# AUTOMORPHISMS OF THE ENDOMORPHISM SEMIGROUP OF A FREE COMMUTATIVE ALGEBRA 

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#### Abstract

We describe the automorphism group of the endomorphism semigroup $\operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ of ring $K\left[x_{1}, \ldots, x_{n}\right]$ of polynomials over an arbitrary field $K$. A similar result is obtained for automorphism group of the category of finitely generated free commutative-associative algebras of the variety $\mathcal{C} \mathcal{A}$ commutative algebras. This solves two problems posed by B. Plotkin ( [18], Problems 12 and 15).

More precisely, we prove that if $\varphi \in \operatorname{AutEnd}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ then there exists a semi-linear automorphism $s: K\left[x_{1}, \ldots, x_{n}\right] \rightarrow K\left[x_{1}, \ldots, x_{n}\right]$ such that $\varphi(g)=s \circ g \circ s^{-1}$ for any $g \in \operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$. This extends the result by A. Berzins obtained for an infinite field $K$.


## 1. Introduction

We describe the group $G=\operatorname{Aut}\left(\operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right.$, where $K$ is an arbitrary field. A similar result is obtained also for automorphism group of the category of finitely generated free commutative-associative algebras of the variety commutative algebras. This solves two problems posed by B. Plotkin ( [18], Problems 12 and 15).

More precisely, we prove that if $\varphi \in \operatorname{AutEnd}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ then there exists a semi-linear automorphism $s: K\left[x_{1}, \ldots, x_{n}\right] \rightarrow K\left[x_{1}, \ldots, x_{n}\right]$ such that $\varphi(g)=s \circ g \circ$ $s^{-1}$ for any $g \in \operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ (see Theorem [3.8). Here "semi-linearity" means that $s$ is a composition of an automorphism of the field $K$ and an automorphism of the ring $K\left[x_{1}, \ldots, x_{n}\right]$. We note that for an infinite ground field $K$ is infinite such result was obtained earlier by A. Berzins [3].

A problem of description of the group $G=\operatorname{Aut}\left(\operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right.$ is also interesting in the context of Universal Algebraic Geometry (UAG). Let $\Theta$ be a variety of algebras over a field $K$ and $F=F(X)$ be a free algebra from $\Theta$ generated by a finite subset $X$ of some infinite universum $X^{0}$. We refer to [17, 18] (see also [8]) for the Universal Algebraic Geometry (UAG) notions used in our work.

If an algebra $G$ belongs to $\Theta$ one can consider the category of algebraic sets $K_{\Theta}(G)$ over $G$. Objects of this category are algebraic sets in affine space over $G$; the category $K_{\Theta}(G)$ defines a geometry of the algebra $G$ in $\Theta$. One of the main problems in UAG is to determine whether two different algebras $G_{1}$ and $G_{2}$ have the same geometry. The coincidence of geometries means that the categories $K_{\Theta}\left(G_{1}\right)$ and $K_{\Theta}\left(G_{2}\right)$ are equivalent. It is known that coincidence of geometries of $G_{1}$ and $G_{2}$ is determined by the structure of the group Aut $\Theta^{0}$, where $\Theta^{0}$ is the category of

[^0]free finitely generated algebras of $\Theta$. On the other hand, there is a natural relation between the structure of the groups Aut End $F$ and Aut $\Theta^{0}$. The structure of the latter is determined by the group Aut End $F$. Should be mentioned that a problem of investigation of the groups Aut $\operatorname{End} F, F \in \Theta$, for different varieties $\Theta$ is quite interesting by itself and has been considered in many papers (see [1]-3], [5] [8] - [11], 13]- 19, ,22]

Let $\mathcal{C} \mathcal{A}$ be the variety of a commutative-associative algebras with 1 over a field $K$, $A=K\left[x_{1}, \ldots, x_{n}\right]$ be a free commutative-associative algebra in $\mathcal{C} \mathcal{A}$ freely generated over $K$ by a set $X=\left\{x_{1}, \ldots, x_{n}\right\}$, i.e., a polynomial algebra in variables $x_{1}, \ldots, x_{n}$. In this work we obtain a description of the group $A u t \mathcal{C} \mathcal{A}^{0}$ of automorphisms of the category $\mathcal{C} \mathcal{A}$. A similar result for a polynomial algebra $A$ over an infinite field $K$ was also obtained earlier in (3).

Our description is based on new characteristics of endomorphisms of $A$ such as rank of endomorphisms of $A$. We discuss external and internal definition of this notation. The former are expressed in terms of the action of the semigroup End $A$ on $A$, while the latter can be written in terms of the semigroup itself. This approach allows us to describe the above mentioned properties of endomorphisms of $A$ in an invariant manner and paves the way for proof of the main assertions in the paper: the group Aut End $A$ is generated by semi-inner of End $A$.

Our approach employs this technique (developed in [5, 9]) supplemented by algebro-geometric methods of investigations

## 2. On THE ENDOMORPHISM SEMIGROUP OF A FREE ASSOCIATIVE-COMMUTATIVE ALGEBRA

2.1. Rank of an endomorphism of polynomial algebra. Let $A=K\left[x_{1}, \ldots, x_{n}\right]$ be a free commutative-associative algebra over a field $K$ generated by $X=\left\{x_{1}, \ldots, x_{n}\right\}$ (below polynomial algebra over $K$ in variables $X$ ). Earlier, in [5], we defined the endomorphism of free associative algebra $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$ of rank 0 and 1 . In this section we introduce a definition of endomorphisms of arbitrary rank $m$ in a free commutative-associative $K\left[x_{1}, \ldots, x_{n}\right]$.

First, we introduce the "external" and "internal" definitions of rank of endomorphism $\varphi$ of algebra $A$ and show their equivalence.

Definition 2.1. ("External" definition of an endomorphism of rank m.) An endomorphism

$$
\varphi: A \rightarrow A
$$

has rank $m$ if $\operatorname{trdeg}(\operatorname{Im} \varphi)=m$, i.e., the transcendence degree of the $K$-algebra $M=\operatorname{Im} \varphi \subseteq A$ is equal to $m$. We denote this as $\operatorname{rk}(\varphi)=m$. It is evident that there exist endomorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$ of arbitrary rank $\leq n$. For instance, the identical mapping on $K\left[x_{1}, \ldots, x_{n}\right]$ is the endomorphism of rank $n$.

For the internal definition of rank $m$ endomorphisms, we need to define a congruence on the semigroup $\operatorname{End}(A)$ with respect to a fixed endomorphism $\varphi$ of $A$.

Definition 2.2. Endomorphisms $\varphi_{1}$ and $\varphi_{2}$ of $A$ are $\varphi$-equivalent if $\varphi \varphi_{1}=\varphi \varphi_{2}$. In this case we write $\varphi_{1} \sim_{\varphi} \varphi_{2}$.

It is clear that $\sim_{\varphi}$ is an equivalence relation on $\operatorname{End} A$. Let $S$ be the set of all $\varphi$-equivalences on $\operatorname{End} A$. We determine the preorder $\unlhd$ on the set $S$ as follows. We say that $\sim_{\phi} \unlhd \sim_{\psi}$, where $\phi, \psi \in \operatorname{End} A$, if

$$
\phi \varphi_{1}=\phi \varphi_{2} \Rightarrow \psi \varphi_{1}=\psi \varphi_{2}
$$

for any $\varphi_{1}, \varphi_{2} \in$ End $A$. The preorder $\unlhd$ can be extended up to the order $\preceq$ on the quotient set $\widetilde{S}=S / R$ under equivalence $R$, where $\sim_{\phi} R \sim_{\psi}$ if and only if $\sim_{\phi} \unlhd \sim_{\psi}$ and $\sim_{\psi} \unlhd \sim_{\phi}$. Denote by $\sim_{\psi_{R}}$ the $R$-equivalence class of a relation $\sim_{\psi}$.

Definition 2.3. We say that $\phi \preceq \psi$ iff $\backsim_{\phi_{R}} \preceq \backsim_{\psi_{R}}$.
Definition 2.4. We say that $\phi \prec \psi$ if $\backsim_{\phi_{R}} \preceq \backsim_{\psi_{R}}$ and $\backsim_{\psi_{R}} \nsim \sim_{\phi_{R}}$.
It is clear that relations $\preceq$ and $\prec$ are an order and a strong order, respectively, on End $A$. Note that the smaller endomorphism $\varphi$ (in the sense of $\preceq$ ) corresponds to stronger equivalence relation $\sim_{\varphi}$. The proof of the following Lemma is straightforward.

Lemma 2.5. Let $\varphi=\left(\varphi_{1}(\vec{x}) \ldots, \varphi_{n}(\vec{x})\right)$ and $\phi=\left(\psi_{1}(\vec{x}), \ldots, \psi_{n}(\vec{x})\right)$ be two endomorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$. Then
(1) $\phi \sim \psi$ iff for all $H(\vec{x}) \in K\left[x_{1}, \ldots, x_{n}\right]$ the condition $H\left(\varphi_{1}(\vec{x}) \ldots, \varphi_{n}(\vec{x})\right)=$ 0 is equivalent to $H\left(\psi_{1}(\vec{x}), \ldots, \psi_{n}(\vec{x})\right)=0$.
(2) $\phi \preceq \psi$ iff for all $H(\vec{x}) \in K\left[x_{1}, \ldots, x_{n}\right]$ the condition $H\left(\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x})\right)=$ 0 implies $H\left(\psi_{1}(\vec{x}), \ldots, \psi_{n}(\vec{x})\right)=0$.
(3) $\phi \prec \psi$ iff for all $H(\vec{x}) \in K\left[x_{1}, \ldots, x_{n}\right]$ the condition $H\left(\varphi_{1}(\vec{x}) \ldots, \varphi_{n}(\vec{x})\right)=$ 0 implies $H\left(\psi_{1}(\vec{x}), \ldots, \psi_{n}(\vec{x})\right)=0$ and there exists $R(\vec{x}) \in K\left[x_{1}, \ldots, x_{n}\right]$ such that $R\left(\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x})\right)=0$ but $H\left(\psi_{1}(\vec{x}), \ldots, \psi_{n}(\vec{x})\right) \neq 0$.

Definition 2.6. ("Internal" definition of an endomorphism of rank m.) An endomorphism $\psi: A \rightarrow A$ is of rank $m$, if maximum of the lengths of all chains of endomorphisms of $A$ of the form

$$
\begin{equation*}
\psi \precsim \psi_{m-1} \precsim \cdots \precsim \psi_{1} \precsim \psi_{0}, \tag{2.1}
\end{equation*}
$$

is equal to $m$. If there is no endomorphism $\psi$ such that $\psi \npreceq \psi_{0}$, then $\psi$ has rank 0 .
Remark 2.7. If $\operatorname{rk}(\varphi)=0$, then image of $\varphi$ is the ground field. The definition of endomorphisms of rank 0 and 1 for associative commutative algebra are in accordance with the definition for a free associative algebra given in 5]. The internal definition of rank 0 is pretty similar.

Proposition 2.8. Definitions 2.6 and 2.1 are equivalent.
We precede the proof of this proposition by several lemmas. Denote by $\mathbf{A}_{K}^{n}$ an n-dimensional affine space over the algebraic closure $\bar{K}$ of the field $K$. It is clear that $\mathbf{A}_{K}^{n} \simeq \operatorname{Specm}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$, where $\operatorname{Specm}\left(K\left[x_{1}, \ldots, x_{n}\right]\right.$ is the set of all maximal ideals. Let us investigate the algebro-geometric properties of polynomial endomorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$ and their relation to polynomial maps of $\mathbf{A}_{K}^{n}$ into itself.

Each endomorphism $\varphi: K\left[x_{1}, \ldots, x_{n}\right] \rightarrow K\left[x_{1}, \ldots, x_{n}\right]$ such that

$$
\varphi\left(x_{i}\right)=\varphi_{i}\left(x_{1}, \ldots, x_{n}\right), \text { where } \varphi_{i}=\varphi_{i}\left(x_{1}, \ldots, x_{n}\right) \in K\left[x_{1}, \ldots, x_{n}\right]
$$

determines a polynomial map $\varphi^{*}=\left(\varphi_{1}, \ldots, \varphi_{n}\right): \mathbf{A}_{K}^{n} \rightarrow \mathbf{A}_{K}^{n}$ of the affine space $\mathbf{A}_{K}^{n}$ into itself of the form

$$
\begin{equation*}
\left(x_{1}, \ldots, x_{n}\right) \rightarrow\left(\varphi_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, \varphi_{n}\left(x_{1}, \ldots, x_{n}\right)\right) \tag{2.2}
\end{equation*}
$$

The converse is also true: to each polynomial map $\varphi^{*}: \mathbf{A}_{K}^{n} \rightarrow \mathbf{A}_{K}^{n}$ of the form (2.2) corresponds the above mentioned endomorphism $\varphi$ of the algebra $K\left[x_{1}, \ldots, x_{n}\right]$. We will make use of this relation below.

Denote by $M_{\varphi}$ the variety $\varphi^{*}\left(\mathbf{A}_{K}^{n}\right)$. We shall say that the variety $M_{\varphi}$ corresponds to the endomorphism $\varphi$ of the polynomial algebra $K\left[x_{1}, \ldots, x_{n}\right]$. The coordinate ring $K\left[M_{\varphi}\right]$ of the variety $M_{\varphi}$ is $K\left[M_{\varphi}\right]=K\left[x_{1}, \ldots, x_{n}\right] / I$, where

$$
I=\left\{H\left(x_{1}, \ldots, x_{n}\right) \mid H\left(\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x})\right)=0\right\}
$$

is the ideal in $K\left[x_{1}, \ldots, x_{n}\right]$ corresponding to the variety $M_{\varphi}$. It is clear that $K\left[M_{\varphi}\right] \simeq K\left[\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x})\right]$ and $\operatorname{dim} M_{\varphi}=\operatorname{trdeg} K\left[\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x})\right]$.
Lemma 2.9. The variety $M_{\varphi}$ is irreducible.
Proof. Since the affine variety $\mathbf{A}_{K}^{n}$ corresponding to the algebra $K\left[x_{1}, \ldots, x_{n}\right]$ is irreducible and the image of an irreducible algebraic variety is also irreducible [6, 21], the variety $M_{\varphi}$ is irreducible. Hint: coordinate ring of an image isomorphic to subring of the coordinate ring of the preimage, hence has no zero divisors.)

Lemma 2.10. Let $\phi_{1}, \phi_{2}$ be endomorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$ and $M_{\phi_{1}}, M_{\phi_{2}}$ be two corresponding varieties, respectively. The following properties hold:
(1) If $\phi_{1} \sim \phi_{2}$, then $M_{\phi_{1}} \cong M_{\phi_{2}}$ and the corresponding coordinate rings are isomorphic.
(2) $\phi_{1} \preceq \phi_{2}$ if and only if the coordinate ring of $M_{\phi_{1}}$ is a quotient ring of the coordinate ring of $M_{\phi_{2}}$. In this case $\operatorname{dim} M_{\phi_{2}} \leq \operatorname{dim} M_{\phi_{1}}$, where $\operatorname{dim} X$ is the Krull dimension of a variety $X$. If the quotient ring is proper, then the inequality is strict.
Proof. (1) By item (3) of Lemma 2.5, the coordinate rings of the varieties $M_{\phi_{1}}$ and $M_{\phi_{2}}$ are isomorphic. Therefore, the above varieties themselves are isomorphic.
(2) By item (2) of Lemma [2.5, the coordinate ring of the variety $M_{\phi_{1}}$ is a quotient ring of the coordinate ring of the variety $M_{\phi_{2}}$ by some its ideal. As a consequence, $\operatorname{dim} M_{\phi_{1}} \leq \operatorname{dim} M_{\phi_{2}}$ (see also [6, 21]).

Let $\psi$ be an endomorphism of $K\left[x_{1}, \ldots, x_{n}\right]$ of "external" rank $m$. The last lemma shows that there exists no chains of endomorphisms $\psi_{i}$ of the form (2.1) of length more than $m$ beginning with $\psi$. It means that the inner rank of $\psi$ is less or equal than the outer its rank. In order to prove the proposition 2.8 we need to establish an opposite inequality, i.e., to prove that there exists a chain (2.1) of length $m$ beginning with $\psi$.
Lemma 2.11. Notations being as above, let $\operatorname{dim} M_{\varphi}=m$. Then there exists an endomorphism $\varphi^{\prime}$ of $K\left[x_{1}, \ldots, x_{n}\right]$ such that $\varphi^{\prime} \prec \varphi$ and $\operatorname{dim} M_{\varphi^{\prime}}=m-1$.

The assertion of this lemma is evident for $m=1$ : in this case it is sufficient to consider specialization $x_{i} \rightarrow \xi_{i}, \xi_{i} \in K$, into ground field $K$.

Now we pass to the general case. We need the following lemma
Lemma 2.12. Let $R$ be a subalgebra of $K\left[x_{1}, \ldots, x_{n}\right]$ of a transcendence degree $m$ $(m \leq n)$. Then there exists an embedding from $R$ into $K\left[x_{1}, \ldots, x_{m}\right]$.

Remark 2.13. A similar statement for field embeddings was established in [4].

Proof. It is known that any transcendence base of a subalgebra A of a algebra B can be extended to a transcendence base of the algebra B. Let $y_{1}, \ldots, y_{m}$ be a transcendence base of $R$. We can complete this base to a base $y_{1}, \ldots, y_{m}, z_{1}, \ldots, z_{n-m}$ of $K\left[x_{1}, \ldots, x_{n}\right]$. It is clear that the elements $z_{1}, \ldots, z_{n-m}$ are algebraically independent over $R$ and they generate a subalgebra $R\left[z_{1}, \ldots, z_{n-m}\right]$ of $K\left[x_{1}, \ldots, x_{n}\right]$. Therefore, the affine domain $R\left[z_{1}, \ldots, z_{n-m}\right]$ can be embedded into an affine domain $K\left[x_{1}, \ldots, x_{m}\right]\left[x_{1}, \ldots, x_{n-m}\right]$. However, it is known that if $A$ and $B$ are two domains such that $A\left[x_{1}, \ldots, x_{s}\right]$ can be embedded into $B\left[x_{1}, \ldots, x_{s}\right]$, then $A$ can be embedded into $B$ (see [4]). Therefore, $R$ can be embedded into the polynomial algebra $K\left[x_{1}, \ldots, x_{m}\right]$.

Now, by Lemma 2.12 one can assume that polynomials $\varphi_{1}, \ldots, \varphi_{n}$ defining the mapping $\varphi$ belong to $K\left[x_{1}, \ldots, x_{m}\right]$ and $\operatorname{trdeg}\left(\varphi_{1}, \ldots, \varphi_{n}\right)=m, m \leq n$.

Lemma 2.14. Let $\varphi_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, \varphi_{n}\left(x_{1}, \ldots, x_{m}\right)$, where $n \geq m$, be a collection of polynomials from $K\left[x_{1}, \ldots, x_{m}\right]$ which generates the subalgebra of $K\left[x_{1}, \ldots, x_{n}\right]$ of transcendence degree $m$. Then for any specialization $x_{m} \rightarrow \xi, \xi \in K$, except a finite set of values of $\xi \in K$, the algebra $K\left[\varphi_{1}\left(x_{1}, \ldots, x_{m-1}, \xi\right), \ldots, \varphi_{n}\left(x_{1}, \ldots, x_{m-1}, \xi\right)\right]$ has the transcendence degree $m-1$.

Proof. Without loss of generality it is sufficient to consider the case when $K$ is an algebraically closed field (tensoring over algebraic closure, if necessary). Consider a mapping $\Phi: \mathbf{A}_{K}^{m} \rightarrow \mathbf{A}_{K}^{n+1}$ such that $\Phi(\vec{x})=\left(\varphi_{1}(\vec{x}), \ldots, \varphi_{n}(\vec{x}), x_{m}\right)$ where $\vec{x}=$ $\left(x_{1}, \ldots, x_{m}\right)$. Denote by $M$ the image of $\Phi$. Since $\operatorname{trdeg}\left(\varphi_{1}, \ldots, \varphi_{n}\right)=m$ and the dimension of image $\Phi$ is at most $m$, we have $\operatorname{dim} M=m$. Now we consider a projection $\pi: \mathbf{A}_{K}^{n+1} \rightarrow \mathbf{A}_{K}^{1}$ such that $\pi\left(z_{1}, \ldots, z_{n}, x_{m}\right)=x_{m}$. Denote by $\pi_{1}$ the restriction of $\pi$ to $M$. It is clear that $\pi_{1}$ is an epimorphic mapping. Further we use the following

Theorem 2.15. [6, 21] If $f: X \rightarrow Y$ is a regular mapping between irreducible varieties $X$ and $Y: f(X)=Y, \operatorname{dim} X=n, \operatorname{dim} Y=m$, then $m \leq n$ and
(1) $\operatorname{dim} f^{-1}(y) \geq n-m$ for every point $y \in Y$.
(2) There exists a non empty set $U \subset Y$ such that $\operatorname{dim} f^{-1}(y)=n-m$ for all $y \in U$.

In our case $Y=\mathbf{A}_{K}^{1}, \operatorname{dim} Y=1, \operatorname{dim} X=m$. Therefore, for all points of $\mathbf{A}_{K}^{1}$, except points of closed subvariety $T$ of $\mathbf{A}_{K}^{1}$, the fiber $\pi^{-1}(\xi)$ has the dimension $m-1$. Therefore,

$$
\operatorname{trdeg} K\left[P_{1}\left(x_{1}, \ldots, x_{m-1}, \xi\right), \ldots, P_{n}\left(x_{1}, \ldots, x_{m-1}, \xi\right)\right]=m-1
$$

except a finite set of $\xi \in K$. This concludes the proof of Lemma 2.14.
Remark 2.16. A proof of Lemma 2.11follows immediately from the above Lemma in the case of an infinite ground field. Indeed, if a field $K$ is infinite, by Lemma 2.14 we can choose $\xi \in K$ such that $\varphi_{1}^{\prime}=\varphi_{1}\left(x_{1}, x_{2}, \ldots, x_{n-1}, \xi\right), \ldots, \varphi_{n}^{\prime}=$ $\varphi_{n}\left(x_{1}, \ldots, x_{n-1}, \xi\right)$ and $\operatorname{trdeg} K\left[\varphi_{1}^{\prime}(\vec{x}), \ldots, \varphi_{n}^{\prime}(\vec{x})\right]=m-1$. As a corollary, we have $\operatorname{dim} M_{\varphi^{\prime}}=k-1$, where $\varphi^{\prime}=\left(\varphi_{1}^{\prime}, \ldots, \varphi_{n}^{\prime}\right)$. Hence, our Lemma 2.11 is proven in the case of an infinite field. This provides a description of the group $\operatorname{Aut}\left(\operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)$ for the case of an infinite ground field $K$ as was obtained earlier by Berzins 3.

However, in the case of a finite ground field there can be no such small jumps from $\varphi_{i}$ to $\varphi_{i}^{\prime}$, such that $\operatorname{dim} M_{\varphi^{\prime}}=\operatorname{dim} M_{\varphi}-1$, for any specialization of variables into a ground field $K$.
Example 2.17. Let $|K|=q$ and $\varphi_{i}=\prod_{k=1}^{n}\left(x_{k}^{q}-x_{k}\right) \cdot x_{i}$. It is evident that $\operatorname{trdeg}\left(\varphi_{1}, \ldots, \varphi_{n}\right)=n$. However, any specialization of $\varphi_{i}$ of the form: $x_{n} \rightarrow \xi, \xi \in$ $K$, yields us $\varphi_{i}^{\prime}=0$.

If a field $K$ is finite instead of specializations of $x_{n}$ into ground field we consider substitutions into polynomials depending on other variables, in particular, on powers of other variables. We need the following

Theorem 2.18. 4] Let $\xi_{1}, \ldots, \xi_{s}$ be algebraic over $K\left[x_{1}, \ldots, x_{m}\right]$, the polynomials $Q_{i}(\vec{t}, \vec{x}, \vec{\xi}), i=1, \ldots, n$, are algebraically independent for some value of set of parameter $\vec{t}=\left(t_{1}, \ldots, t_{n}\right)$ in some extension field $k_{1}$ of the ground field $k$. Then there exists polynomials $R_{i} \in \Phi\left[x_{1}\right], i=1,2, \ldots, r, \vec{R}=\left(R_{1}, \ldots, R_{r}\right)$ such that the set of polynomial

$$
\left\{Q_{1}(\vec{t}, \vec{x}, \vec{\xi}), \ldots, Q_{n}(\vec{t}, \vec{x}, \vec{\xi})\right\}
$$

is algebraically independent. Moreover, if the growth of the sequence

$$
n_{1} \ll n_{2} \ll \cdots \ll n_{r}
$$

is sufficiently large, we may be assume $R_{i}=x_{1}^{n_{i}}$. The above statement is still valid if we replace " $k\left[x_{1}, \ldots, x_{m}\right]$ " by " $k\left(x_{1}, \ldots, x_{m}\right)$ " and "polynomial" for rational function. In this case we can put $R_{i}=x_{1}^{-n_{i}}$.

Instead of $x_{1}$ one can take any other variable $x_{i} ; \Phi=\mathbb{Z}_{p}$ if char $K=p$ and $\Phi=\mathbb{Z}$ if $\operatorname{char} K=0$.

We use a special case of this Theorem for $r=1$ and $s=0$, i.e, a variant of this Theorem without $\xi_{i}$. The next Assertion is also needed for the proof of Lemma 2.11 in the case of a finite ground field $K$.

Assertion 2.19. Let $Q_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{m}\right)$ be a set of polynomials from $K\left[x_{1}, \ldots, x_{m}\right],|K|<\infty$, and the transcendence degree of the algebra

$$
K\left[Q_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{m}\right)\right]
$$

equal to $m$, where $m>1$ and $m \leq n$. If $r \in \mathbb{N}$ is sufficiently large, then

$$
\operatorname{trdeg}\left(K\left[Q_{1}\left(x_{1}, \ldots, x_{1}^{r}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{1}^{r}\right)\right]\right)=m-1
$$

Proof. Denote by $A=K\left[Q_{1}\left(x_{1}, \ldots, x_{m-1}, x_{1}^{r}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{m-1}, x_{1}^{r}\right)\right]$. It is clear that $A \subseteq K\left[x_{1}, \ldots, x_{m-1}\right]$, i.e., $\operatorname{trdeg}(A) \leq m-1$. We have to prove that the opposite inequality is also fulfilled for sufficiently large $r$. Since

$$
\operatorname{trdeg}\left(K\left[Q_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{m}\right)\right]\right)=m
$$

we can choose $m$ algebraically independent polynomials between $Q_{i}$. Without loss of generality, we can set that these polynomials are $Q_{1}, \ldots, Q_{m}$. By Lemma 2.14, there exists $\eta \in \bar{K}$, where $\bar{K}$ is the algebraic closure of field $K$, such that

$$
\operatorname{trdeg}\left(\bar{K}\left[Q_{1}\left(x_{1}, \ldots, x_{m-1}, \eta\right), \ldots, Q_{m}\left(x_{1}, \ldots, x_{m-1}, \eta\right)\right]\right)=m-1
$$

Without loss of generality, we can suppose that the first $m-1$ polynomials $Q_{i}\left(x_{1}, \ldots, x_{m-1}, \eta\right), 1 \leq i \leq m-1$, are algebraically independent over $\bar{K}$. By Theorem 2.18 there exists a natural $r_{0}$, such that the polynomials

$$
Q_{1}\left(x_{1}, \ldots, x_{m-1}, x^{r}\right), \ldots, Q_{m-1}\left(x_{1}, \ldots, x_{m-1}, x^{r}\right)
$$

are algebraically independence over $K$ for any $r \geq r_{0}$. Since the dimension of the subring $K\left[Q_{1}\left(x_{1}, \ldots, x_{m-1}, x^{r}\right), \ldots, Q_{m-1}\left(x_{1}, \ldots, x_{n}, x^{r}\right)\right]$ is not less than the dimension of its subring $K\left[Q_{1}\left(x_{1}, \ldots, x_{m-1}, x^{r}\right), \ldots, Q_{n}\left(x_{1}, \ldots, x_{m-1}, x^{r}\right)\right]$, the proof is complete.

We summarize our results in the following
Assertion 2.20. Let $\varphi=\left(\varphi_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, \varphi_{n}\left(x_{1}, \ldots, x_{n}\right)\right)$ be an endomorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$ of "internal" rank $m$. Then there exists an endomorphism $\psi=\left(\psi_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, \psi_{n}\left(x_{1}, \ldots, x_{m}\right)\right), \psi_{i}\left(x_{1}, \ldots, x_{m}\right) \in K\left[x_{1}, \ldots, x_{m}\right]$, such that $\varphi \sim \psi$. In addition, an endomorphism

$$
\psi_{(r)}^{\prime}=\left(\psi_{1}\left(x_{1}, \ldots, x_{m-1}, x_{1}^{r}\right), \ldots, \psi_{n}\left(x_{1}, \ldots, x_{m-1}, x_{1}^{r}\right)\right)
$$

has the rank at most $m-1$ for any $r \in \mathbb{N}$. Moreover, there exists $r_{0} \in \mathbb{N}$ such that for all $r \geq r_{0}$ holds: $\psi_{(r)}^{\prime} \prec \psi$. As consequence, $\psi_{(r)}^{\prime} \prec \varphi$ and an "internal" rank of $\psi_{(r)}^{\prime}$ is equal to $m-1$ for all $r \geq r_{0}$.

With these Assertion, the proof of Lemma 2.11 is straightforward. Now we ready to prove Proposition 2.8

Proof of Proposition 2.8 Suppose that $\varphi$ has an "internal" rank $m$, i.e., there exists a maximal chain of length $m$ beginning with $\varphi$ :

$$
\begin{equation*}
\varphi \precsim \varphi_{m-1} \precsim \cdots \precsim \varphi_{1} \prec \varphi_{0}, \tag{2.3}
\end{equation*}
$$

We have a descending chain of the corresponding varieties $M_{\varphi_{i}}$ :

$$
\begin{equation*}
M_{\varphi_{0}} \subseteq M_{\varphi_{1}} \subseteq \cdots \subseteq M_{\varphi_{m-1}} \subseteq M_{\varphi} \tag{2.4}
\end{equation*}
$$

The induction argument on the length $m$ of the chain (2.4) leads us to the case $m=0$ for which our assertion is evident. Therefore, the "external" rank of $\varphi$ is also equal to $m$.

Conversely, let an endomorphism $\varphi$ be of "external" rank $m$, i.e., $\operatorname{trdeg} \operatorname{Im} \varphi=m$. By Lemma 2.11 there exists an endomorphism $\psi_{m-1}$ of $K\left[x_{1}, \ldots, x_{n}\right]$ such that $\psi_{m-1} \prec \varphi$ and $\operatorname{dim} M_{\psi_{m-1}}=m-1$. In the same way, we can construct a chain of the form (2.3) beginning with $\varphi$. It is clear that this chain has the length $m$, as desired.

Since the chain (2.1) is invariant under automorphisms of End $K\left[x_{1}, \ldots, x_{n}\right]$, we have

Corollary 2.21. Let $\Phi \in \operatorname{Aut}(\operatorname{End}(A)), \psi \in \operatorname{End}(A)$, and $\operatorname{rk}(\psi)=m$. Then $\operatorname{rk}(\Phi(\psi))=m$.

Remark 2.22. Below we need endomorphisms of rank zero and one. By Definition 2.1, an endomorphism $\psi$ of $A$ is of rank zero if $\psi(A)=K$. An endomorphism $\varphi$ of $A$ is of rank one if $\operatorname{trdeg}(\operatorname{Im} \varphi)=1$. It is known [4], 20, that every integrally closed subalgebra $B$ of $A=K\left[x_{1}, \ldots, x_{n}\right]$ of transcendence degree 1 is isomorphic to a polynomial algebra $K[t]$ in variable $t$. Taking into account that the integer closure $B$ of the algebra $\varphi(A)$ in $A$ is an algebra of the same transcendence degree as $\varphi(A)$, we conclude that the algebra $B$ is isomorphic to a polynomial algebra $K[t]$ in variable $t$. As a consequence, the algebra $\varphi(A)$ is a polynomial algebra $K[y]$, where $y$ is an element in $K\left[x_{1}, \ldots, x_{n}\right]$.
2.2. Representations of Kronecker semigroup of rank $n$. Recall the definition of Kroneker endomorphisms of the free associative algebra $A$.

Definition 2.23. (cf. [9, 11]) Kroneker endomorphisms of $A$ in the base $X=$ $\left\{x_{1}, \ldots, x_{n}\right\}, x_{i} \in A$, are the endomorphisms $e_{i j}, i, j \in[1 n]$, of $A$ which are determined on free generators $x_{k} \in X$ by the rule: $e_{i j}\left(x_{k}\right)=\delta_{j k} x_{i}, x_{i} \in X, i, j, k \in[1 n]$ and $\delta_{j k}$ is the Kronecker delta.

It is clear that any Kronecker endomorphism of $A$ has rank 1.
Definition 2.24. A semigroup $\Gamma_{n}$ with an adjoint zero element 0 generated by $b_{i j}, i j \in[1 n]$, with defining relations

$$
b_{i j} \cdot b_{k m}=\delta_{j k} b_{i m}, b_{i j} \cdot 0=0 \cdot b_{i j}=0
$$

is called a Kronecker semigroup of rank $n$.
Denote by $E_{n}$ a semigroup generated by $e_{i j}, i, j \in[1 n]$, and an adjoint zero. Clearly, the semigroup $E_{n}$ is a Kronecker semigroup of rank $n$.
Remark 2.25. We have a notion of the rank of a Kronecker semigroup $\Gamma$. Don't confuse it with the rank of an endomorphism of $A$.

Definition 2.26. A representation of a semigroup $T$ in the semigroup End $A$ is a homomorphism $\nu: T \rightarrow$ End $A$.

Definition 2.27. Let $\rho: \Gamma_{n} \rightarrow$ End $A$ be a representation of the Kronecker semigroup $\Gamma$ of rank $n$ in $\operatorname{End} A$. We say that the representation $\rho$ is singular if $\operatorname{rk} \rho\left(b_{i j}\right)=0$ for any $i, j \in[1 n]$.

In fact, it is sufficient to require that $\operatorname{rk} \rho\left(b_{11}\right)=0$.
Proposition 2.28. Let $\rho: \Gamma_{n} \rightarrow$ End $A$ be a singular representation of the Kronecker semigroup $\Gamma$ of rank $n$ in End $A$ and $q=\rho \cdot \rho^{-1}$ the kernel congruence on $\Gamma_{n}$. Then $\Gamma_{n} / q \cong A$, where $A=\langle\varphi\rangle$ is a one-element semigroup such that $\rho(0)=\varphi, \varphi \in \operatorname{End} A$, and $\operatorname{rk}(\varphi)=0$. Conversely, if $\varphi \in \operatorname{End} A$ is an endomorphism of rank 0 , then there exists a representation $\rho: \Gamma_{n} \rightarrow$ End $A$ such that $\rho(0)=\varphi$.
Proof. From $0 \cdot b_{i j}=0, i, j \in[1 n]$, it follows $\varphi \rho\left(b_{i j}\right)=\varphi$, where $\rho(0)=\varphi$. Since $\varphi$ is the identical mapping on $K$ and $\operatorname{rk}\left(\rho\left(b_{i j}\right)\right)=0$, we have $\rho\left(b_{i j}\right)=\varphi$ for any $i, j \in[1 n]$. Thus, $\Gamma_{n} / q \cong A$, where $A=\langle\varphi\rangle$.

Conversely, if $\varphi$ is an endomorphism of $\operatorname{End} A$ such that $\operatorname{rk}(\varphi)=0$. Define a representation $\rho: \Gamma_{n} \rightarrow$ End $A$ by the rule $\rho(0)=\rho\left(b_{i j}\right)=\varphi$ for all $i, j \in[1 n]$. It is clear that we obtained a required representation $\rho$.

Remark 2.29. Let $\rho: \Gamma_{n} \rightarrow$ End $A$ be a singular representation of the Kronecker semigroup $\Gamma_{n}$ of rank $n$ in End $A$ such that $\rho(0)=\varphi, \varphi \in \operatorname{End} A$, and $\operatorname{rk}(\varphi)=0$. We can set $\varphi\left(x_{i}\right)=\alpha_{i}, \alpha_{i} \in K$. Denote by $\psi: K^{n} \rightarrow K^{n}$ the mapping on $K^{n}$ such that $\psi\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}-\alpha_{1}, \ldots, x_{n}-\alpha_{n}\right)$. Define a representation $\widehat{\rho}: \Gamma_{n} \rightarrow \operatorname{End} A$ of $\Gamma_{n}$ in End $A$ by the rule $\widehat{\rho}(0)=\widehat{\rho}\left(b_{i j}\right)=\varphi \psi$ for all $i, j \in[1 n]$. Then $\varphi \psi=\widehat{O}$ and $\widehat{\rho}(0)=\widehat{O}$.

Proposition 2.30. Let $\rho: \Gamma_{n} \rightarrow$ End $A$ be a non-singular representation of a Kronecker semigroup $\Gamma_{n}$. Then, $\operatorname{rk}\left(\rho\left(b_{i j}\right)\right)=1$ for all $i, j \in[1 n]$.

Proof. We will make use below relations between polynomial map $\varphi: K^{n} \rightarrow K^{n}$ and endomorphisms of the polynomial algebra $K\left[x_{1}, \ldots, x_{n}\right]$, described on the page 3.

Denote $\rho\left(b_{i j}\right)$ by $\varphi_{i j}, i, j \in[1 n]$. Let $\bar{\varphi}_{i j}$ be the endomorphisms of the algebra $B=K\left[x_{1}, \ldots, x_{n}\right]$ of commutative polynomials in variables $x_{1}, \ldots, x_{n}$ induced by the endomorphisms $\varphi_{i j}$ of the algebra $A$. Clearly, $\bar{\varphi}_{i j} \bar{\varphi}_{k m}=\delta_{j k} \bar{\varphi}_{i m}$. Let us note $o \cdot \varphi_{i m}=\widehat{O}$. For a fix $j \in[1 n]$ consider $\bar{\varphi}_{j j}$ as a polynomial mapping from $K^{n}$ into $K^{n}$, i.e., $\bar{\varphi}_{j j}\left(x_{1}, \ldots, x_{n}\right)=\left(\bar{\varphi}_{j j}\left(x_{1}\right), \ldots, \bar{\varphi}_{j j}\left(x_{n}\right)\right)$. Since $\bar{\varphi}_{j j}^{2}=\bar{\varphi}_{j j}$, the mapping $\bar{\varphi}_{j j}$ has a fixed point in $K^{n}$. This point $d=\left(d_{1}, \ldots, d_{n}\right), d_{i} \in K$, can be chosen arbitrarily from the image of $\bar{\varphi}_{j j}$. Therefore, we have $\bar{\varphi}_{j j}\left(d_{1}, \ldots, d_{n}\right)=\left(d_{1}, \ldots, d_{n}\right)$.

Denote by $T: K^{n} \rightarrow K^{n}$ the polynomial mapping on $K^{n}$ such that $T\left(x_{1}, \ldots, x_{n}\right)=$ $\left(x_{1}+d_{1}, \ldots, x_{n}+d_{n}\right)$. Let $\tilde{\varphi}_{i j}=T^{-1} \bar{\varphi}_{i j} T$ be a mapping $K^{n}$ into itself. Denote by $p_{i j}^{(k)}$ the element $T^{-1} \bar{\varphi}_{i j} T\left(x_{k}\right)$. Since the mapping $\tilde{\varphi}_{i i}$ has the fixed point $0 \in K^{n}$, the elements $p_{i i}^{(k)}$ do not have constant terms for any $i, k \in[1 n]$. Now we will prove that the elements $p_{i j}^{(k)}, i, j, k \in[1 n]$, also do not have constant terms. Assume, on the contrary, that there exist $i, j, k \in[1 n], i \neq j$, such that the element $p_{i j}^{(k)}$ has a constant term. Since the elements $p_{j j}^{(m)}=T^{-1} \bar{\varphi}_{j j} T\left(x_{m}\right)$ do not have a constant term for any $m, j \in[1 n]$, we obtain

$$
\left(T^{-1} \bar{\varphi}_{j j} T\right)\left(T^{-1} \bar{\varphi}_{i j} T\right)\left(x_{k}\right)=\left(T^{-1} \bar{\varphi}_{j j} T\right) p_{i j}^{(k)} \neq 0
$$

On the other hand, since $i \neq j$

$$
\left(T^{-1} \bar{\varphi}_{j j} T\right)\left(T^{-1} \bar{\varphi}_{i j} T\right)\left(x_{k}\right)=\left(T^{-1} \bar{\varphi}_{j j} \bar{\varphi}_{i j} T\right)\left(x_{k}\right)=0 .
$$

This contradiction proves that the elements $p_{i j}^{(k)}=T^{-1} \bar{\varphi}_{i j} T\left(x_{k}\right)$ do not have a constant term for any $i, j, k \in[1 n]$. As a consequence, the elements $T^{-1} \varphi_{i j} T\left(x_{k}\right)$ do not have constant terms for any $i, j, k \in[1 n]$, too.

Denote the mapping $T^{-1} \varphi_{i j} T: A \rightarrow A$ by $\hat{\varphi}_{i j}$. We now prove that $\hat{\varphi}_{i j}(A)$ is a subalgebra of $K[w]$ for some $w \in A$. Let $I$ be the ideal of $A$ generated by $x_{1}, \ldots, x_{n}$. Since the elements $\hat{\varphi}_{i j}\left(x_{k}\right), i, j, k \in[1 n]$, do not have a constant term, $\hat{\varphi}_{i j}\left(I^{s}\right) \subseteq I^{s}$ for any $s \geq 1$. Now we fix some $i, j \in[1 n]$ and consider induced $\operatorname{maps} \tilde{\varphi}_{i j}^{(s)}: I^{s} / I^{s+1} \rightarrow I^{s} / I^{s+1}$ for any $s \geq 1$. We intend to prove that $\operatorname{Im} \tilde{\varphi}_{i j}^{(s)}$ are one-dimensional vector spaces over $K$. Let $s=1$. Then $\tilde{\varphi}_{i j}^{(1)}: I / I^{2} \rightarrow I / I^{2}$ is a linear mapping from the vector space $I / I^{2}$ into itself. Since $\tilde{\varphi}_{i j}^{(1)} \tilde{\varphi}_{m k}^{(1)}=\delta_{j m} \tilde{\varphi}_{i k}^{(1)}$, by Lemma 4.7 [11] there exists a basis $\bar{z}_{r 1}=z_{r}+I^{2}$, where $z_{r} \in I, r \in[1 n]$, of $I / I^{2}$ such that $\tilde{\varphi}_{i j}^{(1)}\left(\bar{z}_{r 1}\right)=\delta_{j r} \bar{z}_{i 1}$. For a fix number $s \geq 2$ denote $\bar{z}_{r s}=z_{r}+I^{s+1}, r \in[1 n]$. We have $\tilde{\varphi}_{i j}^{(s)}\left(\bar{z}_{i_{1} s} \cdots \bar{z}_{i_{s} s}\right)=\delta_{j i_{1}} \cdots \delta_{j i_{s}} \bar{z}_{i s}^{s}$. Thus, $\tilde{\varphi}_{i j}^{(s)}\left(I^{s} / I^{s+1}\right)$ is a one-dimensional vector space with a basis $\left\{\bar{z}_{i s}^{s}\right\}$. The latter assertion holds for any $s \geq 2$. As a consequence, we have $\hat{\varphi}_{i j}(A) \subseteq K\left[z_{i}\right]$. Hence, $\varphi_{i j}(A)$ is a subalgebra of $K[w]$, where $w=T z_{i}$. Since the representation $\rho$ of $\Gamma$ is non-singular, $K \subset \varphi_{i j}(A)$. Thus, $\operatorname{rk}\left(\varphi_{i j}\right)=\operatorname{rk} \rho\left(b_{i j}\right)=1$ for all $i, j \in[1 n]$.
2.3. Bases and subbases of the semigroup End $A$. We need the following

Definition 2.31. A set of endomorphisms $\mathcal{B}_{e}=\left\{e_{i j}^{\prime} \mid e_{i j}^{\prime} \in \operatorname{End} A\right.$ and $e_{i j}^{\prime} \neq$ $\widehat{O}, \forall i, j \in[1 n]\}$ of $A$ is called a subbase of End $A$ if $e_{i j}^{\prime} e_{k m}^{\prime}=\delta_{j k} e_{i m}^{\prime}, \forall i, j, k, m \in$ [1n].

Let us note that $0 \cdot e_{i j}^{\prime}=\widehat{O}$. Denote by $E^{\prime}$ a semigroup of End $A$ generated by endomorphisms $e_{i j}^{\prime}$ and the endomorphism $\widehat{O}$. By Theorem 2.30, we obtain the following

Corollary 2.32. $\operatorname{rk}\left(e_{i j}^{\prime}\right)=1$ for any $i, j \in[1 n]$.
We can assume that $e_{i j}^{\prime}(A)$ is a subalgebra of $K\left[z_{i j}\right], i, j \in[1 n]$, where $z_{i j} \in A$. For the sake of simplicity we write $z_{i i}=z_{i}, i \in[1 n]$.

Definition 2.33. ("External" definition of a base collection of End A.) We say that the subbase $\mathcal{B}_{e}$ is a base collection of endomorphisms of $A$ (or a base of End $A$, for short) if $Z=\left\{z_{i} \mid z_{i} \in A\right.$ such that $\left.e_{i i}^{\prime}(A) \subseteq K\left[z_{i}\right], i \in[1 n]\right\}$ is a base of $A$.

Now we show that there exists a subbase of $\operatorname{End} A$ that is not its base.
Example 2.34. Let $\varphi_{i j}: K\left[x_{1}, x_{2}\right] \rightarrow K\left[x_{1}, x_{2}\right]$, where $i, j \in\{1,2\}$, be endomorphisms of the free associative-commutative algebra $A=K\left[x_{1}, x_{2}\right]$ such that

$$
\begin{align*}
& \varphi_{11}\left(x_{1}\right)=x_{1}+x_{1} x_{2}, \varphi_{11}\left(x_{2}\right)=0, \varphi_{22}\left(x_{1}\right)=0, \varphi_{22}\left(x_{2}\right)=x_{2}  \tag{2.5}\\
& \varphi_{12}\left(x_{1}\right)=0, \varphi_{12}\left(x_{2}\right)=x_{1}+x_{1} x_{2}, \varphi_{21}\left(x_{1}\right)=x_{1}, \varphi_{21}\left(x_{2}\right)=0
\end{align*}
$$

It is easy to see that $\operatorname{rk}\left(\varphi_{i j}\right)=1$ and $\varphi_{i j} \varphi_{k m}=\delta_{j k} \varphi_{i m}$ for any $i, j, k, m \in\{1,2\}$, i.e., the set of endomorphisms $B_{\varphi}=\left\{\varphi_{i j} \mid \varphi_{i j} \in \operatorname{End} A, i, j \in\{1,2\}\right\}$ is a subbase of the semigroup End $A$. We will prove that $B_{\varphi}$ is not its base. It is clear that $\varphi_{11}(A)=K[u]$, where $u=x_{1}+x_{1} x_{2}$, and $\varphi_{22}(A)=K\left[x_{1}\right]$. We can take $z_{1}=u$ and $z_{2}=x_{1}$. The elements $z_{1}$ and $z_{2}$ generate the algebra $K\left[x_{1}+x_{1} x_{2}, x_{1}\right]$. Let us show that $K\left[x_{1}+x_{1} x_{2}, x_{2}\right] \neq K\left[x_{1}, x_{2}\right]$. If, on the contrary, $K\left[x_{1}+x_{1}, x_{2}, x_{2}\right]=K\left[x_{1}, x_{2}\right]$ then $x_{1}=\alpha\left(x_{1}+x_{1} x_{2}\right)+\beta x_{2}+P\left(u, x_{2}\right)$, where $\operatorname{deg} P\left(u, x_{2}\right) \geq 2$ and $\alpha, \beta \in K$. Hence $\beta=0, \alpha=1$ and $P\left(u, x_{2}\right)=0$. We come to a contradiction. Therefore, the subbase $B_{\varphi}$ is not a base of End $A$.
"Internal" definition of a base collection of End $A$ is a bit tricky (see [11, [9). It was inspired by G.Zhitomirski (see [22]).
Definition 2.35. ("Internal" definition of a base collection of End A.) The subbase of endomorphisms $\mathcal{B}_{e}=\left\{e_{i j}^{\prime} \mid e_{i j}^{\prime} \in\right.$ End $\left.A, i, j \in[1 n]\right\}$ of End $A$ is its base if for any collection of endomorphisms $\alpha_{i}: A \rightarrow A, \forall i \in[1 n]$, and any subbase $\mathcal{B}_{f}=\left\{f_{i j}^{\prime} \mid\right.$ $i, j \in[1 n]\}$ of End $A$ there exist endomorphisms $\varphi, \psi \in \operatorname{End} A$ such that

$$
\begin{equation*}
\alpha_{i} \circ f_{i i}^{\prime}=\psi \circ e_{i i}^{\prime} \circ \varphi, \text { for all } i \in[1 n] \tag{2.6}
\end{equation*}
$$

Our aim is to prove statement similar to the proposition 2.27 in [5].
Proposition 2.36. Internal and external definitions of a base collection of End $A$ are equivalent.

Proof. Let a subbase of endomorphisms $\mathcal{B}_{e}$ be a base according Definition 2.33 Since $\operatorname{rk}\left(f_{i j}^{\prime}\right)=1, \forall i, j \in[1 n]$, there exist elements $y_{i j} \in A, i, j \in[1 n]$, such that $K \subset f_{i j}^{\prime}(A(X)) \subseteq K\left[y_{i j}\right]$ for all $i, j \in[1 n]$. Define endomorphisms $\psi$ and $\varphi$ of $A$ as follows:

$$
\varphi\left(x_{i}\right)=z_{i} \text { and } \psi\left(z_{i}\right)=\alpha_{i}\left(y_{i}\right), \text { for all } i \in[1 n]
$$

where $e_{i i}^{\prime}(A) \subseteq K\left[z_{i}\right], z_{i} \in A$, and $y_{i}=y_{i i}, \forall i \in[1 n]$. Since $Z=\left\langle z_{i}\right| z_{i} \in A, i \in$ $[1 n]\rangle$ is a base of $A$, the endomorphism $\psi$ is well-defined. Now it is easy to check that the condition (2.6) with the given $\varphi$ and $\psi$ is fulfilled.

Conversely, assume that the condition (2.6) is fulfilled for the subbase $\mathcal{B}_{e}$. Let us prove that $Z=\left\langle z_{i} \mid z_{i} \in A, i \in[1 n]\right\rangle$ is a base of $A$. Choosing $\alpha_{i}=e_{i i}$ and $f_{i j}^{\prime}=e_{i j}, i, j \in[1 n]$, in (2.6), we obtain

$$
e_{i i}=\psi \circ e_{i i}^{\prime} \circ \varphi,
$$

i.e., $\psi\left(e_{i i}^{\prime} \varphi\left(x_{i}\right)\right)=x_{i}$ for any $i \in[1 n]$. Denote by $t_{i}=e_{i i}^{\prime} \varphi\left(x_{i}\right)$. We have $\psi\left(t_{i}\right)=x_{i}$. Since $A$ is Hopfian, i.e., any surjective endomorphism of A into itself is isomorphism, the elements $t_{i}, i \in[1 n]$, form the base of $A$. By Corollary and Remark 2.22 2.32, $K \subset e_{i i}^{\prime}(A) \subseteq K\left[z_{i}\right]$. Therefore, there exists a non-scalar polynomial $\chi_{i}\left(z_{i}\right) \in K\left[z_{i}\right]$ such that $t_{i}=\chi_{i}\left(z_{i}\right)$. Since $t_{i}=\chi_{i}\left(z_{i}\right), i=1, \ldots, n$, forms the base of $A$, the elements $z_{i}, i=1, \ldots, n$, forms a base of $A$ as claimed.

Now we deduce
Corollary 2.37. Let $\Phi \in$ Aut End $A$ and $E$ be the subsemigroup of End $A$ generated by the Kronecker endomorphisms $e_{i j}, i, j \in[1 n]$ (see Definition [2.23). Then $\mathcal{C}=\left\{\Phi\left(e_{i j}\right) \mid i, j \in[1 n]\right\}$ is a base of End $A$.

Proof. Assume that $\operatorname{rk}\left(\Phi\left(e_{i j}\right)\right)=0$ for some $i, j \in[1 n]$. By Corollary 2.21, we obtain $\operatorname{rk}\left(e_{i j}\right)=0$. We arrived at a contradiction. Thus, $\operatorname{rk}\left(\Phi\left(e_{i j}\right)\right) \neq 0$. Since $\Phi\left(e_{i j}\right) \Phi\left(e_{k m}\right)=\delta_{j k} \Phi\left(e_{i m}\right)$, the set $\mathcal{C}$ is a subbase of End $A$. It is easy to check that the condition (2.6) is fulfilled for the subbase $\mathcal{C}$. Thus, $\mathcal{C}$ is a base of End $A$.

Lemma 2.38. Let $\mathcal{B}_{e}=\left\{e_{i j}^{\prime} \mid e_{i j}^{\prime} \in \operatorname{End} A, i, j \in[1 n]\right\}$ be a base collection of endomorphisms of End $A$. Then there exists a base $Z^{\prime}=\left\{z_{k}^{\prime} \mid z_{k}^{\prime} \in A, k \in[1 n]\right\}$ of $A$ such that the endomorphisms $e_{i j}^{\prime}$ from $\mathcal{B}_{e}$ are Kronecker ones of $A$ in $Z^{\prime}$.

Proof. With the preceding notation from Definition 2.33 we have that the equality $\left(e_{i i}^{\prime}\right)^{2}=e_{i i}^{\prime}$ implies $e_{i i}^{\prime}\left(z_{i}\right)=z_{i}, i \in[1 n]$. Since $e_{i i}^{\prime} e_{i j}^{\prime}\left(z_{j}\right)=e_{i j}^{\prime}\left(z_{j}\right)$ and $K \subset e_{i i}^{\prime}(A) \subseteq K\left[z_{i}\right]$, there exists a non-scalar polynomial $f_{j}\left(z_{i}\right) \in K\left[z_{i}\right]$ such that $e_{i j}^{\prime}\left(z_{j}\right)=f_{j}\left(z_{i}\right)$. Similarly, there exists a non-scalar polynomial $g_{i}\left(z_{j}\right) \in K\left[z_{j}\right]$ such that $e_{j i}^{\prime}\left(z_{i}\right)=g_{i}\left(z_{j}\right)$. We have

$$
z_{j}=e_{j j}^{\prime}\left(z_{j}\right)=e_{j i}^{\prime} e_{i j}^{\prime}\left(z_{j}\right)=e_{j i}^{\prime}\left(f_{j}\left(z_{i}\right)\right)=f_{j}\left(g_{i}\left(z_{j}\right)\right) \text { for all } i, j \in[1 n]
$$

and, in a similar way, $z_{i}=g_{i}\left(f_{j}\left(z_{i}\right)\right)$ for all $i, j \in[1 n]$. Thus $f_{j}$ and $g_{i}$ are linear polynomials over $K$ in variables $z_{i}$ and $z_{j}$, respectively. Therefore,

$$
\begin{equation*}
e_{i j}^{\prime}\left(z_{j}\right)=a_{i} z_{i}+b_{i}, a_{i}, b_{i} \in K \text { and } a_{i} \neq 0 \tag{2.7}
\end{equation*}
$$

Note that $e_{i j}^{\prime}\left(z_{k}\right)=e_{i j}^{\prime}\left(e_{k k}^{\prime}\left(z_{k}\right)=0\right.$ if $k \neq j$. Now we have for $i \neq j$

$$
0=e_{i j}^{\prime}{ }^{2}\left(z_{j}\right)=e_{i j}^{\prime}\left(a_{i} z_{i}+b_{i}\right)=e_{i j}^{\prime}\left(b_{i}\right)=b_{i}
$$

i.e., $e_{i j}^{\prime}\left(z_{j}\right)=a_{i} z_{i}, a_{i} \neq 0$. Let $z_{i}^{\prime}=a_{i}^{-1} z_{i}$. We obtain a base $Z=\left\{z_{k}^{\prime} \mid z_{k}^{\prime} \in\right.$ $A, k \in[1 n]\}$ of $A$ such that $e_{i j}^{\prime}\left(z_{k}^{\prime}\right)=\delta_{j k} z_{k}^{\prime}, i, j, k \in[1 n]$, i.e., $e_{i j}^{\prime}$ are Kronecker endomorphisms of $A$ in the base $Z^{\prime}$. The proof is completed.

## 3. Automorphisms of the semigroup End $A$

3.1. On the group Aut End $A$. We need the following notion.

Definition 3.1. ([7]) Let $A_{1}$ and $A_{2}$ be algebras over $K$ from a variety $\mathcal{A}, \delta$ be an automorphism of $K$ and $\varphi: A_{1} \rightarrow A_{2}$ be a ring homomorphism of these algebras. A pair $(\delta, \varphi)$ is called a semi-linear homomorphism from $A_{1}$ to $A_{2}$ if

$$
\varphi(\alpha \cdot u)=\delta(\alpha) \cdot \varphi(u), \quad \forall \alpha \in K, \forall u \in A_{1}
$$

Definition 3.2. 17] An automorphism $\Phi$ of the semigroup End $A$ of endomorphisms of $A$ is called quasi-inner if there exists an adjoined bijection $s: A \rightarrow A$ such that $\Phi(\nu)=s \nu s^{-1}$, for any $\nu \in \operatorname{End} A$
Definition 3.3. 17 A quasi-inner automorphism $\Phi$ of End $A$ is called semi-inner if there exists a field automorphism $\delta: K \rightarrow K$ such that $(\delta, s)$ is a semi-linear automorphism of $A$, i.e., for any $\alpha \in K$ and $a, b \in A$ the following conditions hold:

1. $s(a+b)=s(a)+s(b)$,
2. $s(a \cdot b)=s(a) \cdot s(b)$,
3. $s(\alpha a)=\delta(\alpha) s(a)$.

We say that the pair $(\delta, s)$ defines the semi-inner automorphism $\Phi$ of $A$ with the adjoined ring automorphism $s$. If $\delta$ is the identity automorphism of $K$, we call the automorphism $\Phi$ inner.

The description of quasi-inner automorphisms of End $A$ is as follows.
Proposition 3.4. [3, 9, 11 Let $\Phi \in$ Aut End $A$ be a quasi-inner automorphism of End $A$. Then $\Phi$ is of semi-inner automorphisms of End $A$.

We will use the following fact:
Proposition 3.5. 9, 11 Let $\Phi \in$ Aut End $A$ and $E$ be the subsemigroup of End $A$ generated by $e_{i j}, i, j \in[1 n]$. Elements of the semigroup $\Phi(E)$ are Kronecker endomorphisms of $A$ in some base $U=\left\{u_{1}, \ldots, u_{n}\right\}, u_{i} \in A$, if and only if $\Phi$ is a quasi-inner automorphism of End $A$.

Now we obtain one of the main result of the paper
Theorem 3.6. Every automorphism of the group Aut End $A$ is semi-inner.
Proof. By Corollary 2.37, the set of endomorphisms $\mathcal{C}=\left\{\Phi\left(e_{i j}\right) \mid \forall i \in[1 n]\right\}$ is a base collection of endomorphisms of $A$. By Lemma 2.38, there exists a base $S=\left\langle s_{k} \mid s_{k} \in A, k \in[1 n]\right\rangle$ such that the endomorphisms $\Phi\left(e_{i j}\right)$ are Kronecker endomorphisms in $S$. According to Proposition 3.5, we obtain that $\Phi$ is quasiinner. By virtue of Proposition [3.4, every automorphism the group Aut End $A$ is semi-inner and as claimed.

Remark 3.7. If $\mathcal{C A}$ is the category of commutative-associative algebras over a field $K$, we take $\mathcal{S C A}$ to be the category with objects all associative algebras from the category $\mathcal{A}$, morphisms all pairs $\psi_{\delta}=(\psi, \delta): A \rightarrow B, A, B \in \mathrm{Ob} \mathcal{S} \mathcal{A}$, such that $\psi: A \rightarrow B$ are ring homomorphisms from $A$ to $B, \delta: K \rightarrow K$ are automorphisms of the field $K$ and $\psi_{\delta}(\lambda a)=\lambda^{\delta} \psi(a), a \in A$. Morphisms $\psi_{\delta}$ of the category $\mathcal{S} \mathcal{A}$ are called semi-linear homomorphisms(or semihomomorphisms) from $A$ to $B$ (cf. Definition 3.1). Denote by SEnd $A$ the semigroup of semiendomorphisms of $A$ with the usual composition of maps in the category $\mathcal{S C} \mathcal{A}$.

Clearly, that the definitions of endomorphisms of rank one and zero can be transfer to the category $\mathcal{S C A}$. All results about bases and subbases from the sections 2.3 are also true. As a consequence, we obtain the following

Theorem 3.8. Every automorphism of the group Aut SEnd $A$ is semi-inner.

## 4. Automorphisms of the category $\mathcal{A}^{\circ}$

Recall the following notions of the category isomorphism and equivalence (cf. [12]). An isomorphism $\varphi: \mathcal{C} \rightarrow \mathcal{M}$ of categories is a functor $\varphi$ from $\mathcal{C}$ to $\mathcal{M}$, which is a bijection both on objects and morphisms. In other words, there exists a functor $\psi: \mathcal{M} \rightarrow \mathcal{C}$ such that $\psi \varphi=1_{\mathcal{C}}$ and $\varphi \psi=1_{\mathcal{M}}$.

Let $\varphi_{1}$ and $\varphi_{2}$ be two functors from $\mathcal{C}_{1}$ to $\mathcal{C}_{2}$. A functor isomorphism $s: \varphi_{1} \longrightarrow$ $\varphi_{2}$ is a collection of isomorphisms $s_{D}: \varphi_{1}(D) \longrightarrow \varphi_{2}(D)$ defined for all $D \in \operatorname{Ob} \mathcal{C}_{1}$ such that for every $\nu: D \longrightarrow B, \nu \in \operatorname{Mor} \mathcal{C}_{1}, B \in \operatorname{Ob} \mathcal{C}_{1}$

$$
s_{B} \cdot \varphi_{1}(\nu)=\varphi_{2}(\nu) \cdot s_{D}
$$

holds, i.e., the following diagram

is commutative. An isomorphism of functors $\varphi_{1}$ and $\varphi_{2}$ is denoted by $\varphi_{1} \cong \varphi_{2}$.
An equivalence of categories $\mathcal{C}$ and $\mathcal{M}$ is a pair of functors $\varphi: \mathcal{C} \rightarrow \mathcal{M}$ and $\psi: \mathcal{M} \rightarrow \mathcal{C}$ such that $\psi \varphi \cong 1_{\mathcal{C}}$ and $\varphi \psi \cong 1_{\mathcal{M}}$. If $\mathcal{C}=\mathcal{M}$, then we get the notions of automorphism and autoequivalence of the category $\mathcal{C}$.

For every small category $\mathcal{C}$, denote the group of all its automorphisms by Aut $\mathcal{C}$. We distinguish the following classes of automorphisms of $\mathcal{C}$.

Definition 4.1. [8, 15] An automorphism $\varphi: \mathcal{C} \rightarrow \mathcal{C}$ is equinumerous if $\varphi(D) \cong D$ for any object $D \in \mathrm{Ob} \mathcal{C} ; \varphi$ is stable if $\varphi(D)=D$ for any object $D \in \mathrm{Ob} \mathcal{C}$; and $\varphi$ is inner if $\varphi$ and $1_{\mathcal{C}}$ are naturally isomorphic, i.e., $\varphi \cong 1_{\mathcal{C}}$.

In other words, an automorphism $\varphi$ is inner if for all $D \in \operatorname{Ob\mathcal {C}}$ there exists an isomorphism $s_{D}: A \rightarrow \varphi(D)$ such that

$$
\varphi(\nu)=s_{B} \nu s_{D}^{-1}: \varphi(D) \rightarrow \varphi(B)
$$

for any morphism $\nu \in \operatorname{Mor}_{\mathcal{C}}(A, B)$.
Denote by EqnAut $\mathcal{C}, \operatorname{StAut} \mathcal{C}$, and $\operatorname{Int} \mathcal{C}$ the collections of equinumerous, stable, and inner automorphisms of the group Aut $\mathcal{C}$, respectively.

Let $\Theta$ be a variety of linear algebras over $K$. Denote by $\Theta^{0}$ the full subcategory of finitely generated free algebras $F(X),|X|<\infty$, of the variety $\Theta$. Consider a constant morphism $\nu_{0}: F(X) \rightarrow F(X)$ such that $\nu_{0}(x)=x_{0}, x_{0} \in F(X)$, for every $x \in X$.

Theorem 4.2. (Reduction Theorem [8, 13, 16, 22]) Let the free algebra $F(X)$ generate a variety $\Theta$, and $\varphi \in \operatorname{StAut} \Theta^{0}$. If $\varphi$ acts trivially on the monoid $\operatorname{Mor}_{\Theta^{0}}\left(F(X), F(X)\right.$ ) and $\varphi\left(\nu_{0}\right)=\nu_{0}$, then $\varphi$ is inner, i.e., $\varphi \in \operatorname{Int} \Theta^{0}$.

Define the notion of a semi-inner automorphism of the category $\Theta^{0}$ of free finitely generated algebras in the category $\Theta$.

Definition 4.3. [15] An automorphism $\varphi \in \operatorname{Aut} \Theta^{0}$ is called semi-inner if there exists a family of semi-isomorphisms $\left\{s_{F(X)}=(\delta, \tilde{\varphi}): F(X) \rightarrow \tilde{\varphi}(F(X)), F(X) \in\right.$ $\left.\mathrm{Ob} \Theta^{0}\right\}$, where $\delta \in$ Aut $K$ and $\tilde{\varphi}$ is a ring isomorphism from $F(X)$ to $\tilde{\varphi}(F(X))$ such that for any homomorphism $\nu: F(X) \longrightarrow F(Y)$ the following diagram

is commutative.
Further, we will need the following
Proposition 4.4. [8, 15] For any equinumerous automorphism $\varphi \in \operatorname{Aut} \mathcal{C}$ there exist a stable automorphism $\varphi_{S}$ and an inner automorphism $\varphi_{I}$ of the category $\mathcal{C}$ such that $\varphi=\varphi_{S} \varphi_{I}$.

Now we give a description of the groups Aut $\mathcal{C} \mathcal{A}^{\circ}$ over any field. Note that a description of this group over infinite fields was given in [2]

Theorem 4.5. All automorphisms of the group Aut $\mathcal{A}^{\circ}$ of automorphisms of the category $\mathcal{C} \mathcal{A}^{\circ}$ are semi-inner automorphisms of the category $\mathcal{C} \mathcal{A}^{\circ}$.

Proof. Let $\varphi \in \operatorname{Aut} \mathcal{A}^{\circ}$. It is clear that $\varphi$ is an equinumerous automorphism. By Proposition 4.4 $\varphi$ can be represented as a composition of a stable automorphism $\varphi_{S}$ and an inner automorphism $\varphi_{I}$. Since stable automorphisms does not change free algebras from $\mathcal{A}^{\circ}$, we obtain that $\varphi_{S} \in$ Aut End $A$. By Theorem 3.6, $\varphi_{S}$ is semi-inner ofEnd $A$. Using this fact and Reduction Theorem4.2, we obtain that all automorphisms of the group $\operatorname{Aut} \mathcal{C} \mathcal{A}^{\circ}$ are semi-inner automorphisms of the category $\mathcal{C} \mathcal{A}^{\circ}$. This completes the proof.
Problem 4.6. Describe the groups Aut $\mathcal{B}^{\circ}$ and Aut End $B$, where $B=B\left(x_{1}, \ldots, x_{n}\right)$, is a free algebra of a non-associative variety $\mathcal{B}$ of linear algebras finitely generated by a set $X=\left\{x_{1}, \ldots, x_{n}\right\}$.

Note that the above mentioned groups were described for some homogeneous varieties of linear algebras in [5, 9, 11]. In particular, a description of these group for the variety of all Lie algebras over any field was obtained there. A corresponding description in the case of Lie algebras over any infinite field was obtained in [15, 22].

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