tensor law has the representation $\mathbf{A}(z,\bar z)=\{(a_1-\mathrm{i}a_2)/2,(a_1+\mathrm{i}a_2)/2\}$. Dimension of the finite-dimensional subspace $\mathbf{H}_N(\mathbb{D};\mathbf{S}_1,\delta=0)$ equals to $(N+1)(N+4)/2$ and singular values are

$$\sigma^{(\pm 1)}_{n,k}=\begin{cases}\sqrt{4\pi/(n+1)}, & n\geq 0\text{ and }k=0\\\sqrt{8\pi/(n+1)}, & n\geq 1\text{ and }k=1,\dots,[(n+1)/2].\end{cases}$$

The polynomial orthogonal basis $\mathbf{s}^{(\pm1)}_{n,k}$ of the space of solenoidal covectors fields $\mathbf{H}(\mathbb{D}\,;\mathbf{S}_1, \delta=0)$ looks as follows

$$\mathbf{s}^{(+1)}_{n,k} \rightarrowtail \mathbf{S}^{(+1)}_{n,k} = \begin{cases} (-1)^n\{Z^{n,k}+\bar{Z}^{n,k-1}, Z^{n,k-1}+\bar{Z}^{n,k}\}, & 2k\neq n+1\\ \mathbf{S}^{(+1)}_{n,(n+1)/2} = \{0,0\}, & 2k=n+1, \end{cases}$$

$$\mathbf{s}^{(-1)}_{n,k} \rightarrowtail \mathbf{S}^{(-1)}_{n,k} = \begin{cases} (1/\mathrm{i})\{Z^{n,k}-\bar{Z}^{n,k-1}, Z^{n,k-1}-\bar{Z}^{n,k}\}, & 2k\neq n+1\\ \mathbf{S}^{(-1)}_{n,(n+1)/2} = (1/\mathrm{i})\{Z^{n,(n+1)/2}, Z^{n,(n+1)/2-1}\}, & 2k=n+1, \end{cases}$$

where $n\geq 0,\ k=0,1,\ldots,[(n+1)/2]$.

Example 3. For $m=2$ we have a symmetric second-order 2D tensor field $\mathbf{a}(x^1,x^2) = \begin{Bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{Bmatrix}$, which in complex variables has components

$$\mathbf{A}(z,\bar{z}) = \begin{Bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} a_{11}-a_{22}-2\mathrm{i}a_{12} & a_{11}+a_{22}\\ a_{11}+a_{22} & a_{11}-a_{22}+2\mathrm{i}a_{12}\end{Bmatrix}.$$

We also have inverse equalities

$$\mathbf{a}(x^1,x^2) = \begin{Bmatrix} 2(A_{12}+\operatorname{Re}A_{11}) & -2\operatorname{Im}A_{11}\\ -2\operatorname{Im}A_{11} & 2(A_{12}-\operatorname{Re}A_{11})\end{Bmatrix}.$$

Singular values for this case are

$$\sigma^{(\pm2)}_{n,k} = \frac{1}{\sqrt{n+1}}\begin{cases} \sqrt{2\pi}, & n\geq 0 \text{ and } k=0\\ \sqrt{4\pi}, & n=0 \text{ and } k=1\\ \sqrt{6\pi}, & n\geq 1 \text{ and } k=1\\ \sqrt{8\pi}, & n\geq 2 \text{ and } k=2,\ldots,[(n+2)/2], \end{cases}$$

where $n\geq 0,\ k=0,1,\ldots,[(n+2)/2]$ and basis tensor fields are

$$\mathbf{s}^{(+2)}_{n,k} \rightarrowtail \mathbf{S}^{(+2)}_{n,k} = \begin{cases} (-1)^n\begin{Bmatrix} Z^{n,k}+\bar{Z}^{n,k-2} & Z^{n,k-1}+\bar{Z}^{n,k-1}\\ Z^{n,k-1}+\bar{Z}^{n,k-1} & Z^{n,k-2}+\bar{Z}^{n,k}\end{Bmatrix}, \\ \qquad 2k\neq n+2\\ \mathbf{S}^{(+2)}_{n,(n+2)/2} = \begin{Bmatrix} Z^{n,(n+2)/2} & Z^{n,(n+2)/2-1}\\ Z^{n,(n+2)/2-1} & Z^{n,(n+2)/2-2}\end{Bmatrix}, \\ \qquad 2k=n+2, \end{cases}$$

$$\mathbf{s}^{(-2)}_{n,k} \rightarrowtail \mathbf{S}^{(-2)}_{n,k} = \begin{cases} \frac{1}{\mathrm{i}}\begin{Bmatrix} Z^{n,k}-\bar{Z}^{n,k-2} & Z^{n,k-1}-\bar{Z}^{n,k-1}\\ Z^{n,k-1}-\bar{Z}^{n,k-1} & Z^{n,k-2}-\bar{Z}^{n,k}\end{Bmatrix}, \\ \qquad 2k\neq n+2\\ \mathbf{S}^{(-2)}_{n,(n+2)/2} = \frac{1}{\mathrm{i}}\begin{Bmatrix} 0 & 0\\ 0 & 0\end{Bmatrix}, \qquad 2k=n+2. \end{cases}$$

Also in this case we have $\dim\mathbf{H}_N(\mathbb{D}\,;\mathbf{S}_2, \delta=0) = (N+1)(N+6)/2$.

6. IMPLEMENTATION

Scalar and vector cases of the inversion formula were numerically implemented and tested. The algorithm consists of 3 parts: solving the direct problem (that emulates the data acquisition in real life), finding coefficients of the polynomial that represents a function being reconstructed and evaluation of this polynomial on a grid for visualization.

In the scalar case, given a test function, defined by its values on a rectangular grid, the direct problem was solved by computing integrals (4.12) for the number of discrete values β_p, φ_q

$$[Da](\beta_p, \beta_p - \pi/2 + \varphi_q) = f_{p,q} \qquad (p, q = 0, 1, \ldots, M+1). \tag{6.1}$$

Bilinear interpolation was used to get the values of the original function between knots. So the obtained data set is an $(M+2) \times (M+2)$ matrix of $(f_{p,q})$ values that serves as an input for the inversion algorithm. Consider the scalar case for instance. Then, the function $a(x, y)$ is approximated by the polynomial of degree $N \leq M$ (note, that in this section we use notations $x \equiv x^1$ and $y \equiv x^2$)

$$a_N(x,y) = 2\sum_{n=0}^{N}\sum_{k=0}^{[n/2]*}\Big[\big(a_{n,k}\cos((n-2k)\psi) - b_{n,k}\sin((n-2k)\psi)\big) \times (-1)^k r^{n-2k} P_k^{(0,n-2k)}(2r^2-1)\Big]. \tag{6.2}$$

Here (r, ψ) are the polar coordinates of the point (x, y) and the sign $*$ means that in the case of even n the coefficient $a_{n,[n/2]}$ should be divided by 2 and $b_{n,[n/2]}$ should be set to 0. Then the fan-beam transform of (6.2) will look like

$$[\mathcal{D}a_N](\beta,\varphi) = \sum_{n=0}^{N}\frac{4}{n+1}\sum_{k=0}^{[n/2]*} a_{n,k} \times \begin{cases}\cos[(n+1)(\beta-\varphi)]\cos(n-2k)\varphi\\ \sin[(n+1)(\varphi-\beta)]\sin(n-2k)\varphi\end{cases} - b_{n,k} \times \begin{cases}\cos[(n+1)(\beta-\varphi)]\sin(n-2k)\varphi\\ \sin[(n+1)(\beta-\varphi)]\cos(n-2k)\varphi,\end{cases} \tag{6.3}$$

where the upper lines in the braces are used for the even n and the lower lines — for the odd n. The sign $*$ means the same as in (6.2). After the substitution of (6.3) into the (6.1) we get a system of linear equations for determining $(N+1)(N+2)/2$ unknown coefficients $a_{n,k}$ and $b_{n,k}$.

In the case of regular scanning scheme $\beta_p = p\varepsilon$, $\varphi_q = q\varepsilon/2$, $\varepsilon = 2\pi/(M+2)$ an explicit formulas for determining coefficients $a_{n,k}$ and $b_{n,k}$ were derived, provided that $M = N$

$$a_{n,k} = (-1)^k \frac{n+1}{(M+2)^2}\sum_{p=0}^{M+1}\sum_{q=0}^{M+1} f_{p,q}\sin\Big[\varepsilon\Big(p(2k-n) + \frac{q}{2}(2k+1)\Big)\Big], \tag{6.4}$$

$$b_{n,k} = (-1)^{k+1} \frac{n+1}{(M+2)^2}\sum_{p=0}^{M+1}\sum_{q=0}^{M+1} f_{p,q}\cos\Big[\varepsilon\Big(p(2k-n) + \frac{q}{2}(2k+1)\Big)\Big]. \tag{6.5}$$

Analogical formulae for the parallel-beam geometry can be found in [16]. The implementation of formulas (6.4) and (6.5) uses FFT and requires $\mathcal{O}(N^2 \log_2 N)$ operations, see [3].

After the coefficients $a_{n,k}$ and $b_{n,k}$ are found, the polynomial that represents the reconstructed function is effectively evaluated using a recurrent formula, see [3].

The inversion algorithm was also tested under the presence of a noise in the input data (sinogram). Uniform and Poisson random distributions were used for this purpose.

A representative set of numerical tests was performed for scalar and vector cases of the inversion algorithm. Some of the results are shown on Figures 2–4.

On the top of Figure 2 there are original function (to be reconstructed) on the left-hand side and its sinogram (an input data set for the inversion algorithm) on the right-hand side. The middle row contains reconstructions from 32 and 256 fan-beam projections (free of noise). The number of terms in SVD was 30 and 254 respectively. The bottom row contains examples of reconstruction from noisy data. A random noise was superimposed on the sinogram. The L_2-norm of the noise was 10 % of the L_2-norm of the sinogram. 1024 noisy fan-beam projections were used. The number of terms in SVD that were taken for reconstruction are 1022 and 254 respectively. It's possible to reduce the noise in the output image by taking less terms in the SVD.

Another example of scalar tomography is shown on Figure 3. Again, the original unknown function (the fast oscillating one, with fine features) is at the top row, on the left-hand side and it's sinogram is on the right-hand side. The middle row contains reconstructions from 32 and 512 fan-beam projections (free of noise). The number of terms in SVD was 30 and 510 respectively. The bottom row contains examples of reconstruction from noisy data. A random noise was superimposed on the sinogram. The L_2-norm of the noise was 10 % of the L_2-norm of the sinogram. 2048 noisy fan-beam projections were used. The number of terms in SVD that were taken for reconstruction are 1022 and 510 respectively. Again, one can observe significant enhancement of reconstruction when only part of terms are taken in SVD.

Figure 4 illustrates the vector case of the inversion algorithm. The first solenoidal vector field (the top row, where the first component a_1 is on the left and the second component a_2 is on the right) is defined by the formulae

$$\begin{aligned} a_1(x,y) &= 2xy\cos(x^2+y^2) + \cos(6xy) - 6xy\sin(6xy), \\ a_2(x,y) &= -\sin(x^2+y^2) - 2x^2\cos(x^2+y^2) + 6y^2\sin(6xy). \end{aligned} \tag{6.6}$$

Another vector field (the middle row) was obtained from the previous solenoidal vector field by adding the potential part

$$\begin{aligned} a_1(x,y) &= 2xy\cos(x^2+y^2) + \cos(6xy) - 6xy\sin(6xy) \\ &\quad + 2\pi x\cos(\pi(x^2+y^2)), \\ a_2(x,y) &= -\sin(x^2+y^2) - 2x^2\cos(x^2+y^2) + 6y^2\sin(6xy) \\ &\quad + 2\pi y\cos(\pi(x^2+y^2)). \end{aligned} \tag{6.7}$$

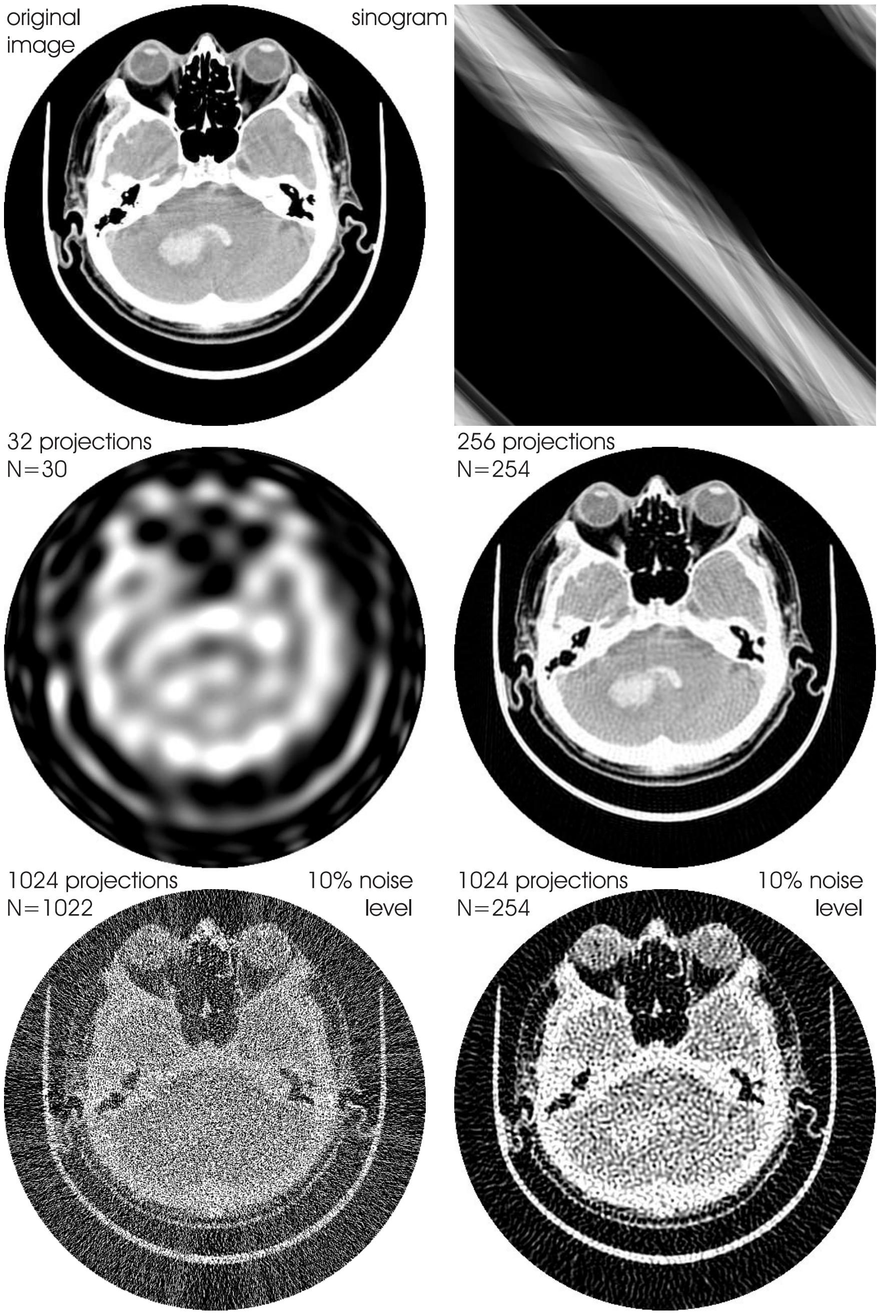

Figure 2. Top row: image of the original function (on the left) and its sinogram (on the right). Middle and bottom rows: reconstructions of the function from different number of fan-projections and under the presence of noise in the sinogram, see the text for detailed explanation

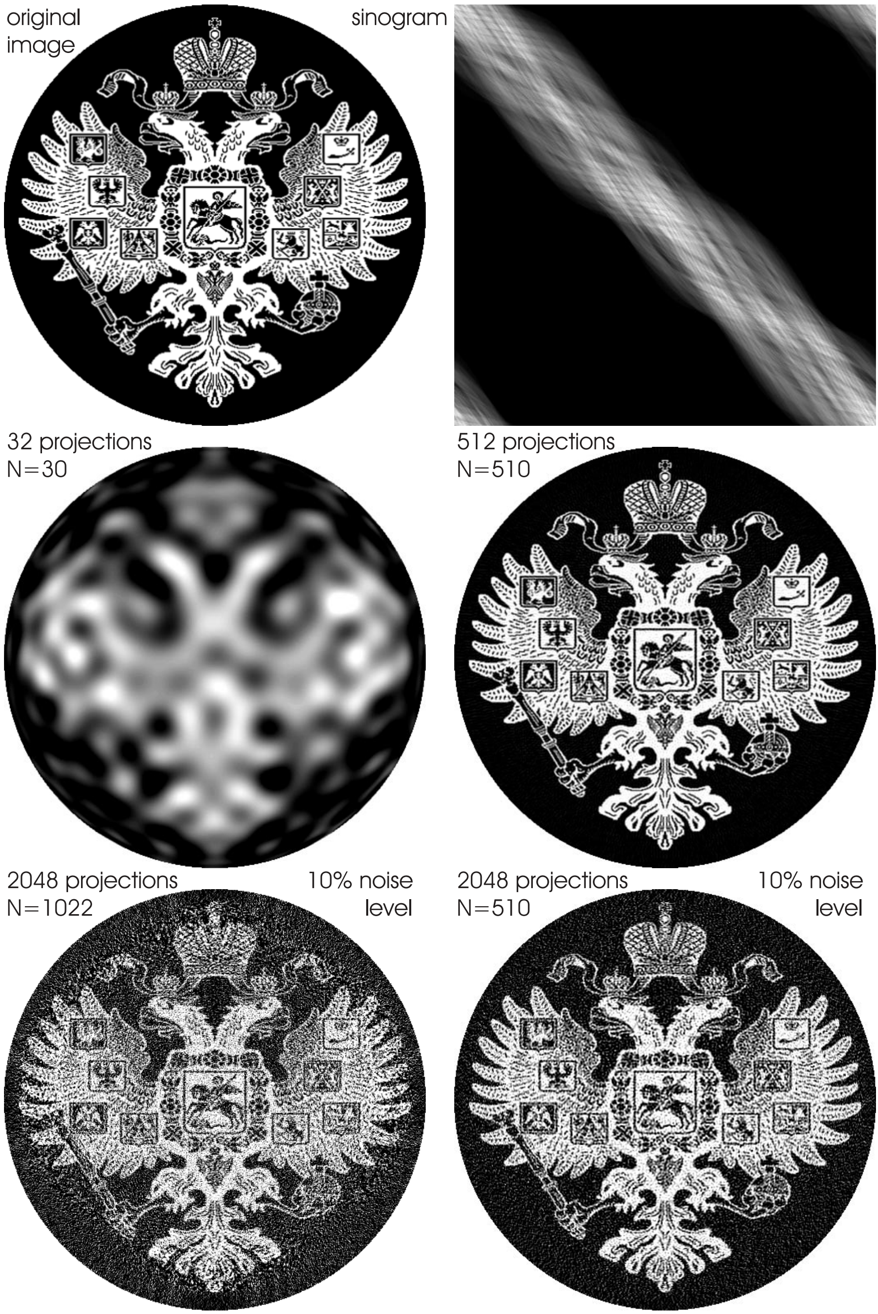

Figure 3. Top row: image of the original function (on the left) and its sinogram (on the right). Middle and bottom rows: reconstructions of the function from different number of fan-projections and under the presence of noise in the sinogram, see the text for detailed explanation

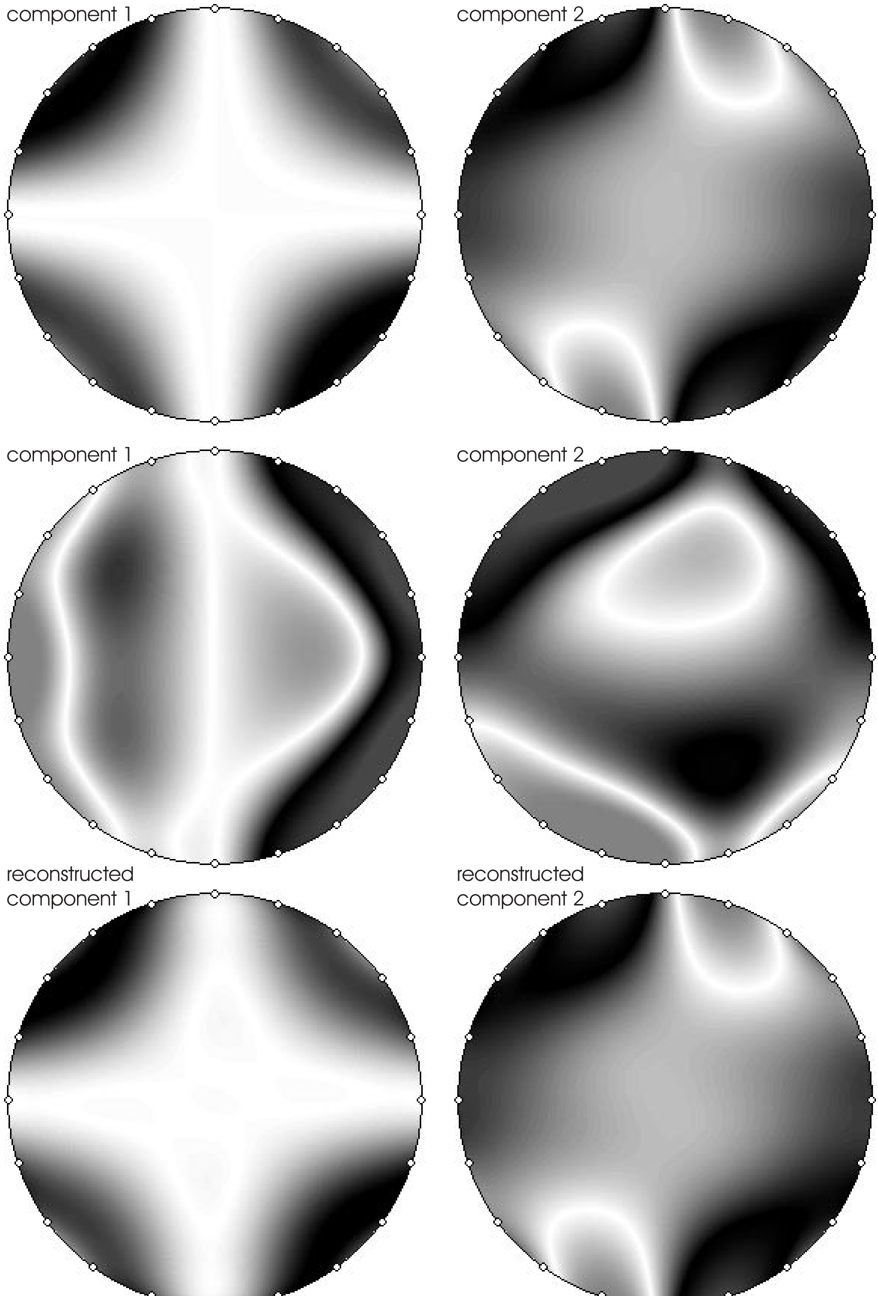

Figure 4. Top row: solenoidal vector field (first component on the left, second component on the right) given by formulae (6.6). Middle row: non-solenoidal vector field given by formulae (6.7). Bottom row: reconstruction from 20 fan-projections is identical in both cases, relative error 0.21 %

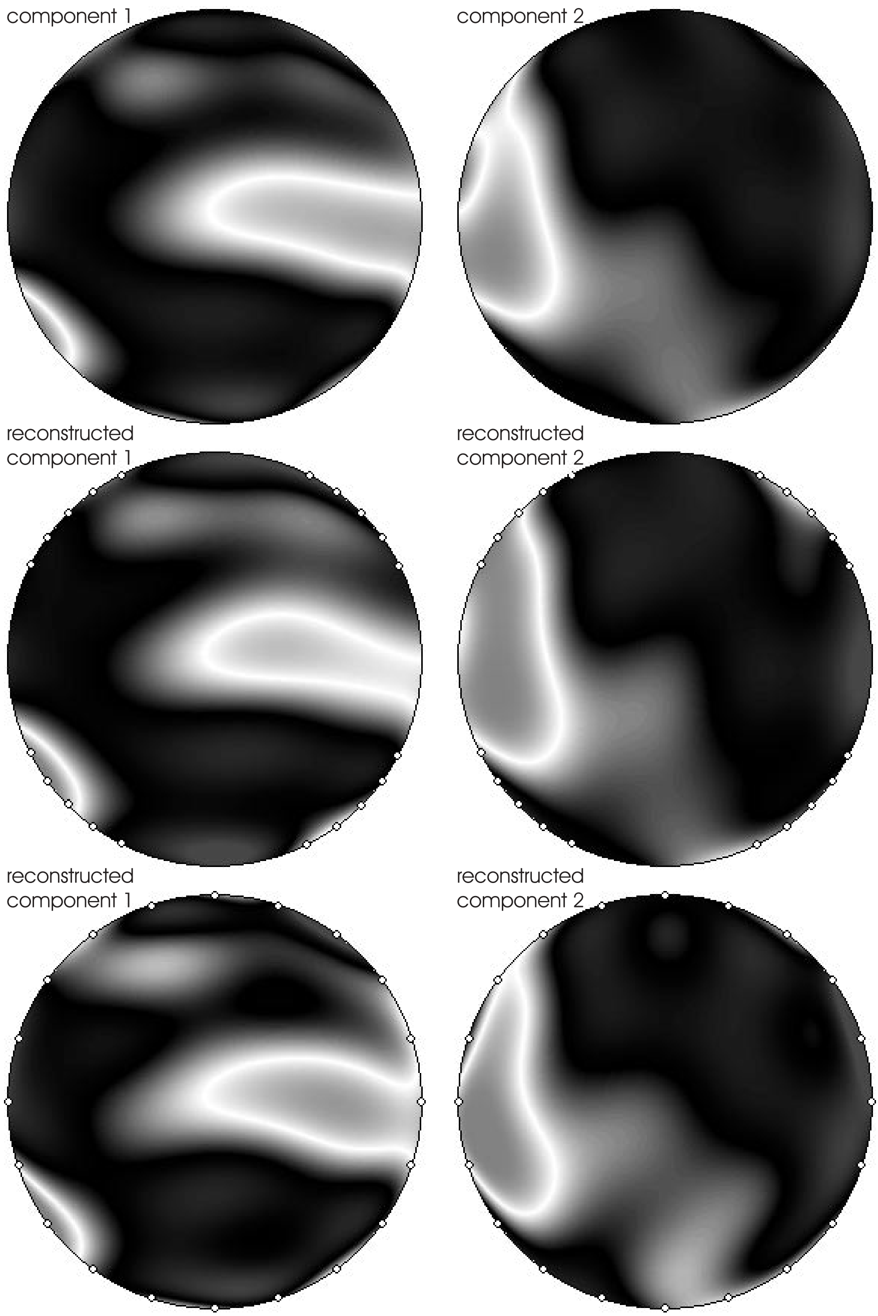

Figure 5. Top row: original solenoidal vector field. Middle row: its reconstruction from 20 irregular fan-projections, relative error 2.6 %. Bottom row: its reconstruction from the 20 regular fan-projections under the presence of noise (3 %) in the sinogram, relative error 1.6 %

As it can be seen (on the bottom row of the figure), reconstruction from these two vector fields is identical and contains only the solenoidal part of the vector field.

Figure 5 illustrates another solenoidal vector field (the top row). Its reconstruction from the 20 irregular fan-projections is in the middle row. Here the positions of the fan-projection vertices are shown as white dots on the boundary of the circle. Scanning was performed only over those lines, whose endpoints belong to this set of 20 points. The last reconstruction (the bottom row) was made under the presence of noise in the sinogram. The noise level was 3 % (again, in L_2-norm).

7. CONCLUSION

The novel inversion algorithm for the tensor tomography problem was developed and numerically implemented. The algorithm is based on the SVD of the tensor Radon transform that allows to characterize the range of the operator, invert it and estimate an incorrectness of the corresponding inverse problem. The algorithm can also be used with noisy measurements.

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