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Inversion of the scalar and vector attenuated X-ray transforms in a unit disc

S. G. Kazantsev and A. A. Bukhgeim

Abstract. This paper summarizes some of the old results obtained for the problems of inverting the two-dimensional attenuated X-ray transform and the attenuated vectorial X-ray transform, both using the fan-beam geometry. These inverse problems are considered on the language of the transport equation and two approaches are described for solving them. The first one, dating back to 1996, gives the inversion formulae on the basis of the theory of so-called $\mathcal{A}$-analytic functions. And the second method (developed by the authors in 2002) yields the inversion formulae for the scalar and vector attenuated X-ray transforms without using the theory of $\mathcal{A}$-analytic functions, but merely by reducing the arising inverse problems to the unattenuated case by the change of variables. Numerical implementation details are also provided.

Key words. $\mathcal{A}$-analytic functions, attenuated X-ray transform, attenuated vectorial X-ray transform, Radon transform, transport equation, Hilbert transform.

AMS classification. 44A12, 92C55, 53C65, 35R25, 65R10.

1. Introduction

The attenuated X-ray transform is a mathematical model for a number of inverse problems, which arise in different fields (for example, in optics, in the analysis of plasma and semi-transparent media, in non-destructive testing, in astronomical measurements and medical applications). The most prevailing problem that is formulated by the attenuated X-ray transform is the problem of single photon emission computerized tomography (SPECT): let the unknown function of radioactive sources distribution (*the emission map*) be given in a unit disc Ω on the plane $\mathbb{R}^2$. An example of such radioactive sources are γ-quanta which propagate in straight lines and are absorbed by the medium. The task is to reconstruct the *unknown emission map* from the known flow of radioactive particles measured on the boundary of a disc, provided that *the attenuation map* of the medium is known. The problems of SPECT are mathematically investigated using the attenuated X-ray transform mostly within the framework of either a *parallel-beam* or a *fan-beam* scanning geometry (the last one is also called a divergent beam or a cone-beam geometry in 3D case). In the first case parallel line integrals are determined for a fixed direction and the process is repeated for a number of different directions. In the second case line integrals emanating from a given source point (also called a *vertex point*) are computed for different directions and then it's repeated for a certain number of source points, see Figure 1a.

In this paper we discuss only the planar (2D) case of tomography. The first result

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on global uniqueness for 2D integral geometry problems of general form had been obtained by R. G. Mukhometov [29] by the method of energy estimates. However, in the presence of attenuation, this result becomes local. For the case of constant attenuation an inversion formula was derived by O. Tretiak and C. Metz in [46]. For the class of (real) analytic attenuation functions an inversion formula was established in [22]. And the first explicit inversion formula for C^2 attenuation functions was obtained in [3]. Both results [3, 22] were based on the Cauchy formula for the so-called $\mathcal{A}$-analytic functions. Then the authors have derived the inversion formulae without using the theory of $\mathcal{A}$-analytic functions in [10] for the scalar and vector attenuated Radon transforms. It was done by reducing the arising inverse problems to the unattenuated case by the change of variables. The singular value decomposition (SVD) of the X-ray transform in a unit disc from our previous work [9, 23] was used.

Another explicit inversion formulae for reconstructing the emission map in the case of a parallel-beam scanning geometry were derived recently by R. Novikov (see [36, 37, 38]) and F. Natterer [32] and numerically evaluated by L. Kunyansky in [26, 19]. Another exact inversion formula for the attenuated X-ray transform, closely related to the Novikov's inversion formula, was derived in [21, 7, 4, 5], using a different derivation. An implementation of the inversion formula very similar to the filtered back-projection algorithm of X-ray tomography is given in [49]. And in [20] authors have shown the equivalence of Novikov's formula to the original approach based on $\mathcal{A}$-analytic functions by using the transformation between parallel-beam and fan-beam coordinates.

A good reference on the developments in the field of the attenuated X-ray transform is the survey articles by D. Finch [17] and P. Kuchment [24]. Some theoretical results based on the language of the transport equation can be found in [1] and [39]. In [2] formulae for the general solution of the transport equation were found. P. Kuchment and I. Shneiberg in [25] have derived an inversion formula of the filtered-backprojection type for the exponential X-ray transform with angle dependent attenuation map. In [35, 40, 41] the problem with incomplete SPECT data was considered and it was shown that an angular range of 180° is sufficient for the parallel-beam geometry with constant attenuation.

Note, that the ideal mathematical solution of the SPECT would be a method for determining the unknown emission source *and* attenuation coefficient *simultaneously using the SPECT data only.* The first contributions to this problem are due to Y. Censor et al in [14]. Then F. Natterer in [31] tried to reconstruct the attenuation map using the SPECT consistency conditions, see also [28]. V. Dicken in [16] formulated this problem as an optimization problem and solved it via nonlinear Tikhonov regularization techniques. More information and references about simultaneous reconstruction of the unknown emission and attenuation maps can be found in [8, 18, 48, 43, 50]. But a satisfactory method for determining both unknown functions from the emission data has not yet been found.

In this paper we summarize some of the old results obtained for the problems of inverting the two-dimensional attenuated X-ray transform and attenuated vectorial X-ray transform in the fan-beam formulation. The attenuation map is assumed to be known from additional transmission measurements.

The structure of this article is as follows. After introduction of scalar and vector attenuated X-ray transforms in Section 2, we consider the problem of inverting them on the language of the transport equation in Section 3 and describe the method of $\mathcal{A}$-analytic functions for solving it. In Section 4 we introduce the angular Hilbert transform and study its properties relevant for inversion of attenuated X-ray transforms. In Sections 5 and 6 we present the inversion formulae for the scalar and vector attenuated X-ray transforms that were developed in our previous work [10]. At the end, we discuss the implementation details and present some numerical results.

2. Statements of the problems and preliminaries

Let $\Omega = \{\mathbf{x} \equiv (x,y) \in \mathbb{R}^2 : |\mathbf{x}|^2 = x^2 + y^2 < 1\}$ be an open unit disc in $\mathbb{R}^2$ with the boundary $\partial\Omega = \{\mathbf{x} \equiv (\cos\beta, \sin\beta),\ \beta \in [0, 2\pi)\}$. Let $a(\mathbf{x}) \in L_2(\Omega)$ be a complex-valued function defined in the disc Ω. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two points from a closed disc $\overline{\Omega}$, then

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} a(\mathbf{x}')|d\mathbf{x}'| := \int_0^{|\mathbf{x}_2-\mathbf{x}_1|} a\Big(\mathbf{x}_1 + s\frac{\mathbf{x}_2-\mathbf{x}_1}{|\mathbf{x}_2-\mathbf{x}_1|}\Big)ds$$

will denote an integral of the function $a(\mathbf{x})$ along a line segment $[\mathbf{x}_1, \mathbf{x}_2]$.

For every point $\mathbf{x} \in \overline{\Omega}$ and a direction vector $\boldsymbol{\theta} = (\cos\varphi, \sin\varphi)$, $\varphi \in [0, 2\pi)$ we define the functions

$$\begin{aligned} \gamma(\mathbf{x},\varphi) &:= \varphi + \pi/2 + \arccos\left(\langle \mathbf{x}, \boldsymbol{\theta}^{\perp}\rangle\right), \\ \boldsymbol{\gamma}(\mathbf{x},\varphi) &:= (\cos\gamma(\mathbf{x},\varphi), \sin\gamma(\mathbf{x},\varphi)), \end{aligned} \tag{2.1}$$

where $\boldsymbol{\theta}^{\perp} = (\cos(\varphi+\pi/2), \sin(\varphi+\pi/2))$. The point $\boldsymbol{\gamma}(\mathbf{x},\varphi)$ gives the intersection of the boundary $\partial\Omega$ with the ray emitted from the point $\mathbf{x}$ in the direction $-\boldsymbol{\theta} = -(\cos\varphi, \sin\varphi)$, see Figure 1b. The function $\gamma(\mathbf{x},\varphi)$ can be defined for any strictly convex region. If $\mathbf{x} = (\cos\beta, \sin\beta)$ represents the boundary point from $\partial\Omega$, then

$$\gamma(\mathbf{x},\varphi) \equiv \gamma(\beta,\varphi) = \begin{cases} 2\varphi - \beta + \pi & \text{for } \varphi \in [\beta - \pi/2, \beta + \pi/2], \\ \beta & \text{otherwise.} \end{cases}$$

Hereinafter we shall use the following notation: if $\mathbf{x} = (\cos\beta, \sin\beta) \in \partial\Omega$ and $v(\,\cdot\,) \in C(\overline{\Omega})$, then we shall imply $v(\beta) \equiv v(\mathbf{x})$ and write $v(\beta)$ instead of $v(\mathbf{x})$.

2.1. The attenuated X-ray transform

Let the attenuation map (tissue density) and the emission map (activity distribution) be described by real-valued functions $\mu(\mathbf{x})$ and $a(\mathbf{x})$ respectively, both defined in Ω.

Definition 2.1. For $\mathbf{x} \in \overline{\Omega}$, $\varphi \in [0, 2\pi)$ we put

$$[\mathcal{D}\mu](\mathbf{x},\varphi) := \int_{\boldsymbol{\gamma}(\mathbf{x},\varphi)}^{\mathbf{x}} \mu(\mathbf{x}')\,|d\mathbf{x}'|. \tag{2.2}$$

The operator $\mathcal{D}$ is known as *the X-ray transform.*

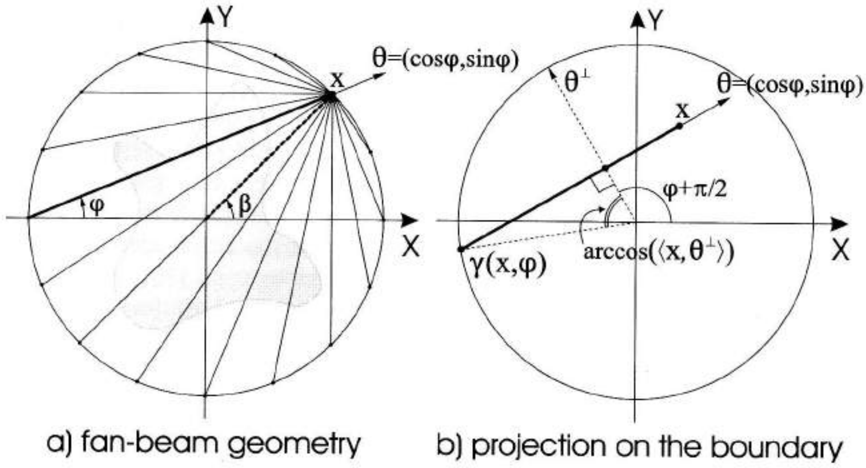


Figure 1. A fan-beam scanning geometry and the projection on the boundary scheme

Remark 2.2. While the classical Radon transform integrates over hyperplanes in $\mathbb{R}^n$ [30], the X-ray transform $\mathcal{D}$ integrates over straight lines. Thus, in our case $n = 2$, Radon transform and X-ray transform differ only in the notation.

Definition 2.3. *The attenuated X-ray transform* $\mathcal{D}_\mu$ is defined for functions with compact support in Ω by

$$[\mathcal{D}_\mu a](\mathbf{x}, \varphi) := \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} a(\mathbf{x}')e^{-\int_{\mathbf{x}'}^{\mathbf{x}} \mu(\mathbf{x}'')\,|d\mathbf{x}''|}\,|d\mathbf{x}'|. \tag{2.3}$$

This integral is the density of particles at a point $\mathbf{x}$ traveling in the direction $\boldsymbol{\theta} = (\cos\varphi, \sin\varphi)$ at a constant speed. In practice we can measure the function (2.3) only for $\mathbf{x} \in \partial\Omega$, so we get SPECT data or *the sinogram* $f(\beta, \varphi)$

$$f(\beta, \varphi) := [\mathcal{D}_\mu a](\beta, \varphi) = \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} a(\mathbf{x}')e^{-[\mathcal{D}\mu](\mathbf{x}',\varphi+\pi)}\,|d\mathbf{x}'|, \tag{2.4}$$

where $\mathbf{x} = (\cos\beta, \sin\beta)$ and $\beta, \varphi \in [0, 2\pi)$. The function $f(\beta, \cdot)$ is also called an attenuated fan-beam projection of $a(\mathbf{x})$ and $\mathbf{x} = (\cos\beta, \sin\beta)$ is *the vertex point* of the fan projection, see Figure 1a. The inverse problem consists in determining the unknown function $a(\mathbf{x})$ from the measured sinogram $f(\beta, \varphi)$, provided that $\mu(\mathbf{x})$ is known.

More information and references about these and related transforms are given in [30, 34].

2.2. Attenuated vectorial X-ray transform

The similar transformation for vector fields was first introduced by K. Stråhlen in [45] for the constant attenuation and was called the exponential vectorial transform.

During the short history of vector tomography, work has been done with a wide variety of applications in mind [44]. Acoustic travel-time measurements is the main field for vector tomography, with a number of applications in medicine and industry. Doppler measurements on flows by continuous ultra sound can be used to detect cancer tumours. Around tumours the blood flow is more irregular and more intense than in normal tissue. It is also used for reconstructing a wind velocity field in the atmosphere. Optical transmission measurements is an analogue of time-of-flight principle but uses laser beam as a source of radiation. Oceanography is a mesoscale acoustic tomography used to estimate fluid velocity. Photoelasticity is a non-destructive method for three-dimensional stress analysis in transparent specimens. Nuclear magnetic resonance also gives a way to measure a vectorial X-ray transform.

Definition 2.4. We define *the attenuated vectorial X-ray transform* $\overrightarrow{\mathcal{D}}_\mu$ for a vector field $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}))$ with compact support in Ω by

$$[\overrightarrow{\mathcal{D}}_\mu \mathbf{a}](\mathbf{x},\varphi) := \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} (a_1(\mathbf{x}')\cos\varphi + a_2(\mathbf{x}')\sin\varphi) e^{-\int_{\mathbf{x}'}^{\mathbf{x}} \mu(\mathbf{x}'')|\mathrm{d}\mathbf{x}''|}\,|\mathrm{d}\mathbf{x}'|$$

$$= \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} (A(\mathbf{x}')e^{\mathrm{i}\varphi} + \overline{A(\mathbf{x}')}e^{-\mathrm{i}\varphi}) e^{-\int_{\mathbf{x}'}^{\mathbf{x}} \mu(\mathbf{x}'')|\mathrm{d}\mathbf{x}''|}\,|\mathrm{d}\mathbf{x}'|, \qquad (2.5)$$

where $A := (a_1 - \mathrm{i}a_2)/2$, $\overline{A} := (a_1 + \mathrm{i}a_2)/2$ and μ is the attenuation function.

Again, in practice we may measure the function (2.5) only for $\mathbf{x} \in \partial\Omega$, so we get *the sinogram* $f(\beta,\varphi)$

$$f(\beta,\varphi) := [\overrightarrow{\mathcal{D}}_\mu \mathbf{a}](\beta,\varphi) = \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} (A(\mathbf{x}')e^{\mathrm{i}\varphi} + \overline{A(\mathbf{x}')}e^{-\mathrm{i}\varphi}) e^{-[\mathcal{D}\mu](\mathbf{x}',\varphi+\pi)}\,|\mathrm{d}\mathbf{x}'|, \quad (2.6)$$

where $\mathbf{x} = (\cos\beta, \sin\beta)$; $\beta, \varphi \in [0, 2\pi)$ and $\mathcal{D}$ is the X-ray transform (2.2). The inverse problem here consists in determining the unknown vector field $\mathbf{a}(\mathbf{x})$ from the measured sinogram $f(\beta,\varphi)$, provided that the attenuation map $\mu(\mathbf{x})$ is a known real-valued function.

It should be stated that in the unattenuated case (when $\mu(\mathbf{x}) \equiv 0$) we can restore only the solenoidal part of the vector field (see [9, 23, 42, 45]). In another words, since for arbitrary square integrable vector field we have the Helmholtz decomposition into the potential and solenoidal parts, $\mathbf{L}_2(\Omega) = \nabla H_0^1(\Omega) \oplus \mathbf{H}(\Omega; \mathrm{div} = 0)$ (see [15, p. 216]), then in the case $\mu(\mathbf{x}) \equiv 0$ we can only determine the component $\mathbf{H}(\Omega; \mathrm{div} = 0)$.

3. Inverse problems for the transport equation

In this section we restate the problems of inverting the transforms (2.4) and (2.6) as the inverse problems for the transport equation and briefly display the formal scheme of the $\mathcal{A}$-analytic functions method for its solution. The idea of this approach was suggested by A. L. Bukhgeim [11] in 1987, see also [3, 12, 13, 22] for more details.

We will utilize the language of complex variables and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual way

$$\mathbf{x} = (x, y) \mapsto z = x + \mathrm{i}y, \qquad \mathrm{i}^2 = -1.$$

Hence,

$$\boldsymbol{\theta} = (\cos\varphi, \sin\varphi) \mapsto e^{\mathrm{i}\varphi}, \qquad \mathbf{x} = (\cos\beta, \sin\beta) \mapsto t = e^{\mathrm{i}\beta}.$$

In moving to the complex variable z we shall keep the old notation for functions, namely that $a(z,\overline{z}) \equiv a(\mathbf{x})$, $u(z,\overline{z},\varphi) \equiv u(\mathbf{x},\boldsymbol{\theta})$ and so on. For brevity we shall write $a(z)$, $u(z,\varphi)$ instead of $a(z,\overline{z})$, $u(z,\overline{z},\varphi)$ and so on. Also, for $t = e^{\mathrm{i}\beta}$ we have $u(\beta,\varphi) \equiv u(t,\varphi)$.

The directional derivative

$$\partial_{\boldsymbol{\theta}} := \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}$$

in complex variables has the form

$$\partial_{\boldsymbol{\theta}} = e^{\mathrm{i}\varphi}\partial + e^{-\mathrm{i}\varphi}\overline{\partial}, \tag{3.1}$$

where ∂ and $\overline{\partial}$ are the brief notations for the formal partial derivatives with respect to variables z and $\overline{z}$ respectively

$$\partial \equiv \frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \qquad \overline{\partial} \equiv \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big).$$

Using the directional derivative (3.1), the problems of inverting the transforms (2.4) and (2.6) are reduced to the following inverse problems for the transport equation

$$e^{\mathrm{i}\varphi}\partial u(z,\varphi) + e^{-\mathrm{i}\varphi}\overline{\partial} u(z,\varphi) + \mu(z)u(z,\varphi) = a(z,\varphi), \tag{3.2}$$

where the right-hand side

$$a(z,\varphi) = \begin{cases} a(z) & \text{for attenuated X-ray transform} \\ a_1(z)\cos\varphi + a_2(z)\sin\varphi & \text{for attenuated vectorial X-ray transform.} \end{cases} \tag{3.3}$$

Inverse problem. *Determine the right-hand side $a(z,\varphi)$ of the transport equation* (3.2), (3.3), *provided that $\mu(z)$ is a known real-valued function in a disc Ω, and the solution $u(z,\varphi)$ which is real-valued and 2π-periodic in φ is defined on the manifold $\Sigma := \partial\Omega \times [0, 2\pi)$, and satisfies the boundary condition*

$$u(t,\varphi) = f(\beta,\varphi), \qquad t = e^{\mathrm{i}\beta} \in \partial\Omega.$$

Note that the condition $f(\gamma(\beta,\varphi),\varphi) \equiv 0$ and consequently $f(\gamma(\mathbf{x},\varphi),\varphi) \equiv 0$ hold for the function $f(\beta,\varphi)$, which means there is no incoming radiation.

3.1. $\mathcal{A}$-analytic functions

In this subsection we briefly review the $\mathcal{A}$-analytic functions method for solving the inverse problems for the 2D transport equations (3.2), (3.3).

3.1.1 Unattenuated case

Firstly we consider the unattenuated case and let us denote the X-ray transform (2.2) of a real-valued function μ by

$$m(\mathbf{x}, \varphi) := [\mathcal{D}\mu](\mathbf{x}, \varphi).$$

Then, the function $m(z, \varphi) \equiv m(\mathbf{x}, \varphi)$ represents the solution of the transport equation in complex form

$$e^{\mathrm{i}\varphi}\partial m(z, \varphi) + e^{-\mathrm{i}\varphi}\overline{\partial} m(z, \varphi) = \mu(z), \qquad (z, \varphi) \in \Omega \times [0, 2\pi) \tag{3.4}$$

with the boundary condition

$$m(t, \varphi) = f(\beta, \varphi), \qquad t = e^{\mathrm{i}\beta}, \qquad (t, \varphi) \in \partial\Omega \times [0, 2\pi).$$

Expanding m in a Fourier series with respect to φ,

$$m(\mathbf{x}, \varphi) = \sum_{k=-\infty}^{\infty} m_k(\mathbf{x}) e^{-\mathrm{i}k\varphi}, \tag{3.5}$$

and substituting it into the transport equation (3.4), we obtain the infinite system of elliptic differential equations after matching terms with the same exponents

$$\overline{\partial} m_{-1} + \partial m_1 = \mu(\mathbf{x}) \qquad \text{(inversion formula)}, \tag{3.6}$$

$$\overline{\partial} m_k + \partial m_{k+2} = 0, \qquad k \neq -1. \tag{3.7}$$

Since m is real-valued, the Fourier coefficients $\{m_k(z)\}$ will be complex-conjugate quantities $m_{-k} = \overline{m}_k$. This implies that for determining m uniquely, it is sufficient to find the complex-valued vector

$$\mathbf{m}(z) := (m_0(z), m_1(z), m_2(z), \ldots).$$

By virtue of (3.7), the vector function $\mathbf{m}$ satisfies the *Beltrami-type* equation

$$\overline{\partial}\mathbf{m} - \mathcal{A}\partial\mathbf{m} = \mathbf{0}, \qquad z \in \Omega \tag{3.8}$$

with the operator coefficient $\mathcal{A}$ acting by the rule

$$\mathcal{A} : (m_0, m_1, m_2, \ldots) \mapsto -(m_2, m_3, m_4, \ldots).$$

Correspondingly, for boundary points we have

$$f(t, \varphi) = \sum_{k=-\infty}^{\infty} f_k(t) e^{-\mathrm{i}k\varphi}, \qquad \mathbf{f}(t) := (f_0(t), f_1(t), f_2(t), \ldots), \qquad t \in \partial\Omega,$$

and thus for $\mathbf{m}(z)$ we have the boundary condition

$$\mathbf{m}(t) = \mathbf{f}(t), \qquad t \in \partial\Omega. \tag{3.9}$$

If we determine the vector function $\mathbf{m}$ uniquely from equation (3.8) and boundary conditions (3.9), then the unknown function $\mu(z)$ can be derived from (3.6) by the formula

$$\mu(z) = 2\Re\big[\partial m_1(z)\big]. \tag{3.10}$$

The operator $\mathcal{A}$ is defined on the Hilbert space l_2 of square summable sequences and its Sobolev subspaces l_2^p

$$l_2^p = \Big\{(m_0, m_1, m_2, \dots) : \sum_{k=0}^{\infty}(1 + k^{2p})|m_k|^2 < \infty\Big\}.$$

Definition 3.1. Solutions $\mathbf{m} \in C(\overline{\Omega}; l_2) \cap C^1(\overline{\Omega}; l_2^1)$ of the equation (3.8) are called *$\mathcal{A}$-analytic functions* and satisfy an analog of Cauchy's integral formula, see [12, 13]

$$\mathbf{m}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \big((t - z)I - \overline{(t - z)}\mathcal{A}\big)^{-1} (\mathrm{d}t + \mathcal{A}\mathrm{d}\bar{t})\mathbf{f}(t), \qquad z \in \Omega,$$

where I is the identity operator.

The inversion formula (3.10) can be further simplified. Let us consider the well-known in the complex analysis *Pompeiu integral operator* [47]

$$[T\mu](z) := -\frac{1}{\pi} \iint_{\Omega} \frac{\mu(\zeta)}{\zeta - z}\, d\xi\, d\eta,$$

where $z = x + \mathrm{i}y$, $\zeta = \xi + \mathrm{i}\eta$. Putting $\zeta - z = \rho e^{\mathrm{i}\varphi}$, we have

$$\begin{aligned}
[T\mu](z) &= -\frac{1}{\pi} \int_0^{2\pi} e^{-\mathrm{i}\varphi} \mathrm{d}\varphi \int_0^{|z - \gamma(z, \varphi + \pi)|} \mu(z + \rho e^{\mathrm{i}\varphi})\, \mathrm{d}\rho \\
&= \frac{1}{\pi} \int_0^{2\pi} e^{-\mathrm{i}\varphi} \mathrm{d}\varphi \int_0^{|z - \gamma(z,\varphi)|} \mu(z - \rho e^{\mathrm{i}\varphi})\, \mathrm{d}\rho = \frac{1}{\pi} \int_0^{2\pi} e^{-\mathrm{i}\varphi} \mathrm{d}\varphi \int_{\gamma(z,\varphi)}^{z} \mu(\zeta)\, |\mathrm{d}\zeta| \\
&= \frac{1}{\pi} \int_0^{2\pi} e^{-\mathrm{i}\varphi} m(z, \varphi)\, \mathrm{d}\varphi = 2m_{-1}(z).
\end{aligned}$$

Substituting the last expression for $[T\mu]$ into the *Cauchy-Pompeiu formula* [47]

$$\mu(z) = \overline{\partial}\, [T\mu](z)$$

we simplify the inversion formula (3.10) into

$$\partial m_1(z) = \overline{\partial} m_{-1}(z) = \mu(z)/2, \tag{3.11}$$

where both $\partial m_1(z)$ and $\overline{\partial} m_{-1}(z)$ are real-valued.

3.1.2 Attenuated case

Following the same scheme, the attenuated X-ray transform

$$u(\mathbf{x},\varphi) := [\mathcal{D}_\mu a](\mathbf{x},\varphi),$$

represents the solution of the transport equation

$$e^{\mathrm{i}\varphi}\partial u(z,\varphi) + e^{-\mathrm{i}\varphi}\overline{\partial} u(z,\varphi) + \mu(z)u(z,\varphi) = a(z), \qquad (z,\varphi) \in \Omega \times [0,2\pi) \quad (3.12)$$

with the boundary condition

$$u(t,\varphi) = f(\beta,\varphi), \qquad t = e^{\mathrm{i}\beta}, \qquad (t,\varphi) \in \partial\Omega \times [0,2\pi).$$

Substituting a Fourier series representation

$$u(\mathbf{x},\varphi) = \sum_{k=-\infty}^{\infty} u_k(\mathbf{x})e^{-\mathrm{i}k\varphi}$$

into (3.12) and matching terms with the same exponents, we get the infinite system of equations

$$\overline{\partial} u_{-1} + \mu u_0 + \partial u_1 = a(z) \qquad \text{(inversion formula)}, \qquad (3.13)$$

$$\overline{\partial} u_k + \mu u_{k+1} + \partial u_{k+2} = 0, \qquad k \neq -1. \qquad (3.14)$$

Due to (3.14), the vector function

$$\mathbf{u}(z) := (u_0(z), u_1(z), u_2(z), \dots)$$

satisfies the *generalized Beltrami-type* equation

$$\overline{\partial}\mathbf{u} - \mathcal{A}\partial\mathbf{u} + \mu\mathcal{B}\mathbf{u} = \mathbf{0}, \qquad z \in \Omega \qquad (3.15)$$

with the operator coefficients $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{B}$ acts by the rule

$$\mathcal{B} : (u_0, u_1, u_2, \dots) \mapsto (u_1, u_2, u_3, \dots).$$

Definition 3.2. Such functions $\mathbf{u}(z)$ are called *generalized $\mathcal{A}$-analytic functions*.

If we determine the vector function $\mathbf{u}$ uniquely from the equation (3.15) and the boundary condition

$$\mathbf{u}(t) = \mathbf{f}(t) := (f_0(t), f_1(t), f_2(t), \dots), \qquad t \in \partial\Omega, \qquad (3.16)$$

then in view of (3.13) the unknown function $a(z)$ will be defined by the formula

$$a(z) = 2\Re\big[\partial u_1(z)\big] + \mu(z)u_0(z).$$

The solution of the boundary problem (3.15), (3.16) was constructed in [22] for the class of (real) analytic functions μ and in [3] for C^2 functions μ and is given by

$$\mathbf{u}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} K_{\mathcal{B}}(t,z)\,(\mathrm{d}t + \mathcal{A}\mathrm{d}\bar{t})\mathbf{f}(t), \qquad z \in \Omega,$$

$$K_{\mathcal{B}}(t,z) = \big((t-z)I - \overline{(t-z)}\,\mathcal{A}\big)^{-1} e^{G(z)-G(t)}, \tag{3.17}$$

where the operator G is defined via the odd-rank Fourier coefficients of $[\mathcal{D}\mu]$

$$G(z) = 2\sum_{k=0}^{\infty} m_{-(2k+1)}(z)\mathcal{B}^{2k+1}.$$

Here we use coefficients m_k from the Fourier expansion (3.5) of $[\mathcal{D}\mu](z,\varphi) \equiv m(z,\varphi)$.

3.1.3 Attenuated vectorial case

Considering the attenuated vectorial X-ray transform

$$u(\mathbf{x},\varphi) := [\overrightarrow{\mathcal{D}}_\mu \mathbf{a}](\mathbf{x},\varphi) \equiv [\mathcal{D}_\mu(e^{\mathrm{i}\varphi}A + e^{-\mathrm{i}\varphi}\overline{A})](\mathbf{x},\varphi)$$

instead of $[\mathcal{D}_\mu a](\mathbf{x},\varphi)$ and applying the same scheme, we get two special equations and a system (3.20) for a generalized $\mathcal{A}$-analytic function $\mathbf{u}(z) = (u_1(z), u_2(z), u_3(z), \ldots)$

$$\overline{\partial} u_{-1} + \mu u_0 + \partial u_1 = 0, \tag{3.18}$$

$$\overline{\partial} u_0 + \mu u_1 + \partial u_2 = \overline{A(z)}, \tag{3.19}$$

$$\overline{\partial} u_k + \mu u_{k+1} + \partial u_{k+2} = 0, \qquad k = 1,2,\ldots. \tag{3.20}$$

Equations (3.18) and (3.19) lead us to the inversion formula. Firstly, we obtain

$$u_0(z) = -\frac{1}{\mu(z)}\, 2\Re[\partial u_1(z)]$$

from (3.18) and then substitute it into (3.19) and get the inversion formula

$$\overline{A(z)} = -\overline{\partial}\Big(\frac{1}{\mu(z)}\, 2\Re[\partial u_1(z)]\Big) + \mu(z)u_1(z) + \partial u_2(z).$$

The functions $u_1(z)$ and $u_2(z)$ are corresponding components of the $\mathcal{A}$-analytic vector $\mathbf{u}(z) = (u_1, u_2, u_3, \ldots)$, which is defined by the Cauchy-type integral

$$\mathbf{u}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} K_{\mathcal{B}}(t,z)\,(\mathrm{d}t + \mathcal{A}\,\mathrm{d}\bar{t})\mathbf{f}(t), \qquad z \in \Omega,$$

where $K_{\mathcal{B}}(t,z)$ is defined as previously by (3.17).

4. The X-ray transform and the angular Hilbert transform

The analysis of the X-ray transform (2.2) is important not only for determining the attenuation map μ from transmission data but also for understanding the attenuated X-ray transform (2.3). In this section we derive some formulae involving the *angular Hilbert transform* of the X-ray transform. The singular value decomposition (SVD) of the X-ray transform in a unit disc from our previous work [9, 23] will be used in proofs of these formulae.

As it is known [31], in the parallel-beam tomography the *Hilbert transform*

$$[Hh](s) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{s-t}\,\mathrm{d}t, \qquad s \in \mathbb{R}, \tag{4.1}$$

plays the basic role. In the fan-beam case we deal with the *angular Hilbert transform*

$$[\Gamma v](\beta) := \frac{1}{2\pi} \int_0^{2\pi} \operatorname{ctg} \frac{\psi - \beta}{2}\, v(\psi)\,\mathrm{d}\psi, \qquad \beta \in [0, 2\pi). \tag{4.2}$$

All integrals in (4.1) and (4.2) are understood in the Cauchy principal-value sense. We will treat the transform Γ as an operator $L_2([0,2\pi)) \to L_2([0,2\pi))$ and for the Fourier basis functions $\{e^{\pm \mathrm{i}n\psi}\}$ the following mappings take place

$$[\Gamma 1](\beta) = 0, \quad [\Gamma e^{\mathrm{i}n\cdot}](\beta) = \mathrm{i}e^{\mathrm{i}n\beta}, \quad [\Gamma e^{-\mathrm{i}n\cdot}](\beta) = -\mathrm{i}e^{-\mathrm{i}n\beta}, \quad n > 0. \tag{4.3}$$

Let us define the following operators

$$
\begin{aligned}
[\mathcal{D}^{(\mathrm{odd})}\mu](\mathbf{x},\varphi) &:= \frac{1}{2}\left([\mathcal{D}\mu](\mathbf{x},\varphi) - [\mathcal{D}\mu](\mathbf{x},\varphi+\pi)\right), \qquad \text{the ``odd'' part of } \mathcal{D},\\
[\mathcal{D}^{(\mathrm{even})}\mu](\mathbf{x},\varphi) &:= \frac{1}{2}\left([\mathcal{D}\mu](\mathbf{x},\varphi) + [\mathcal{D}\mu](\mathbf{x},\varphi+\pi)\right), \qquad \text{the ``even'' part of } \mathcal{D},\\
[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi) &:= \left[\frac{1}{2}(I \mp \mathrm{i}\Gamma)[\mathcal{D}^{(\mathrm{odd})}\mu](\mathbf{x},\cdot)\right](\varphi),
\end{aligned} \tag{4.4}
$$

where I is the identity operator.

If we expand the function $[\mathcal{D}\mu](\mathbf{x},\varphi)$ into the Fourier series with respect to the angular variable φ

$$[\mathcal{D}\mu](\mathbf{x},\varphi) = \sum_{k=-\infty}^{\infty} m_k(\mathbf{x}) e^{-k\mathrm{i}\varphi},$$

then from the definitions we get

$$
\begin{aligned}
[\mathcal{D}^{(\mathrm{odd})}\mu](\mathbf{x},\varphi) &= \sum_{k=-\infty}^{\infty} m_{2k+1}(\mathbf{x}) e^{-(2k+1)\mathrm{i}\varphi},\\
[\mathcal{D}^{(\mathrm{even})}\mu](\mathbf{x},\varphi) &= \sum_{k=-\infty}^{\infty} m_{2k}(\mathbf{x}) e^{-2k\mathrm{i}\varphi},\\
[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi) &= \sum_{k=0}^{\infty} m_{\mp(2k+1)}(\mathbf{x}) e^{\pm(2k+1)\mathrm{i}\varphi}.
\end{aligned} \tag{4.5}
$$

We've used (4.4) and (4.3) for obtaining (4.5). The following evident properties take place for these operators

$$
\begin{aligned}
[\mathcal{D}\mu](\mathbf{x},\varphi) &= [\mathcal{D}^{(\text{even})}\mu](\mathbf{x},\varphi) + [\mathcal{D}^{(\text{odd})}\mu](\mathbf{x},\varphi), \\
[\mathcal{D}^{(\text{odd})}\mu](\mathbf{x},\varphi) &= [\mathcal{D}^{(+)}\mu](\mathbf{x},\varphi) + [\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
[\mathcal{D}^{(\text{even})}\mu](\mathbf{x},\varphi) &= \frac{1}{2}\,[\mathcal{D}\mu](\gamma(\mathbf{x},\varphi+\pi),\varphi), \\
[\mathcal{D}^{(\text{odd})}\mu](\mathbf{x},\varphi) &= [\mathcal{D}\mu](\mathbf{x},\varphi) - \frac{1}{2}\,[\mathcal{D}\mu](\gamma(\mathbf{x},\varphi+\pi),\varphi),
\end{aligned}
\tag{4.7}
$$

$$
[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi) = \overline{[\mathcal{D}^{(+)}\mu](\mathbf{x},\varphi)} \qquad \text{(for real-valued } \mu). \tag{4.8}
$$

Other important properties that will be used later are formulated in the following lemmata.

Lemma 4.1. *Let $\mu(\mathbf{x}) \in L_2(\boldsymbol{\Omega})$, then*

$$
[\Gamma[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\,\cdot\,)](\varphi) = \pm\mathbf{i}[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi), \tag{4.9}
$$

$$
[\Gamma[\mathcal{D}^{(\pm)}\mu](\,\cdot\,,\varphi)](\beta) = \mp\mathbf{i}[\mathcal{D}^{(\pm)}\mu](\beta,\varphi). \tag{4.10}
$$

Here Γ is the angular Hilbert transform (4.2).

Proof. Formula (4.9) follows directly from (4.5) and (4.3). Formula (4.10) follows from the singular value decomposition (SVD) of the operator $\mathcal{D}^{(\text{odd})}$ being considered as an operator from $L_2(\Omega)$ to $L_2([0,2\pi)\times[0,2\pi))$, see [9, 23]. If we expand the function $\mu(\mathbf{x})$ into the series of Zernike polynomials $Z^{n,k}$

$$
\mu(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{n,k} Z^{n,k}(\mathbf{x}), \tag{4.11}
$$

then the SVD yields the representation

$$
\begin{aligned}
&[\mathcal{D}^{(\text{odd})}\mu](\beta,\varphi) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_{n,k}}{2(n+1)}\left(e^{-\mathrm{i}(2k+1)\varphi}e^{\mathrm{i}(n+1)\beta} + (-1)^n e^{\mathrm{i}(2(n-k)+1)\varphi}e^{-\mathrm{i}(n+1)\beta}\right).
\end{aligned}
\tag{4.12}
$$

As soon as the function $\mu(\mathbf{x})$ is real-valued, the coefficients $c_{n,k}$ satisfy the relation $\overline{c_{n,k}} = (-1)^n c_{n,n-k}$ due to the property of Zernike polynomials $\overline{Z^{n,k}(\mathbf{x})} = (-1)^n Z^{n,n-k}(\mathbf{x})$. Using (4.5) with (4.12) we obtain

$$
[\mathcal{D}^{(+)}\mu](\beta,\varphi) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_{n,k}}{2(n+1)}(-1)^n\, e^{\mathrm{i}(2(n-k)+1)\varphi}\, e^{-\mathrm{i}(n+1)\beta}, \tag{4.13}
$$

$$
[\mathcal{D}^{(-)}\mu](\beta,\varphi) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_{n,k}}{2(n+1)}\, e^{-\mathrm{i}(2k+1)\varphi}\, e^{\mathrm{i}(n+1)\beta}. \tag{4.14}
$$

Then (4.10) follows from properties (4.3) and representations (4.13), (4.14). □

Remark 4.2. Combining (4.9) applied to $\mathbf{x} = (\cos\beta, \sin\beta)$ and (4.10), we get

$$\big[\Gamma[\mathcal{D}^{(\pm)}\mu](\beta,\,\cdot\,)\big](\varphi) = -\big[\Gamma[\mathcal{D}^{(\pm)}\mu](\,\cdot\,,\varphi)\big](\beta). \tag{4.15}$$

If we know decompositions (4.11) or (4.12) then we can analytically evaluate the angular Hilbert transforms of (4.12), (4.13) and (4.14) using formulae (4.3).

Lemma 4.3. *Let* $\mu(\mathbf{x}) \in L_2(\Omega)$*, then the functions* $[\mathcal{D}^{(odd)}\mu]$*,* $[\mathcal{D}^{(\pm)}\mu]$*, and* $e^{[\mathcal{D}^{(\pm)}\mu]}$ *represent the solutions of the transport equations*

$$\partial_{\boldsymbol{\theta}}\big([\mathcal{D}^{(\mathrm{odd})}\mu]\big)(\mathbf{x},\varphi) = \mu(\mathbf{x}), \tag{4.16}$$

$$\partial_{\boldsymbol{\theta}}\big([\mathcal{D}^{(\pm)}\mu]\big)(\mathbf{x},\varphi) = \frac{1}{2}\,\mu(\mathbf{x}), \tag{4.17}$$

$$\partial_{\boldsymbol{\theta}}\big(e^{2[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi)}\big)(\mathbf{x},\varphi) = \mu(\mathbf{x})\,e^{2[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi)}, \qquad \mu(\mathbf{x}) \in C^2(\Omega), \tag{4.18}$$

where $\boldsymbol{\theta} = (\cos\varphi, \sin\varphi)$*. Also we have*

$$\int_{\mathbf{x}'}^{\mathbf{x}} \mu(\mathbf{x}'')\,|d\mathbf{x}''| = 2[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi) - 2[\mathcal{D}^{(\pm)}\mu](\mathbf{x}',\varphi), \qquad \mathbf{x}',\mathbf{x} \in \overline{\Omega}, \tag{4.19}$$

$$[\mathcal{D}\mu](\mathbf{x},\varphi) = 2[\mathcal{D}^{(\pm)}\mu](\mathbf{x},\varphi) - 2[\mathcal{D}^{(\pm)}\mu](\gamma(\mathbf{x},\varphi),\varphi). \tag{4.20}$$

Proof. Formula (4.16) is obtained from definitions of $\partial_{\boldsymbol{\theta}}$ and $[\mathcal{D}^{(\mathrm{odd})}\mu]$ and the fact that $\partial_{\boldsymbol{\theta}}([\mathcal{D}\mu]) = \mu$. Let's prove (4.17) for $[\mathcal{D}^{(-)}\mu]$ first. Applying $\partial_{\boldsymbol{\theta}}$ to the representation of $[\mathcal{D}^{(-)}\mu]$ (4.5), we get

$$\begin{aligned}\partial_{\boldsymbol{\theta}}\big([\mathcal{D}^{(-)}\mu]\big) &= (e^{i\varphi}\partial + e^{-i\varphi}\overline{\partial})(m_1 e^{-1i\varphi} + m_3 e^{-3i\varphi} + m_5 e^{-5i\varphi} + \dots)\\ &= \partial m_1 + (\overline{\partial} m_1 + \partial m_3)e^{-2i\varphi} + (\overline{\partial} m_3 + \partial m_5)e^{-4i\varphi} + \dots = \frac{1}{2}\,\mu.\end{aligned} \tag{4.21}$$

Here $\partial m_1 = \frac{1}{2}\mu$ due to (3.11) and expressions of the form $(\overline{\partial} m_k + \partial m_{k+2})$ are all equal to zero according to (3.7). By analogy, property (4.17) is proved for $[\mathcal{D}^{(+)}\mu]$. Then (4.18) is obtained from (4.17) by applying the chain rule and (4.19) is obtained by integrating (4.17). At last, substituting $\mathbf{x}' = \gamma(\mathbf{x},\varphi)$ into (4.19) we get (4.20). □

5. Inversion formula for the attenuated X-ray transform

In this section we derive an explicit inversion formula for the attenuated X-ray transform in a unit disc without using the theory of $\mathcal{A}$-analytic functions.

The essence of this approach (developed in our previous work [10]) can be described in a couple of sentences. Firstly, we perform the change of variables and reduce the inverse problem for the attenuated X-ray transform to the unattenuated case. We consider it on the language of the transport equation. Then we take the $\mathcal{D}^{(-)}$ part of the X-ray transform $\mathcal{D}$ and get the special form (containing only negative harmonics) of the solution to the same transport equation without attenuation. At last, we perform the inverse change of variables and come to the original attenuated problem, but with the special form of the solution to the corresponding transport equation (again, containing only negative harmonics). This special form then yields a simple inversion formula for the attenuated X-ray transform.

This derivation relies on the results obtained in the previous section.

Theorem 5.1. *Let the real-valued functions* $a(\mathbf{x}) \in L_2(\Omega)$ *and* $\mu(\mathbf{x}) \in C^2(\overline{\Omega})$ *be respectively the emission map and the attenuation map for the attenuated X-ray transform* (2.4) *and let*

$$f(\beta,\varphi) := [\mathcal{D}_\mu a](\beta,\varphi)$$

be a known function. Then the following inversion formula takes place

$$a(\mathbf{x}) = \frac{\partial}{\partial z}\Big(\frac{\mathrm{i}}{\pi}\int_0^{2\pi} e^{\mathrm{i}\varphi}\Im\Big[v^*(\gamma(\mathbf{x},\varphi),\varphi)\, e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}\Big]\,\mathrm{d}\varphi\Big), \tag{5.1}$$

where

$$v^*(\beta,\varphi) = \Big[(I - \mathrm{i}\Gamma)\Big(v(\,\cdot\,,\varphi) - \frac{1}{2}\, v(\gamma(\,\cdot\,,\varphi+\pi),\varphi)\Big)\Big](\beta), \tag{5.2}$$

$$v(\beta,\varphi) = e^{2[\mathcal{D}^{(-)}\mu](\beta,\varphi)} f(\beta,\varphi), \tag{5.3}$$

and Γ *is the angular Hilbert transform* (4.2)*. I denotes the identity operator.*

Proof. Let's denote $u(\mathbf{x},\varphi) := [\mathcal{D}_\mu a(\,\cdot\,)](\mathbf{x},\varphi)$. Substituting (4.19) into the definition of $\mathcal{D}_\mu$ (2.3) we get

$$\begin{aligned} u(\mathbf{x},\varphi) &= \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} a(\mathbf{x}')\, e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)+2[\mathcal{D}^{(-)}\mu](\mathbf{x}',\varphi)}\, |\mathrm{d}\mathbf{x}'| \\ &= e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} a(\mathbf{x}')\, e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x}',\varphi)}\, |\mathrm{d}\mathbf{x}'|. \end{aligned}$$

Thus, introducing the change of variables

$$v(\mathbf{x},\varphi) := e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} u(\mathbf{x},\varphi), \tag{5.4}$$

$$b(\mathbf{x},\varphi) := e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} a(\mathbf{x}) \tag{5.5}$$

we obtain

$$v(\mathbf{x},\varphi) = \int_{\gamma(\mathbf{x},\varphi)}^{\mathbf{x}} b(\mathbf{x}',\varphi)\, |\mathrm{d}\mathbf{x}'| = [\mathcal{D}b(\,\cdot\,,\varphi)](\mathbf{x},\varphi), \tag{5.6}$$

and consequently

$$\partial_{\boldsymbol{\theta}} v(\mathbf{x},\varphi) = b(\mathbf{x},\varphi). \tag{5.7}$$

Note, that substituting the representation of $\mathcal{D}^{(-)}$ (4.5) into the Taylor series for the exponent $e^x = \sum_{k=0}^{\infty}(x^k/k!)$ we get a series expansion for $e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}$, which contains only non-positive harmonics

$$e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} = \sum_{k=0}^{\infty} M_k(\mathbf{x}) e^{-\mathrm{i}k\varphi}. \tag{5.8}$$

The series expansion for $e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}$ also contains only non-positive harmonics. Substituting (5.8) into (5.5) and then into (5.6), we get the following expansions

$$b(\mathbf{x},\varphi) = \sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi} b_k(\mathbf{x}), \qquad v(\mathbf{x},\varphi) = \sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi} [\mathcal{D}b_k](\mathbf{x},\varphi). \tag{5.9}$$

Using (4.20) and denoting

$$v^*(\mathbf{x},\varphi) := 2\sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi} [\mathcal{D}^{(-)}b_k](\mathbf{x},\varphi), \tag{5.10}$$

we rewrite (5.9) as

$$v(\mathbf{x},\varphi) = v^*(\mathbf{x},\varphi) - v^*(\gamma(\mathbf{x},\varphi),\varphi). \tag{5.11}$$

From (5.10) and (4.5) it follows that the Fourier series decomposition of $v^*(\mathbf{x},\varphi)$ contains only negative harmonics

$$v^*(\mathbf{x},\varphi) = \sum_{k=1}^{\infty} v_k^*(\mathbf{x}) e^{-\mathrm{i}k\varphi}. \tag{5.12}$$

Since $v^*(\gamma(\mathbf{x},\varphi),\varphi)$ is fully determined by its boundary values, it satisfies the equation $\partial_{\boldsymbol{\theta}} v^*(\gamma(\mathbf{x},\varphi),\varphi) = 0$. Thus, from (5.7) and (5.11) we get

$$\partial_{\boldsymbol{\theta}} v^*(\mathbf{x},\varphi) = b(\mathbf{x},\varphi). \tag{5.13}$$

Applying the inverse change of variables

$$u^*(\mathbf{x},\varphi) := e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} v^*(\mathbf{x},\varphi), \tag{5.14}$$

$$a(\mathbf{x}) = e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} b(\mathbf{x},\varphi)$$

we come to the new function $u^*(\mathbf{x},\varphi)$, which by virtue of (5.13) and (4.18) satisfies the equation

$$\partial_{\boldsymbol{\theta}} u^*(\mathbf{x},\varphi) + \mu u^*(\mathbf{x},\varphi) = a(\mathbf{x}) \tag{5.15}$$

and has a Fourier series expansion

$$u^*(\mathbf{x},\varphi) = \sum_{k=1}^{\infty} u_k^*(\mathbf{x}) e^{-\mathrm{i}k\varphi}. \tag{5.16}$$

The latter follows from (5.14), (5.12) and (5.8).

Substituting the expansion (5.16) into the equation (5.15) and matching terms with the same exponents, we come to the system of equations

$$\partial u_1^* = a, \tag{5.17}$$

$$\mu u_1^* + \partial u_2^* = 0, \tag{5.18}$$

$$\overline{\partial} u_k^* + \mu u_{k+1}^* + \partial u_{k+2}^* = 0, \qquad k = 1, 2, \dots. \tag{5.19}$$

The equation (5.17) gives the inversion formula for obtaining the unknown $a(\mathbf{x})$ from the first Fourier coefficient of u^*

$$u_1^*(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}\varphi} u^*(\mathbf{x},\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}\varphi}\big(u^*(\mathbf{x},\varphi) - \overline{u^*(\mathbf{x},\varphi)}\big)\,\mathrm{d}\varphi.$$

Here we've used the fact that $\overline{u^*(\mathbf{x},\varphi)}$ does not contain negative harmonics due to (5.16), thus $\int_0^{2\pi} e^{\mathrm{i}\varphi}\overline{u^*(\mathbf{x},\varphi)}\,\mathrm{d}\varphi = 0$. So, for obtaining $a(\mathbf{x})$ we get the inversion formula

$$a(\mathbf{x}) = \partial\Big(\frac{\mathrm{i}}{\pi}\int_0^{2\pi} e^{\mathrm{i}\varphi}\Im\big[u^*(\mathbf{x},\varphi)\big]\,\mathrm{d}\varphi\Big). \tag{5.20}$$

The last step is to express $\Im[u^*(\mathbf{x},\varphi)]$ from the sinogram $f(\beta,\varphi) \equiv u(\beta,\varphi)$ and transmission measurements $[\mathcal{D}\mu(\,\cdot\,)](\beta,\varphi)$. According to (5.11), (5.14) and (5.4), we have

$$v^*(\gamma(\mathbf{x},\varphi),\varphi) = v^*(\mathbf{x},\varphi) - v(\mathbf{x},\varphi) = \big(u^*(\mathbf{x},\varphi) - u(\mathbf{x},\varphi)\big)\,e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}.$$

Since the function $u(\mathbf{x},\varphi)$ is real-valued, we can eliminate it by taking imaginary part

$$\Im\big[u^*(\mathbf{x},\varphi)\big] = \Im\big[v^*(\gamma(\mathbf{x},\varphi),\varphi)\,e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}\big]. \tag{5.21}$$

Substituting (5.21) into (5.20), we get the required inversion formula (5.1).

The boundary values of v^* in (5.21) can be expressed in terms of the boundary values of v. Firstly, we apply the definition (4.4) of $\mathcal{D}^{(-)}$ to (5.10)

$$v^*(\beta,\varphi) = \sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi}\big[(I + \mathrm{i}\Gamma)[\mathcal{D}^{(\mathrm{odd})}b_k](\beta,\,\cdot\,)\big](\varphi).$$

Then, using properties (4.6) and (4.15), we change the action of Γ from φ to β

$$v^*(\beta,\varphi) = \sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi}\big[(I - \mathrm{i}\Gamma)[\mathcal{D}^{(\mathrm{odd})}b_k](\,\cdot\,,\varphi)\big](\beta),$$

which allows us to factor it out of the sum

$$v^*(\beta,\varphi) = \Big[(I - \mathrm{i}\Gamma)\Big(\sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi}[\mathcal{D}^{(\mathrm{odd})}b_k](\,\cdot\,,\varphi)\Big)\Big](\beta).$$

Applying (4.7) we finally get

$$v^*(\beta,\varphi) = \Big[(I - \mathrm{i}\Gamma)\Big(\sum_{k=0}^{\infty} e^{-\mathrm{i}k\varphi}\Big([\mathcal{D}b_k](\,\cdot\,,\varphi) - \frac{1}{2}\,[\mathcal{D}b_k](\gamma(\,\cdot\,,\varphi+\pi),\varphi)\Big)\Big)\Big](\beta),$$

which yields (5.2) due to (5.9). The values of $v(\beta,\varphi)$ in (5.2) are evaluated by (5.3) due to (5.4) and the boundary condition $f(\beta,\varphi) \equiv u(\beta,\varphi)$. Theorem 5.1 is proved. □

Remark 5.2. One can derive **infinitely many different inversion formulae** from the system of equations (5.17)–(5.19), which are not necessarily better for numerical computations though. For instance, we can first express $u_1^*(\mathbf{x})$ from (5.18)

$$u_1^*(\mathbf{x}) = -\frac{1}{\mu(\mathbf{x})}\,\partial u_2^*(\mathbf{x})$$

and then substitute it into (5.17) thus getting an inversion formula

$$a(\mathbf{x}) = \partial u_1^*(\mathbf{x}) = -\partial\Big(\frac{1}{\mu(\mathbf{x})}\,\partial u_2^*(\mathbf{x})\Big) = -\partial\Big(\frac{1}{\mu(\mathbf{x})}\,\partial\Big(\frac{\mathrm{i}}{\pi}\int_0^{2\pi} e^{2\mathrm{i}\varphi}\Im\big[u^*(\mathbf{x},\varphi)\big]\,\mathrm{d}\varphi\Big)\Big).$$

6. Inversion formula for the attenuated vectorial X-ray transform

In this section we derive an explicit inversion formula (first obtained in [10]) for the attenuated vectorial X-ray transform in a unit disc using the same approach as in the previous section. As a matter of fact, the derivation will closely follow the proof of Theorem 5.1. Another exact inversion formulas for the attenuated vectorial Radon transform was derived by G. Bal [4] and by F. Natterer [33]. In [6] J. Boman investigated the injectivity for a weighted vectorial Radon transform.

Recall, that in the unattenuated case (when $\mu(\mathbf{x}) \equiv 0$) we can restore only the solenoidal part of the vector field. In the next theorem we derive the inversion formula for the full reconstruction of the vector field from its attenuated vectorial X-ray transform.

Theorem 6.1. *Let the real-valued vector functions* $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x})) \in \mathbf{L}_2(\Omega)$ *and* $\mu(\mathbf{x}) \in C^2(\overline{\Omega})$ *be the vectorial "emission map" and the scalar attenuation map for the attenuated vectorial X-ray transform* (2.6) *respectively. The SPECT data*

$$f(\beta,\varphi) := [\overrightarrow{\mathcal{D}}_\mu \mathbf{a}](\beta,\varphi)$$

is known. Then the following inversion formula takes place for points $\mathbf{x}$ *such that* $\mu(\mathbf{x}) \neq 0$

$$\begin{aligned} A(\mathbf{x}) &= \frac{a_1(\mathbf{x}) - \mathrm{i}a_2(\mathbf{x})}{2} \\ &= -\partial\Big(\frac{1}{\mu(\mathbf{x})}\,\partial\Big(\frac{\mathrm{i}}{\pi}\int_0^{2\pi} e^{\mathrm{i}\varphi}\Im\big[v^*(\gamma(\mathbf{x},\varphi),\varphi)\,e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}\big]\,\mathrm{d}\varphi\Big)\Big), \end{aligned} \tag{6.1}$$

where

$$v^*(\beta,\varphi) = \big[(I - \mathrm{i}\Gamma)\big(v(\,\cdot\,,\varphi) - \frac{1}{2}\,v(\gamma(\,\cdot\,,\varphi+\pi),\varphi)\big)\big](\beta), \tag{6.2}$$

$$v(\beta,\varphi) = e^{2[\mathcal{D}^{(-)}\mu](\beta,\varphi)} f(\beta,\varphi), \tag{6.3}$$

and Γ *is the angular Hilbert transform* (4.2). I *denotes the identity operator.*

Proof. By analogy with Theorem 5.1, let's denote

$$u(\mathbf{x},\varphi) := [\overrightarrow{\mathcal{D}}_\mu \mathbf{a}(\,\cdot\,)](\mathbf{x},\varphi) \equiv \big[\mathcal{D}_\mu(e^{\mathrm{i}\varphi}A(\,\cdot\,) + e^{-\mathrm{i}\varphi}\overline{A(\,\cdot\,)}\,)\big](\mathbf{x},\varphi).$$

After the change of variables

$$v(\mathbf{x},\varphi) := e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} u(\mathbf{x},\varphi), \tag{6.4}$$

$$b(\mathbf{x},\varphi) := e^{2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} (e^{\mathrm{i}\varphi}A(\mathbf{x}) + e^{-\mathrm{i}\varphi}\overline{A(\mathbf{x})}) \tag{6.5}$$

we obtain that $v(\mathbf{x},\varphi)$ satisfies

$$v(\mathbf{x},\varphi) = [\mathcal{D}b(\,\cdot\,,\varphi)](\mathbf{x},\varphi), \tag{6.6}$$

and consequently

$$\partial_{\boldsymbol{\theta}} v(\mathbf{x},\varphi) = b(\mathbf{x},\varphi). \tag{6.7}$$

Substituting (5.8) into (6.5) and then into (6.6), we get the following expansions

$$b(\mathbf{x},\varphi) = \sum_{k=-1}^{\infty} e^{-\mathrm{i}k\varphi} b_k(\mathbf{x}), \qquad v(\mathbf{x},\varphi) = \sum_{k=-1}^{\infty} e^{-\mathrm{i}k\varphi}[\mathcal{D}b_k](\mathbf{x},\varphi).$$

Then, following the same arguments as in Theorem 5.1, the function

$$v^*(\mathbf{x},\varphi) := 2\sum_{k=-1}^{\infty} e^{-\mathrm{i}k\varphi}[\mathcal{D}^{(-)}b_k](\mathbf{x},\varphi), \tag{6.8}$$

also satisfies

$$\partial_{\boldsymbol{\theta}} v^*(\mathbf{x},\varphi) = b(\mathbf{x},\varphi) \tag{6.9}$$

and, due to (6.8) and (4.5), has a Fourier series expansion that contains only non-positive harmonics

$$v^*(\mathbf{x},\varphi) = \sum_{k=0}^{\infty} v_k^*(\mathbf{x})\, e^{-\mathrm{i}k\varphi}. \tag{6.10}$$

Then, applying the inverse change of variables

$$u^*(\mathbf{x},\varphi) := e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} v^*(\mathbf{x},\varphi), \tag{6.11}$$

$$e^{\mathrm{i}\varphi}A(\mathbf{x}) + e^{-\mathrm{i}\varphi}\overline{A(\mathbf{x})} = e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)} b(\mathbf{x},\varphi)$$

we come to the new function $u^*(\mathbf{x},\varphi)$, which by virtue of (6.9) and (4.18) satisfies the equation

$$\partial_{\boldsymbol{\theta}} u^*(\mathbf{x},\varphi) + \mu u^*(\mathbf{x},\varphi) = e^{\mathrm{i}\varphi} A(\mathbf{x}) + e^{-\mathrm{i}\varphi}\overline{A(\mathbf{x})} \tag{6.12}$$

and has a Fourier series expansion with only non-positive harmonics

$$u^*(\mathbf{x},\varphi) = \sum_{k=0}^{\infty} u_k^*(\mathbf{x})\, e^{-\mathrm{i}k\varphi}. \tag{6.13}$$

The latter follows from (6.11), (6.10) and (5.8).

Substituting the expansion (6.13) into the equation (6.12) and matching terms with the same exponents, we come to the system of equations

$$\partial u_0^* = A, \tag{6.14}$$

$$\mu u_0^* + \partial u_1^* = 0, \tag{6.15}$$

$$\overline{\partial} u_0^* + \mu u_1^* + \partial u_2^* = \overline{A}, \tag{6.16}$$

$$\overline{\partial} u_k^* + \mu u_{k+1}^* + \partial u_{k+2}^* = 0, \qquad k = 1, 2, \ldots. \tag{6.17}$$

Unfortunately we can't use the first equation (6.14) of this system for determining $A(\mathbf{x})$ immediately, because it would require the knowledge of $u_0^*(\mathbf{x})$, whereas we can only find $\Im[u_0^*(\mathbf{x})]$. By analogy with Theorem 5.1, the following relation takes place

$$\Im\big[u^*(\mathbf{x},\varphi)\big] = \Im\big[v^*(\gamma(\mathbf{x},\varphi),\varphi)\, e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)}\big].$$

So, in order to get an inversion formula, we first express $u_0^*(\mathbf{x})$ from (6.15)

$$u_0^* = -\frac{1}{\mu}\,\partial u_1^*$$

and then substitute it into (6.14)

$$A(\mathbf{x}) = -\partial\Big(\frac{1}{\mu(\mathbf{x})}\,\partial u_1^*(\mathbf{x})\Big).$$

This gives us an inversion formula (6.1). Theorem 6.1 is proved. □

7. Reconstruction algorithm

We have developed an application in C++ for solving the two-dimensional SPECT problem in scalar and vector cases. We reconstruct a scalar emission map $a(\mathbf{x})$ (or a vectorial "emission map" $\mathbf{a}(\mathbf{x})$) while the attenuation map $\mu(\mathbf{x})$ is assumed to be known. In practice it can be determined from additional transmission measurements, i.e. from the X-ray transform $\mathcal{D}\mu$.

Using our previous works [9, 23] we represent the attenuation map μ as a polynomial of degree N using the series of Zernike polynomials

$$\mu(\mathbf{x}) = \sum_{n=0}^{N} \sum_{k=0}^{n} c_{n,k} Z^{n,k}(\mathbf{x}), \qquad c_{n,k} = a_{n,k} + \mathrm{i} b_{n,k}. \tag{7.1}$$

Note, that this is a truncated version of the representation (4.11). Hereinafter, we shall need to evaluate functions $[\mathcal{D}^{(\pm)}\mu](\beta,\varphi)$. Due to property (4.8) we can denote

$$2[\mathcal{D}^{(-)}\mu](\beta,\varphi) := U(\beta,\varphi) + \mathrm{i}V(\beta,\varphi), \tag{7.2}$$

$$2[\mathcal{D}^{(+)}\mu](\beta,\varphi) := U(\beta,\varphi) - \mathrm{i}V(\beta,\varphi). \tag{7.3}$$

The functions $U(\beta,\varphi)$ and $V(\beta,\varphi)$ can be computed from the corresponding truncated versions of representations (4.13–4.14) by formulae

$$\begin{aligned} U(\beta,\varphi) = 2\sum_{n=0}^{N} \frac{1}{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor *} a_{n,k} \times &\left\{ \begin{array}{l} \cos\left[(n+1)(\beta-\varphi)\right] \cos\left[(n-2k)\varphi\right] \\ -\sin\left[(n+1)(\beta-\varphi)\right] \sin\left[(n-2k)\varphi\right] \end{array} \right. \\ - b_{n,k} \times &\left\{ \begin{array}{l} \cos\left[(n+1)(\beta-\varphi)\right] \sin\left[(n-2k)\varphi\right] \\ \sin\left[(n+1)(\beta-\varphi)\right] \cos\left[(n-2k)\varphi\right] \end{array} \right. , \end{aligned} \tag{7.4}$$

$$\begin{aligned} V(\beta,\varphi) = 2\sum_{n=0}^{N} \frac{1}{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor *} a_{n,k} \times &\left\{ \begin{array}{l} \sin\left[(n+1)(\beta-\varphi)\right] \cos\left[(n-2k)\varphi\right] \\ \cos\left[(n+1)(\beta-\varphi)\right] \sin\left[(n-2k)\varphi\right] \end{array} \right. \\ - b_{n,k} \times &\left\{ \begin{array}{l} -\sin\left[(n+1)(\beta-\varphi)\right] \sin\left[(n-2k)\varphi\right] \\ \cos\left[(n+1)(\beta-\varphi)\right] \cos\left[(n-2k)\varphi\right] \end{array} \right. , \end{aligned} \tag{7.5}$$

where the top line should be used in the case of even n, and bottom line — in the case of odd n, and the sign $*$ near by the second sum denotes that in the case of even n, coefficient $a_{n,\lfloor n/2 \rfloor}$ should be divided by 2 and $b_{n,\lfloor n/2 \rfloor}$ should be set to 0. Computation of functions U and V on a regular 2D grid can be performed in $\mathcal{O}(N^2 \log_2 N)$ operations.

An implementation of the inversion formulae (5.1), (6.1) consists of the following steps.

- Computation of the modified sinogram $v(\beta,\varphi)$ by the change of variables (5.3), (6.3).
- Computation of its $\mathcal{D}^{(-)}$ part $v^*(\beta,\varphi)$ by formulae (5.2), (6.2), which involve the angular Hilbert transform.
- Performing the inverse change of variables in the integrands (5.1), (6.1), and subsequent numerical integration and differentiation.

Computation of the modified sinogram. At the first step, we transform a real-valued sinogram $f(\beta,\varphi)$ into the complex-valued modified sinogram $v(\beta,\varphi)$ using (7.2) and previously determined functions $U(\beta,\varphi)$ and $V(\beta,\varphi)$. This corresponds to formulae (5.3) and (6.3).

$$v(\beta,\varphi) = e^{2[\mathcal{D}^{(-)}\mu](\beta,\varphi)} f(\beta,\varphi) = e^{U(\beta,\varphi)}\big(\cos(V(\beta,\varphi)) + \mathrm{i}\sin(V(\beta,\varphi))\big) f(\beta,\varphi).$$

Evaluation of the angular Hilbert transform. Evaluation of the function $v^*(\beta,\varphi)$ in (5.2) and (6.2) employs the angular Hilbert transform (4.2)

$$[\Gamma v](\beta) := \frac{1}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{\psi-\beta}{2}\, v(\psi)\,\mathrm{d}\psi, \qquad \beta\in[0,2\pi). \tag{7.6}$$

We use the quadrature formula from [27] for evaluation of this singular integral. It is based on the fact that a periodic function $v(\psi)$ can be approximated by the trigonometric polynomial of degree n

$$v_n(\psi) = \sum_{k=0}^{2n} \frac{v(\psi_k)}{2n+1}\,\frac{\sin\big[(2n+1)(\psi-\psi_k)/2\big]}{\sin\big[(\psi-\psi_k)/2\big]},$$

which satisfies the conditions

$$v_n(\psi_k) = v(\psi_k), \qquad \psi_k = \frac{2k}{2n+1}\,\pi, \qquad k = 0,1,\ldots,2n.$$

In order to verify this fact it is sufficient to notice that $\sin[(2n+1)\alpha/2] = 0$ when $\alpha = 2k\pi/(2n+1)$, $k = 0,1,\ldots,2n$ and

$$\lim_{\alpha\to 0}\frac{\sin(2n+1)\alpha/2}{\sin(\alpha/2)} = 2n+1.$$

Substituting $v_n(\psi)$ instead of $v(\psi)$ into (7.6) and making use of trigonometric identities

$$\frac{\sin\big[(2n+1)\psi/2\big]}{\sin\big[\psi/2\big]} \equiv 1 + 2(\cos\psi + \ldots + \cos n\psi),$$

$$\operatorname{ctg}\frac{\psi}{2} - \frac{\cos\big[(2n+1)\psi/2\big]}{\sin\big[\psi/2\big]} \equiv 2\big(\sin\psi + \ldots + \sin n\psi\big)$$

and the relation (4.3) in the form

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\psi-\beta}{2}\cos(k\psi)\,\mathrm{d}\psi = -\sin(k\beta), \qquad k = 0,1,\ldots,$$

one gets the quadrature formula

$$\begin{aligned}[Sv](\beta) &:= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\psi-\beta}{2}\, v_n(\psi)\,\mathrm{d}\psi \\ &= \sum_{k=0}^{2n}\frac{v(\psi_k)}{2n+1}\left[\operatorname{ctg}\frac{\psi_k-\beta}{2} - \frac{\cos\big[(2n+1)(\psi_k-\beta)/2\big]}{\sin\big[(\psi_k-\beta)/2\big]}\right].\end{aligned} \tag{7.7}$$

This formula is exact for every trigonometric polynomial of degree n, as in this case $v_n(\psi) \equiv v(\psi)$ and formula (7.7) gives the exact value of the integral (7.6). When this formula is considered at the points

$$\beta_m = \psi_m + \frac{\pi}{2n+1}, \qquad m = 0, 1, \ldots, 2n,$$

then (7.7) simplifies into

$$[Sv](\beta_m) = \sum_{k=0}^{2n} \frac{v(\psi_k)}{2n+1} \operatorname{ctg} \frac{\psi_k - \beta_m}{2}, \tag{7.8}$$

as in this case $\cos\big[(2n+1)(\psi_k - \beta_m)/2\big] = 0$ and $\sin\big[(\psi_k - \beta_m)/2\big] \neq 0$ for all $k, m = 0, 1, \ldots, 2n$.

For implementation of computations by this formula it is significant that the circle should be partitioned into the odd number $(2n+1)$ of angles and the values of $v(\psi_k)$ in (7.8) should be taken right in the middle between the knots β_m where the function $[Sv]$ (which is an approximation of $[\Gamma v]$) is evaluated.

Evaluation of the integrand. In the integrals (5.1), (6.1) we encounter an expression of the form

$$v^*(\gamma(\mathbf{x}, \varphi), \varphi)\, e^{-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)},$$

where $\gamma(\mathbf{x}, \varphi)$ is given by the formula (2.1). Recall, that from (7.2), (7.3) we have

$$[\mathcal{D}^{(-)}\mu](\beta, \varphi) + [\mathcal{D}^{(+)}\mu](\beta, \varphi) = U(\beta, \varphi), \tag{7.9}$$

$$[\mathcal{D}^{(-)}\mu](\beta, \varphi) - [\mathcal{D}^{(+)}\mu](\beta, \varphi) = \mathrm{i}V(\beta, \varphi). \tag{7.10}$$

In order to evaluate $-2[\mathcal{D}^{(-)}\mu](\mathbf{x}, \varphi)$ at the *interior point* $\mathbf{x}$ of a disc, we make use of the Lemma 4.3. The formula (4.20) yields

$$[\mathcal{D}\mu](\mathbf{x}, \varphi) = 2[\mathcal{D}^{(+)}\mu](\mathbf{x}, \varphi) - 2[\mathcal{D}^{(+)}\mu](\gamma(\mathbf{x}, \varphi), \varphi), \tag{7.11}$$

$$[\mathcal{D}\mu](\mathbf{x}, \varphi) = 2[\mathcal{D}^{(-)}\mu](\mathbf{x}, \varphi) - 2[\mathcal{D}^{(-)}\mu](\gamma(\mathbf{x}, \varphi), \varphi). \tag{7.12}$$

Subtracting (7.12) from (7.11), we get

$$[\mathcal{D}^{(+)}\mu](\mathbf{x}, \varphi) - [\mathcal{D}^{(-)}\mu](\mathbf{x}, \varphi) = [\mathcal{D}^{(+)}\mu](\gamma(\mathbf{x}, \varphi), \varphi) - [\mathcal{D}^{(-)}\mu](\gamma(\mathbf{x}, \varphi), \varphi)$$

or, using (7.10), it can be rewritten as

$$[\mathcal{D}^{(+)}\mu](\mathbf{x}, \varphi) - [\mathcal{D}^{(-)}\mu](\mathbf{x}, \varphi) = -\mathrm{i}V(\gamma(\mathbf{x}, \varphi), \varphi)). \tag{7.13}$$

Applying the angular Hilbert transform Γ with respect to the angle φ to the both sides of (7.13)

$$\big[\Gamma[\mathcal{D}^{(+)}\mu](\mathbf{x}, \cdot)\big](\varphi) - \big[\Gamma[\mathcal{D}^{(-)}\mu](\mathbf{x}, \cdot)\big](\varphi) = -\mathrm{i}\big[\Gamma V(\gamma(\mathbf{x}, \cdot), \cdot)\big](\varphi)$$

and using (4.9) from the Lemma 4.1 for the left hand side, we get

$$\mathrm{i}\big([\mathcal{D}^{(+)}\mu](\mathbf{x},\varphi)+[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)\big)=-\mathrm{i}\big[\Gamma V(\gamma(\mathbf{x},\,\cdot\,),\,\cdot\,)\big](\varphi),$$

or, dividing by i,

$$[\mathcal{D}^{(+)}\mu](\mathbf{x},\varphi)+[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)=-\big[\Gamma V(\gamma(\mathbf{x},\,\cdot\,),\,\cdot\,)\big](\varphi). \tag{7.14}$$

Subtracting (7.14) from (7.13) we get a required expression for $-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)$

$$-2[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)=\big[\Gamma V(\gamma(\mathbf{x},\,\cdot\,),\,\cdot\,)\big](\varphi)-\mathrm{i}V(\gamma(\mathbf{x},\varphi),\varphi)). \tag{7.15}$$

In the same manner we can derive an expression for $[\mathcal{D}\mu](\mathbf{x},\varphi)$ by adding (7.11) together with (7.12) and using (7.14) and (7.9)

$$\begin{aligned}[\mathcal{D}\mu](\mathbf{x},\varphi)&=\big([\mathcal{D}^{(+)}\mu](\mathbf{x},\varphi)+[\mathcal{D}^{(-)}\mu](\mathbf{x},\varphi)\big)\\&\quad-\big([\mathcal{D}^{(+)}\mu](\gamma(\mathbf{x},\varphi),\varphi)+[\mathcal{D}^{(-)}\mu](\gamma(\mathbf{x},\varphi),\varphi)\big)\\&=-\big[\Gamma V(\gamma(\mathbf{x},\,\cdot\,),\,\cdot\,)\big](\varphi)-U(\gamma(\mathbf{x},\varphi),\varphi).\end{aligned} \tag{7.16}$$

For computation of (7.15) and (7.16) we use previously evaluated functions $U(\beta,\varphi)$ and $V(\beta,\varphi)$.

7.1. Numerical examples

Figures 2–3 illustrate reconstructions of scalar source functions (emission maps) under the presence of a variable attenuation. Both, in Figures 2 and 3, panel (a) represents the unknown source function, panel (b) shows the known attenuation map, panels (c)–(d) show reconstructions of the source function using different number of fan projections without adding noise, and panels (e)–(f) show the reconstruction from the sinogram contaminated by noise (with Poisson distribution). Noise levels (5% for panel (e) and 10% for panel (f)) are given in the L_2-norm. Black color in the images corresponds to the value 0, and white color – to the value 1, both for the attenuation map and for the source distribution.

Figures 4–5 show the full (not only solenoidal part) reconstructions of vector fields (smooth in Figure 4 and discontinuous in Figure 5) under the presence of a constant nonzero attenuation. Panels (a) depict original vector fields (vectorial "emission maps") and panels (b) show reconstructions. The vector field being reconstructed in Figure 4 is given by formulae

$$\mathbf{a}(x,y)=\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}2xy\cos(x^2+y^2)+\cos(6xy)-6xy\sin(6xy)\\-\sin(x^2+y^2)-2x^2\cos(x^2+y^2)+6y^2\sin(6xy)\end{pmatrix} \tag{7.17}$$

$$+\begin{pmatrix}2\pi x\cos\big(\pi(x^2+y^2)\big)\\2\pi y\cos\big(\pi(x^2+y^2)\big)\end{pmatrix} \tag{7.18}$$

Figure 2. Reconstructions of a scalar function (a) from its attenuated Radon transform with a known attenuation map (b) without (c,d) and with (e,f) added noise

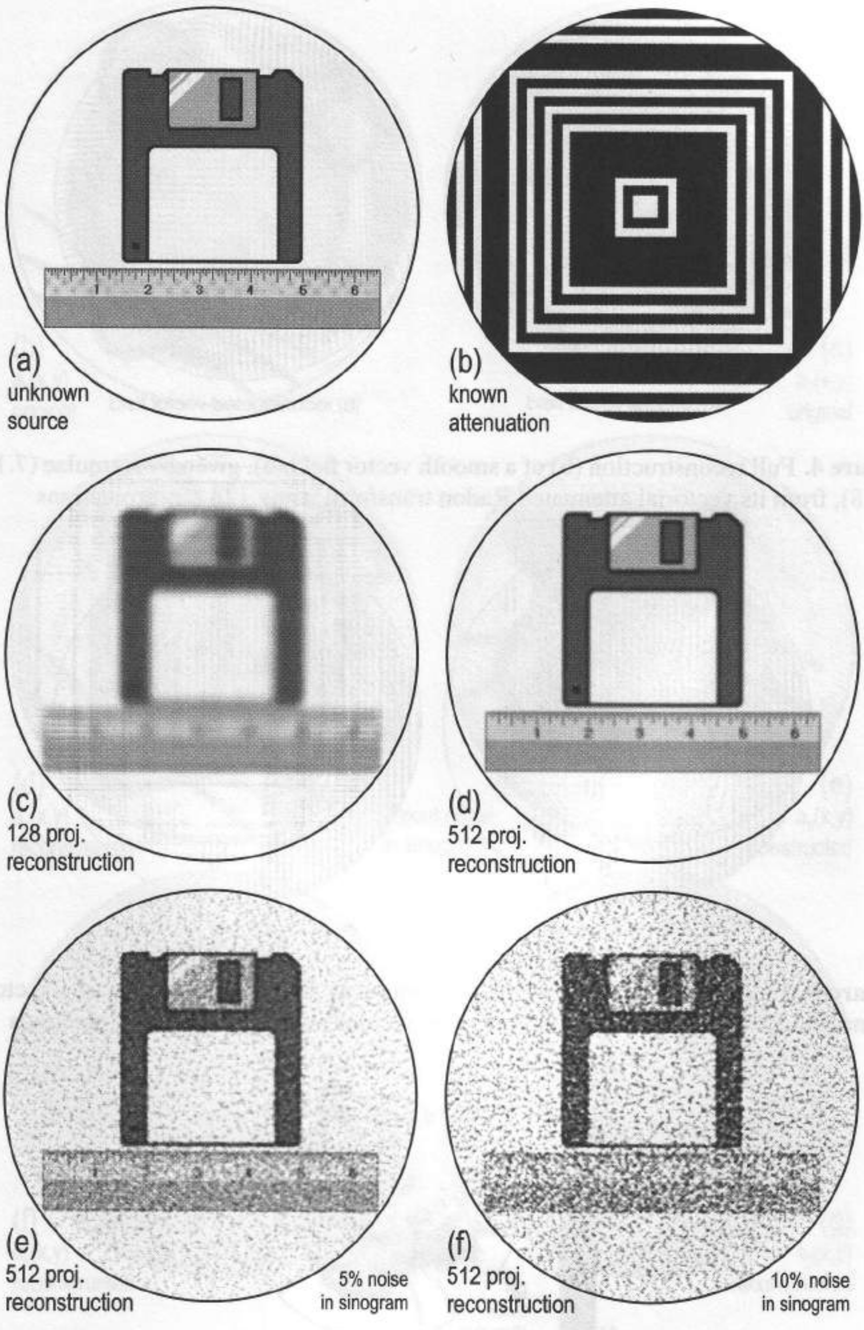
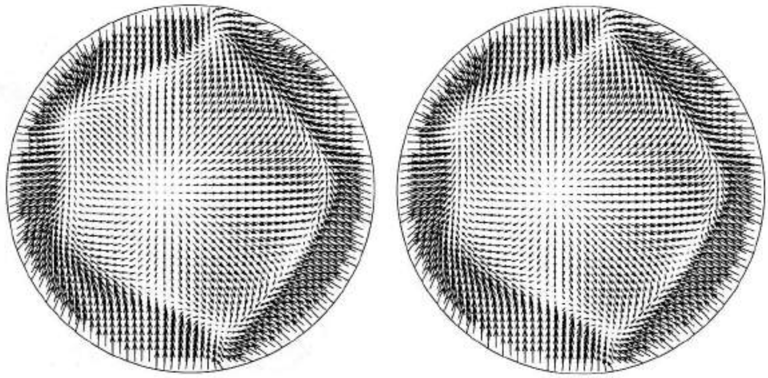
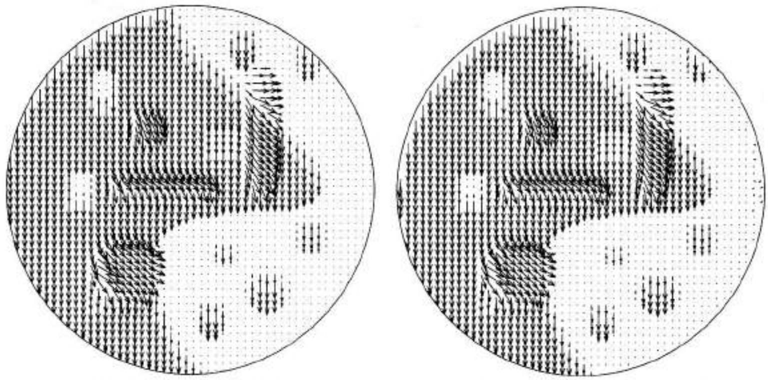


Figure 3. Reconstructions of a scalar function (a) from its attenuated Radon transform with a known attenuation map (b) without (c,d) and with (e,f) added noise



(a) original vector field (b) reconstructed vector field

Figure 4. Full reconstruction (b) of a smooth vector field (a), given by formulae (7.17)–(7.18), from its vectorial attenuated Radon transform using 128 fan-projections



(a) original vector field (b) reconstructed vector field

Figure 5. Full reconstruction (b) of a discontinuous vector field (a) from its vectorial attenuated Radon transform with a constant attenuation using 256 fan-projections

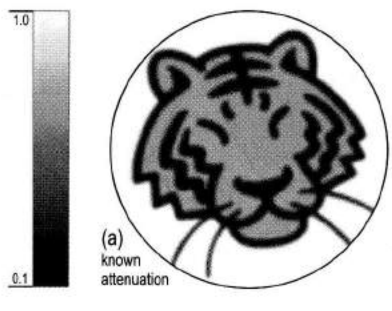


Figure 6. Attenuation map for Figure 7

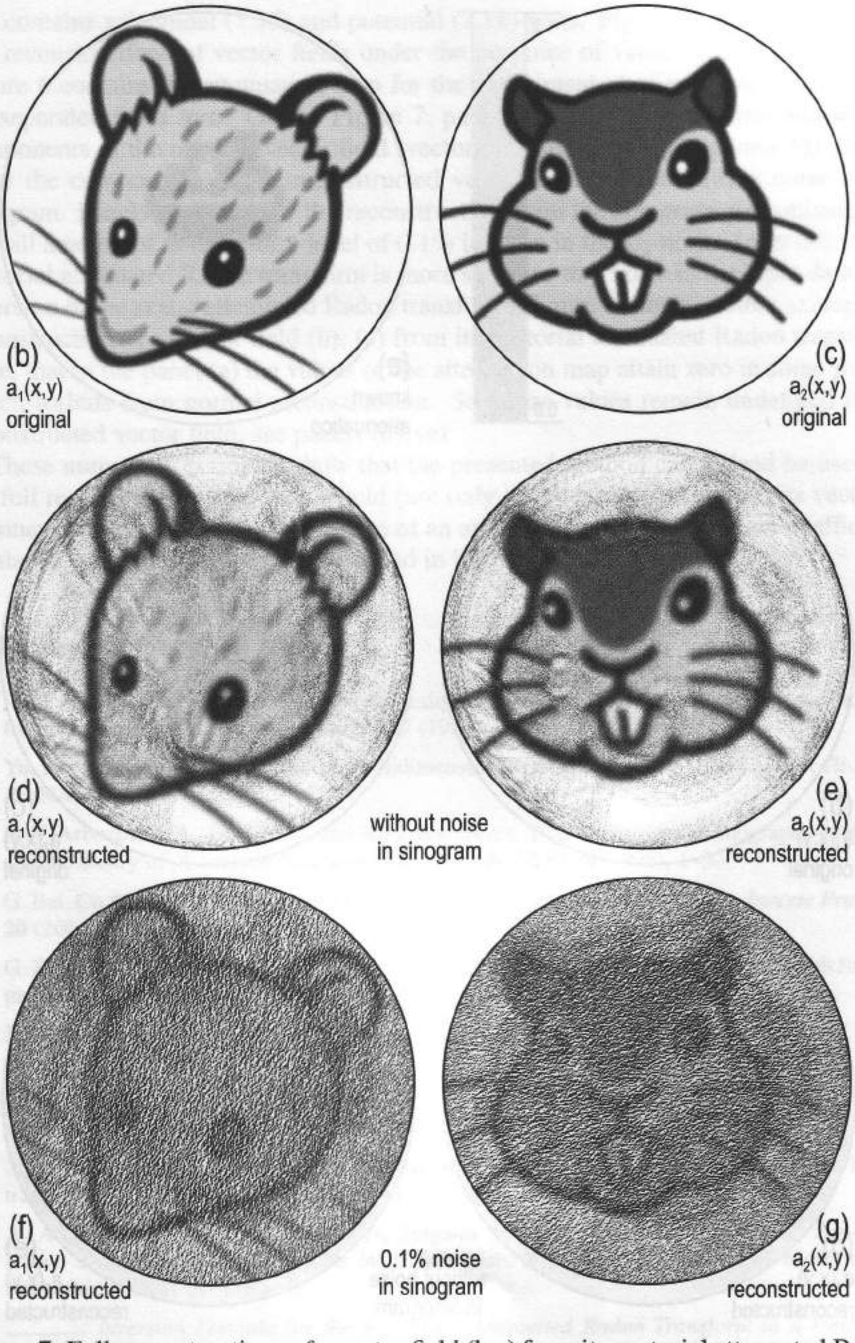


Figure 7. Full reconstructions of a vector field (b,c) from its vectorial attenuated Radon transform with a known attenuation map (Figure 6) without (d,e) and with (f,g) added noise

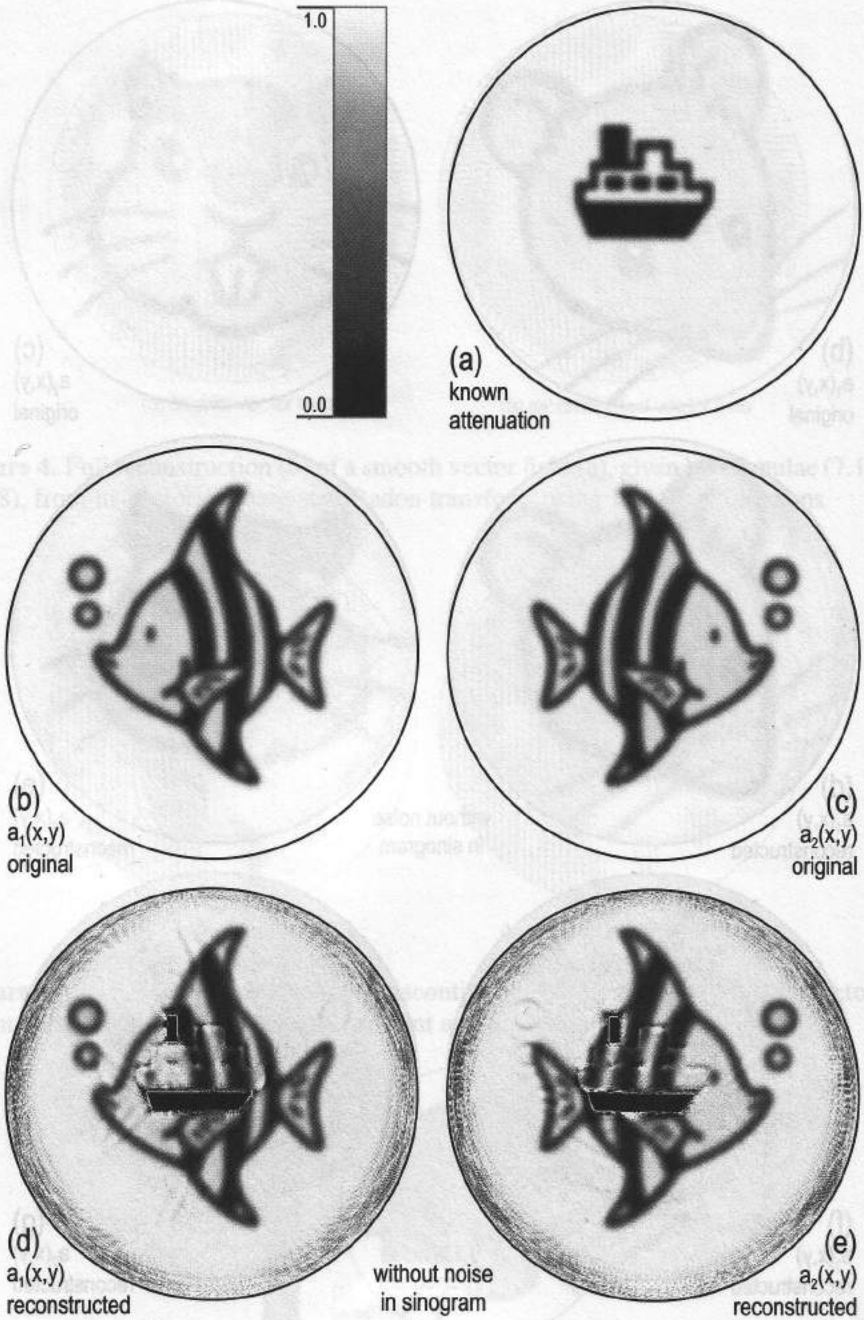


Figure 8. Full reconstruction (d,e) of a vector field (b,c) from its vectorial attenuated Radon transform with a known attenuation map containing zero values (a)

and contains solenoidal (7.17) and potential (7.18) parts. Figures 6, 7 and 8 illustrate full reconstructions of vector fields under the presence of variable attenuation maps. Figure 6 contains the attenuation map for the experiment on the Figure 7. Its values are separated from zero. On the Figure 7, panels (b) and (c) depict first and second components of the original vector field (vectorial "emission map"), panels (d) and (e) show the components of the reconstructed vector field without adding noise to the sinogram. Panels (f), (g) show the reconstruction from the sinogram contaminated by a small amount of noise. Noise level of 0.1% is given in the L_2-norm. Inversion of the vectorial attenuated Radon transform is more sensitive to a noise in the input data than inversion of the scalar attenuated Radon transform. Figure 8 shows another example of reconstruction of a vector field (b), (c) from its vectorial attenuated Radon transform. Note, that in the panel (a) the values of the attenuation map attain zero in some places, which forbids from normal reconstruction. So, some values remain undefined in the reconstructed vector field, see panels (d), (e).

These numerical examples show that the presented method can indeed be used for the full reconstruction of a vector field (not only its solenoidal part) from its vectorial attenuated Radon transform in the case of an arbitrary nonzero attenuation coefficient. Further numerical examples can be found in [10].

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Author information

S. G. Kazantsev, Sobolev Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, Acad. Koptyug prosp. 4, 630090 Novosibirsk, Russia.
Email: Kazan@math.nsc.ru

A. A. Bukhgeim, Schlumberger, Aslakveien 14E, 0753 Oslo, Norway.
Email: ABoukhgueim@slb.com