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FUNK–MINKOWSKI TYPE TRANSFORMS OF VECTOR FIELDS ON THE SPHERE $\mathbb{S}^2$

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Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Throughout the paper we adopt the convention to denote in bold type the vectors in $\mathbb{R}^3$, and in simple type the scalars in $\mathbb{R}$. By the greek letters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and so on we denote the units vectors on the sphere $\mathbb{S}^2$. The Funk–Minkowski transform $\mathcal{F}$ associates a function u or vector field $\mathbf{f}$ on the sphere $\mathbb{S}^2$ with its mean values (integrals) along all great circles of the sphere,

$$\{\mathcal{F} \begin{matrix} u \\ \mathbf{f} \end{matrix} \}(\boldsymbol{\eta}) \equiv \mathcal{F}_{\boldsymbol{\eta}} \begin{matrix} u \\ \mathbf{f} \end{matrix} = \frac{1}{2\pi} \int_{\mathbb{S}^2} \begin{matrix} u(\boldsymbol{\theta}) \\ \mathbf{f}(\boldsymbol{\theta}) \end{matrix} \, \delta(\boldsymbol{\eta} \cdot \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}, \tag{1}$$

where δ is the Dirac delta function and the $\mathrm{d}\boldsymbol{\theta}$ is the surface measure on $\mathbb{S}^2$ with normalization $\int_{\mathbb{S}^2} \mathrm{d}\boldsymbol{\theta} = 4\pi$. In the second case the Funk–Minkowski transform $\mathcal{F}$ is applied to vector function $\mathbf{f}$ by componentwise.

The spherical convolution operator $\mathcal{S}$ of Hilbert type is defined by,

$$\{\mathcal{S} \begin{matrix} u \\ \mathbf{f} \end{matrix} \}(\boldsymbol{\theta}) \equiv \mathcal{S}_{\boldsymbol{\theta}} \begin{matrix} u \\ \mathbf{f} \end{matrix} = \frac{\text{p.v.}}{4\pi} \int_{\mathbb{S}^2} \begin{matrix} u(\boldsymbol{\eta}) \\ \mathbf{f}(\boldsymbol{\eta}) \end{matrix} \, \frac{\mathrm{d}\boldsymbol{\eta}}{\boldsymbol{\theta} \cdot \boldsymbol{\eta}}, \ \ \boldsymbol{\theta} \in \mathbb{S}^2.$$

In addition, we also consider the following Funk-Minkowski type transforms of vector fields on the the sphere

$$\{\mathcal{F}^{(\boldsymbol{\tau})}\mathbf{f}\}(\boldsymbol{\eta}) \equiv \mathcal{F}^{(\boldsymbol{\tau})}_{\boldsymbol{\eta}}\mathbf{f} = \frac{\boldsymbol{\eta} \cdot}{2\pi} \int_{\mathbb{S}^2} \boldsymbol{\theta} \times \mathbf{f}(\boldsymbol{\theta}) \delta(\boldsymbol{\eta} \cdot \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}, \tag{2}$$

$$\{\mathcal{F}^{(\boldsymbol{\beta})}\mathbf{f}\}(\boldsymbol{\eta}) \equiv \mathcal{F}^{(\boldsymbol{\beta})}_{\boldsymbol{\eta}}\mathbf{f} = \frac{\boldsymbol{\eta} \cdot}{2\pi} \int_{\mathbb{S}^2} \mathbf{f}(\boldsymbol{\theta}) \delta(\boldsymbol{\eta} \cdot \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}. \tag{3}$$

The transform (2) will be an analog of the longitudinal ray transform of vector fields in the Euclidean case. In the physical sense, the quantity $\mathcal{F}^{(\boldsymbol{\tau})}_{\boldsymbol{\eta}}\mathbf{f}$ is equal to the circulation (work) of vector field $\mathbf{f}$ along the closed contour (big circle) $\boldsymbol{\theta} \cdot \boldsymbol{\eta} = 0$ on the sphere.

The tangential gradient or the surface gradient, denoted by $\nabla \equiv \nabla_{\boldsymbol{\xi}}$ and the tangential rotated gradient (the surface curl-gradient), denoted by $\nabla^{\perp} \equiv \nabla^{\perp}_{\boldsymbol{\xi}}$, are defined accordingly as

$$\nabla_{\boldsymbol{\xi}} u = \frac{\partial u}{\partial \theta}\mathbf{e}_1(\boldsymbol{\xi}) + \frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_2(\boldsymbol{\xi}), \quad \nabla^{\perp}_{\boldsymbol{\xi}} u = \boldsymbol{\xi} \times \nabla_{\boldsymbol{\xi}} u, \tag{4}$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the orthonormal basis in the tangent plane $\boldsymbol{\xi}^{\perp} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \boldsymbol{\xi} = 0\}$,

$$\mathbf{e}_1(\boldsymbol{\xi}) = \frac{\partial \boldsymbol{\xi}}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \ \mathbf{e}_2(\boldsymbol{\xi}) = \frac{1}{\sin\theta}\frac{\partial \boldsymbol{\xi}}{\partial \varphi} = (-\sin\varphi, \cos\varphi, 0),$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\theta, \varphi) = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

The surface divergence $\mathrm{div}_{\boldsymbol{\xi}}$ of vector-valued function $\mathbf{v}(\boldsymbol{\xi}) = v^1\mathbf{e}_1(\boldsymbol{\xi}) + v^2\mathbf{e}_2(\boldsymbol{\xi}) + v^3\boldsymbol{\xi}$ on the sphere $\mathbb{S}^2$ is written as,

$$\mathrm{div}_{\boldsymbol{\xi}}\mathbf{v} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(v^1\sin\theta) + \frac{\partial}{\partial\varphi}v^2\right) + 2v^3\,. \tag{5}$$

Finally, we define the Laplace–Beltrami operator $\Delta \equiv \Delta_{\boldsymbol{\xi}}$ as $\Delta_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \mathrm{div}_{\boldsymbol{\xi}}\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi})\,.$

Theorem 1. *For any function $f(\boldsymbol{\theta}) \in H^1(\mathbb{S}^2)$ the following identity takes place*

(6)
$$f(\boldsymbol{\theta}) = \underbrace{\frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}_{=f_{00}} + \frac{\text{p.v.}}{4\pi}\int_{\mathbb{S}^2}\frac{(\boldsymbol{\eta}+\boldsymbol{\theta})\,\boldsymbol{\cdot}\left\{\Big[\mathcal{F},\nabla\Big]f\right\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} = f_{00} + \mathcal{S}_{\boldsymbol{\theta}}(\boldsymbol{\eta}+\boldsymbol{\theta})\,\boldsymbol{\cdot}\,\Big[\mathcal{F},\nabla\Big]_{\boldsymbol{\eta}} f.$$

Here operators $\mathcal{F}$ and ∇ are the Funk–Minkowski transform (1) and the surface gradient (4), respectively. Through the square brackets $[.,.]$ we, as usual, denoted the commutator $\Big[\mathcal{F},\nabla\Big]f = \mathcal{F}\nabla f - \nabla\mathcal{F}f$.

We see that by using formula (6) the unknown function f completely reconstruct, if two Funk–Minkowski transforms, $\mathcal{F}f$ and $\mathcal{F}\nabla f$, are known.

Another result of this article is related to the problem of Helmholtz–Hodge decomposition for tangent vector field on the sphere $\mathbb{S}^2$. The Helmholtz–Hodge decomposition says that we can write any vector field tangent to the surface of the sphere as the sum of a curl-free component and a divergence-free component

(7) $$\mathbf{f}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta}) + \nabla^{\perp}_{\boldsymbol{\theta}} v(\boldsymbol{\theta}).$$

Here $\nabla_{\boldsymbol{\theta}} u$ is called also as inrrotational, poloidal, electric or potential field and $\nabla^{\perp}_{\boldsymbol{\theta}} v$ is called as incompressible, toroidal, magnetic or stream vector field. Scalar functions u and v are called velocity potential and stream functions, respectively.

In the next theorem we show that decomposition (7) is obtained by use of Funk–Minkowski-transform $\mathcal{F}$ and spherical convolution transform $\mathcal{S}$.

Theorem 2. *Any vector field $\mathbf{f} \in \mathbf{L}_{2,tan}(\mathbb{S}^2)$ that is tangent to the sphere can be uniquely decomposed into a sum (7) of a surface curl-free component and a surface divergence-free component with scalar valued functions $u, v \in H^1(\mathbb{S}^2)/\mathbb{C}$. Functions u and v are velocity potential and stream functions that are calculated unique up to a constant by the formulas*

$$u(\boldsymbol{\theta}) = \Big[\mathcal{S}, \boldsymbol{\eta}\,\boldsymbol{\cdot}, \mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \Big\{\mathcal{S}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \Big\{\mathcal{F}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta}) = \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f},$$

$$v(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\Big[\mathcal{S}, \boldsymbol{\eta}\times, \mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\Big\{\mathcal{S}\boldsymbol{\eta}\times\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\Big\{\mathcal{F}\boldsymbol{\eta}\times\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f},$$

where through $[\mathcal{A},\mathcal{B},\mathcal{C}]$ we denote the generalized commutator, $[\mathcal{A},\mathcal{B},\mathcal{C}] = \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{C}\mathcal{B}\mathcal{A}$.

Theorem 3. *For any functions $u, v \in H^1(\mathbb{S}^2)$ the following identities take place*

(8) $$\nabla u(\boldsymbol{\theta}) = \underbrace{\frac{\nabla}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\{\mathcal{F}^{(\boldsymbol{\beta})}\nabla u\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{even\,part} + \underbrace{\frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\Delta\{\mathcal{F}u\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{odd\,part},$$

(9) $$\nabla^{\perp} v(\boldsymbol{\theta}) = \underbrace{-\frac{\nabla^{\perp}_{\boldsymbol{\theta}}}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\{\mathcal{F}^{(\boldsymbol{\tau})}\nabla^{\perp} v\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{odd\,part} + \underbrace{\frac{\boldsymbol{\theta}\times}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\Delta\{\mathcal{F}v\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{even\,part}.$$

The analytic inversion formulas for operators $\mathcal{F}^{(\boldsymbol{\tau})}$ and $\mathcal{F}^{(\boldsymbol{\beta})}$ follow from the Theorem 3. Let $\mathbf{f}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta}) + \nabla^{\perp}_{\boldsymbol{\theta}} v(\boldsymbol{\theta})$ is an odd vector field, $\mathbf{f}(-\boldsymbol{\eta}) = -\mathbf{f}(\boldsymbol{\eta})$. It is obvious that for even vector fields $\mathbf{f}(-\boldsymbol{\eta}) = \mathbf{f}(\boldsymbol{\eta})$ the $\mathcal{F}^{(\boldsymbol{\tau})}\mathbf{f}$ will be zero. We also know that $\mathcal{F}^{(\boldsymbol{\tau})}\nabla u = 0$, so the original vector field is not completely determined by its transformation $\mathcal{F}^{(\boldsymbol{\tau})}$. We see that the first term in the formula (9) gives the inversion formula. So we define only the stream function v_{odd} and, accordingly, only the solenoidal part $\nabla^{\perp} v_{odd}(\boldsymbol{\theta})$ of the vector field $\mathbf{f}$.

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