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Representability and Specht problem for G-graded algebras ☆

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Abstract

Let W be an associative PI-algebra over a field F of characteristic zero, graded by a finite group G. Let $\mathrm{id}_G(W)$ denote the T-ideal of G-graded identities of W. We prove: 1. [G-graded PI-equivalence] There exists a field extension K of F and a finite-dimensional $\mathbb{Z}/2\mathbb{Z} \times G$-graded algebra A over K such that $\mathrm{id}_G(W) = \mathrm{id}_G(A^*)$ where A^* is the Grassmann envelope of A. 2. [G-graded Specht problem] The T-ideal $\mathrm{id}_G(W)$ is finitely generated as a T-ideal. 3. [G-graded PI-equivalence for affine algebras] Let W be a G-graded affine algebra over F. Then there exists a field extension K of F and a finite-dimensional algebra A over K such that $\mathrm{id}_G(W) = \mathrm{id}_G(A)$.

Keywords: Graded algebra; Polynomial identity

1. Introduction

The Specht problem (see [44]) is considered as one of the main problems in the theory of algebras satisfying polynomial identities. The (generalized) Specht problem asks whether for a

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given class of algebras (associative, Lie, Jordan, superalgebras, etc.), the corresponding T-ideals of identities are finitely based (i.e. finitely generated as a T-ideal). For associative algebras over fields of characteristic zero we refer the reader to [25,26,30,38,42]. For Lie algebras the reader is referred to [14,22,28,29,48] whereas for alternative and Jordan algebras the reader is referred to [20,21,31,46]. As for applications of "Specht type problems" in other topics we refer the reader to [50] (in pro-p groups), [24,43,49] (in superalgebras), [13,27,34–36,39,51] (in invariant theory and the theory of representations) and [32,33] (in noncommutative geometry). For more comprehensive expositions on polynomial identities the reader is referred to [5,23,15,16,18,30, 37,40].

Polynomial identities were also studied in the context of G-graded algebras (again, associative, Lie, Jordan). Here we refer the reader to [2,3,6–8,47]. More generally one may consider polynomial identities for H-comodule algebras (see [4,11]) and use them to construct versal objects which specialize into k-forms (in the sense of "Galois descent") of a given H-comodule algebra over the algebraic closure of k. H-comodule algebras may be viewed as the noncommutative analogues of principal fibre bundles where H plays the role of the structural group (see [12,19,41]).

One of the main results of this paper is a solution to the Specht problem for G-graded *PI*-algebras over a field of characteristic zero where the group G is finite.

Let W be an associative *PI*-algebra over a field F of characteristic zero. Assume $W = \bigoplus_{g\in G} W_g$ is G-graded where $G = \{g_1 = e, g_2, \ldots, g_r\}$ is a finite group. For every $g \in G$ let $X_g = \{x_{1,g}, x_{2,g}, \ldots\}$ be a countable set of variables of degree g and let $\Omega_{F,G} = F\langle\{X_{g_1}, \ldots, X_{g_r}\}\rangle$ be the free G-graded algebra on these variables (the G-degree of $x_{i_1,g_{i_1}} x_{i_2,g_{i_2}} \cdots x_{i_k,g_{i_k}}$ is the element $g_{i_1} g_{i_2} \cdots g_{i_k} \in G$). We refer to the elements of $\Omega_{F,G}$ as graded polynomial or G-graded polynomials. An evaluation of a graded polynomial $f \in \Omega_{F,G}$ on W is *admissible* if the variables $x_{i,g}$ of f are substituted (only) by elements $\widehat{x_{i,g}} \in W_g$. A graded polynomial f is a *graded identity* of W if f vanishes upon any admissible evaluation on W. Let $\mathrm{id}_G(W) \leqslant \Omega_{F,G}$ be the T-ideal of G-graded identities of W (an ideal I of $\Omega_{F,G}$ is a T-ideal if it is closed under all G-graded endomorphisms of $\Omega_{F,G}$). As in the classical case, also here, the T-ideal of identities is generated by multilinear polynomials. Moreover, we can assume the identities are *strongly homogeneous*, that is every monomial in f has the same G-degree. In order to state our main results we consider first the affine case. Let W be a *PI* G-graded affine algebra.

Theorem 1.1 *(G-graded PI-equivalence-affine). There exists a field extension K of F and a finite-dimensional G-graded algebra A over K such that* $\mathrm{id}_G(W) = \mathrm{id}_G(A)$ *(in $\Omega_{F,G}$).*

Theorem 1.2 *(G-graded Specht problem-affine). The ideal* $\mathrm{id}_G(W)$ *is finitely generated as a T-ideal.*

In order to state the results for arbitrary G-graded algebras (i.e. not necessarily affine) recall that the *Grassmann algebra E* over an unspecified infinite-dimensional K-vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra where the components of degree zero and one, denoted by E_0 and E_1, are spanned by products of even and odd number of vectors respectively. For any $\mathbb{Z}/2\mathbb{Z}$-graded algebra $B = B_0 \oplus B_1$ over K, we let $B^* = B_0 \otimes_K E_0 \oplus B_1 \otimes_K E_1$ be the *Grassmann envelope* of B. If the algebra B has an additional (compatible) G-grading, that is B is $\mathbb{Z}/2\mathbb{Z} \times G$-graded, then B_0 and B_1 are G-graded and we obtain a natural G-grading on B^*.

Following Kemer's approach (see [24]), the results above for G-graded affine algebras together with a general result of Berele and Bergen in [11] give:

Theorem 1.3 *(G-graded PI-equivalence). Let W be a PI G-graded algebra over F. Then there exists a field extension K of F and a finite-dimensional $\mathbb{Z}/2\mathbb{Z} \times G$-graded algebra A over K such that* $\mathrm{id}_G(W) = \mathrm{id}_G(A^*)$ *where A^* is the Grassmann envelope of A.*

Theorem 1.4 *(G-graded Specht problem). The ideal* $\mathrm{id}_G(W)$ *is finitely generated as a T-ideal.*

In the last section of the paper we show how to "pass" from the affine case to the general case using Berele and Bergen result.

Note that the G-graded algebra W mentioned above (affine or non-affine) is assumed to be (ungraded) *PI*, i.e. it satisfies an ungraded polynomial identity (an algebra may be G-graded *PI* even if it is not *PI*; for instance take W a free algebra over a field F generated by two or more indeterminates with trivial G-grading where $G \neq \{e\}$ (i.e. $W_g = 0$ for $g \neq \{e\}$)). Note that Theorems 1.1 and 1.3 are false if W is not *PI*. The G-graded Specht problem remains open for G-graded *PI* but non-*PI*-algebras (affine or non-affine).

Remark 1.5. All algebras considered in this paper are algebras over a fixed field F of characteristic zero. Some of the algebras will be finite-dimensional over field extensions of F. Whenever we say that "there exists a finite-dimensional algebra A such that..." (without specifying the field over which this occurs) we mean that "there exists a field extension K of F and a finite-dimensional algebra A over K such that...". Since for fields of characteristic zero (see, e.g., [18]) $\mathrm{id}_G(W) = \mathrm{id}_G(W \otimes_F L)$ (in $\Omega_{F,G}$) where L is any field extension of F it is easy to see (and well known) that for the proofs of the main theorems of the paper we can (and will) always assume that the field F (as well as its extensions) is algebraically closed. It is convenient to do so since over algebraically closed fields it is easier to describe the possible structures of G-graded, finite-dimensional simple algebras.

In our exposition we will follow (at least partially) the general idea of the proofs in the ungraded case as they appear in [23]. In the final steps of the proof of Theorem 1.1 we apply the Zubrilin–Razmyslov identity, an approach which is substantially different from the exposition in [23]. It should be mentioned however that the authors of [23] hint that the Zubrilin–Razmyslov identity could be used to finalize the proof. In addition to the above mentioned difference, there are several substantial obstacles which should be overcome when generalizing from the ungraded case to the G-graded case (and especially to the case where the group G is non-abelian). Let us mention here the main steps of the proof.

As in the ungraded case, we "approximate" the T-ideal $\mathrm{id}_G(W)$ by $\mathrm{id}_G(A) \subseteq \mathrm{id}_G(W)$ where A is a G-graded finite-dimensional algebra. Then by induction we get "closer" to $\mathrm{id}_G(W)$. The first step is therefore the statement which "allows the induction to get started" namely showing the existence of a G-graded finite-dimensional algebra A with $\mathrm{id}_G(A) \subseteq \mathrm{id}_G(W)$. We point out that already in this step we need to assume that G is finite.

In order to apply induction we represent the T-ideal Γ by a certain finite set of parameters $\{(\alpha, s)\} \subset (\mathbb{Z}^{\geqslant 0})^r \times \mathbb{Z}^{\geqslant 0}$ which we call *Kemer points* (here $\mathbb{Z}^{\geqslant 0}$ denotes the set of non-negative integers). To each Kemer point (α, s) we attach a certain set of polynomials, called *Kemer polynomials*, which are *outside* Γ. These polynomials are G-graded, multilinear and have alternating sets of cardinalities as prescribed by the point (α, s). This is the point where our proof differs substantially from the ungraded case and moreover where the noncommutativity of the group G comes into play. The alternation of G-graded variables where G is non-abelian yields monomials which belong to different G-graded components. This basic fact led us to consider alternating sets which are homogeneous, i.e. of variables that correspond to the same g-component.

As mentioned above, the Kemer polynomials that correspond to the point (α, s) do not belong to the T-ideal Γ and hence, if we add them to Γ we obtain a larger T-ideal Γ' of which (α, s) is not a Kemer point. Consequently the Kemer points of Γ' are "smaller" compared to those of Γ (with respect to a certain ordering). By induction, there is a G-graded finite-dimensional algebra A' with $\Gamma' = \mathrm{id}_G(A')$.

Let us sketch the rest of proof of Theorem 1.1. Let $\mathcal{W}_\Gamma = F\langle\{X_{g_1}, \ldots, X_{g_r}\}\rangle/\Gamma$ be the relatively free algebra of the ideal Γ. Clearly $\mathrm{id}_G(\mathcal{W}_\Gamma) = \Gamma$. We construct a representable algebra $B_{(\alpha,s)}$ (i.e. an algebra which can be G-graded embedded in a G-graded matrix algebra over a large enough field K) which is on one hand a G-graded homomorphic image of the relatively free algebra $\mathcal{W}_\Gamma$ and on the other hand the ideal I in $\mathcal{W}_\Gamma$, generated by the Kemer polynomials that correspond to the point (α, s), is mapped isomorphically. Then we conclude that $\Gamma = \mathrm{id}_G(B_{(\alpha,s)} \oplus A')$.

The exposition above does not reveal a fundamental feature of the proof. One is able to prove, using Zubrilin–Razmyslov theory, that elements of I which correspond to (rather than generated by) Kemer polynomials are mapped isomorphically into $B_{(\alpha,s)}$. Clearly this is not sufficient. In order to show that the "entire" ideal I in $\mathcal{W}_\Gamma$ is mapped isomorphically one needs to show that any non-zero element of I generates another element of I which corresponds to a Kemer polynomial. This is the so-called *Phoenix property*. It is fair to say that a big part (if not the main part) of the proof of Theorem 1.1 is devoted to the proof of the Phoenix property of Kemer polynomials. This is achieved by establishing a fundamental connection between Kemer points, Kemer polynomials and the structure of G-graded finite-dimensional algebras. Here we use a key result of Bahturin, Sehgal and Zaicev in which they fully describe the structure of G-graded, finite-dimensional G-simple algebras in terms of fine and elementary gradings (see [7] and Theorem 4.3 below).

Remark 1.6. This is a second place where the noncommutativity of the group G comes into play. It is not difficult to show that if G is abelian then a G-graded, finite-dimensional simple algebra (over an algebraically closed field F of characteristic zero) is the direct product of matrix algebras of the *same degree*. This is not the case in general (although not impossible) if G is non-abelian.

After completing the proof of Theorem 1.1 we turn to the proof of Theorem 1.2 (Specht problem). This is again based on the above mentioned result of Bahturin, Sehgal and Zaicev. The main point is that one can deduce from their result that if F is algebraically closed then the number of non-isomorphic G-gradings which can be defined on a given semisimple algebra A is finite. Interestingly, this is in contrast to the case where the "grading" is given by other type of Hopf algebras (see [1,4]). For instance if H is the Sweedler algebra of dimension 4 over the field of complex numbers, then there exist infinitely many non-isomorphic H-comodule structures on $M_2(\mathbb{C})$.

Remark 1.7. It is important to mention that the proof of Theorem 1.1 (as, in fact, all proofs known to us of "Specht type problems") can be viewed as an applications of the Grothendieck approach to noncommutative polynomials. Indeed one has to translate properties of finite-dimensional algebras B (dimension of the g-homogeneous component of the semisimple part of B, index of nilpotency of the radical $J(B)$) which we call "geometric" to a "functional" or "combinatorial" language of polynomial identities (see Sections 3, 4, 5, 6, Appendix A and Theorems 11.2, 13.1).

Remark 1.8. As in the ungraded case also here the solution of the Specht problem does not yield explicit generating sets of the T-ideals of identities. Nevertheless in some special cases such generating sets were found and in particular the Specht problem was solved (see [3,6,47]).

Remark 1.9 *(Codimension growth).* In [2] the codimension growth of G-graded algebras where G is a finite *abelian* group was considered. It is proved that if A is a G-graded finite-dimensional algebra then

$$\exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$$

exists and is an integer. Here $c_n^G(A)$ denotes the dimension of the subspace of multilinear elements in n free generators in the relatively free G-graded algebra of A.

Applying Theorem 1.1 we obtain:

Corollary 1.10. *Let W be a PI, G-graded affine algebra where G is a finite abelian group. Then*

$$\exp^G(W) = \lim_{n\to\infty} \sqrt[n]{c_n^G(W)}$$

exists and is an integer.

Before embarking into the proofs (Sections 2–13) the reader is advised to read Appendix A in which we present some of the basic ideas which show the connection between the structure of polynomial and finite-dimensional algebras. These ideas are fundamental and being used along the entire paper for G-graded algebras. For simplicity, in Appendix A, we present them for ungraded algebras. Of course the reader who is familiar with Kemer's proof of the Specht problem may skip the reading of Appendix A.

Remark 1.11. It came to our attention that Irina Sviridova recently obtained similar results in case the grading group G is finite *abelian*. Sviridova' results and the results of this paper were obtained independently.

2. Getting started

In this short section we show that $\mathrm{id}_G(W)$ contains the T-ideal of identities of a finite-dimensional graded algebra.

Proposition 2.1. *Let G be a finite group and W a G-graded affine algebra. Assume W is (ungraded) PI. Then there exists a finite-dimensional G-graded algebra A_G with* $\mathrm{id}_G(A_G) \subset \mathrm{id}_G(W)$.

Proof. From the classical theory (see [23, Corollary 4.9]) we know that there exists a finite-dimensional algebra A such that $\mathrm{id}(A) \subset \mathrm{id}(W)$. Consider the algebra $A_G = A \otimes_F FG$, G-graded via FG.

Claim 2.2. $\mathrm{id}_G(A_G) \subset \mathrm{id}_G(W)$.

To see this let $f(\{x_{i,g}\}) \in \mathrm{id}_G(A_G)$ be a multilinear polynomial. As mentioned in Section 1 we can assume that $f = f_{\widehat{g}}$ is $\widehat{g}$-strongly homogeneous (i.e. consisting of monomials with total degree $\widehat{g} \in G$). Now observe that if we ignore the G-degree of each variable in $f_{\widehat{g}}$ we obtain an identity of A. This is easily seen since any G-graded evaluation of $f(\{x_{i,g}\})$ in $A_G = A \otimes_F FG$ has the form $f(\{a_i\}) \otimes F\widehat{g}$, where $\{a_i\} \subset A$.

But $\mathrm{id}(A) \subset \mathrm{id}(W)$ and since any G-graded polynomial which is an ordinary identity of W (by ignoring the G-degrees of the variables) is also a G-graded identity of W the claim follows. Finally, since the group G is finite, the G-graded algebra A_G is finite-dimensional and the proposition is proved. $\square$

Remark 2.3.
This is the point where we need to assume that the grading group G is finite. We do not know whether the statement holds for infinite groups.

3. The index of G-graded T-ideals

Let Γ be a G-graded T-ideal. As noted in Section 1 since the field F is of characteristic zero the T-ideal Γ is generated by multilinear graded polynomials which are strongly homogeneous.

Definition 3.1. Let $f(X_G) = f(\{x_{i,g}\})$ be a multilinear G-graded polynomial which is strongly homogeneous. Let $g \in G$ and let X_g be the set of all g-variables in X_G. Let $S_g = \{x_{1,g}, x_{2,g}, \dots, x_{m,g}\}$ be a subset of X_g and let $Y_G = X_G \setminus S_g$ be the set of the remaining variables. We say that $f(X_G)$ is *alternating in the set* S_g (or that the variables of S_g *alternate in* $f(X_G)$) if there exists a (multilinear, strongly homogeneous) G-graded polynomial $h(S_g; Y_G) = h(x_{1,g}, x_{2,g}, \dots, x_{m,g}; Y_G)$ such that

$$f(x_{1,g}, x_{2,g}, \dots, x_{m,g}; Y_G) = \sum_{\sigma \in \mathrm{Sym}(m)} \mathrm{sgn}(\sigma) h(x_{\sigma(1),g}, x_{\sigma(2),g}, \dots, x_{\sigma(m),g}; Y_G).$$

Following the notation in [23], if $S_{g_{i_1}}, S_{g_{i_2}}, \dots, S_{g_{i_p}}$ are p disjoint sets of variables of X_G (where $S_{g_{i_j}} \subset X_{g_{i_j}}$) we say that $f(X_G)$ is alternating in $S_{g_{i_1}}, S_{g_{i_2}}, \dots, S_{g_{i_p}}$ if $f(X_G)$ is alternating in each set $S_{g_{i_j}}$ (note that the polynomial $x_1 x_2 y_1 y_2 - x_2 x_1 y_2 y_1$ is not alternating in the x's nor in the y's).

As in the classical theory we will consider polynomials which alternate in ν disjoint sets of the form S_g for every $g \in G$. If for every g in G, the sets S_g have the same cardinality (say d_g) we will say that $f(X_G)$ is ν-fold $(d_{g_1}, d_{g_2}, \dots, d_{g_r})$-alternating. Further we will need to consider polynomials (again, in analogy to the classical case) which in addition to the alternating sets mentioned above it alternates in t disjoint sets $K_g \subset X_g$ (and also disjoint to the previous sets) such that $\mathrm{ord}(K_g) = d_g + 1$. Note that the g's in G that correspond to the K_g's need not be different.

We continue with the definition of the *graded index* of a T-ideal Γ. For this we need the notion of g-Capelli polynomial.

Let $X_{n,g} = \{x_{1,g}, \dots, x_{n,g}\}$ be a set of n variables of degree g and let $Y = \{y_1, \dots, y_n\}$ be a set of n ungraded variables.

Definition 3.2. The *g-Capelli polynomial* $c_{n,g}$ (of degree $2n$) is the polynomial obtained by alternating the set $x_{i,g}$'s in the monomial $x_{1,g}y_1x_{2,g}y_2\cdots x_{n,g}y_n$. That is

$$c_{n,g}=\sum_{\sigma\in \mathrm{Sym}(n)}\mathrm{sgn}(\sigma)x_{\sigma(1),g}y_1x_{\sigma(2),g}y_2\cdots x_{\sigma(n),g}y_n.$$

Remark 3.3. Note that a g-Capelli polynomial yields a set of G-graded polynomials by specifying degrees in G to the y's. So when we say that the g-Capelli polynomial $c_{n,g}$ is in Γ we mean that all the G-graded polynomials obtained from $c_{n,g}$ by substitutions of the form $y_i\mapsto y_{i,g}$, some $g\in G$, are in Γ.

Lemma 3.4. *For every $g\in G$, there exists an integer n_g such that the T-ideal Γ contains $c_{n_g,g}$.*

Proof. Let A_G be a finite-dimensional G-graded algebra with $\mathrm{id}_G(A_G)\subset\Gamma$. Let A_g be the g-homogeneous component of A_G and let $n_g=\dim(A_g)+1$. Clearly, $c_{n_g,g}$ is contained in $\mathrm{id}_G(A_G)$ and hence in Γ. □

Corollary 3.5. *If $f(X_G)=f(\{x_{i,g}\})$ is a multilinear G-graded polynomial, strongly homogeneous and alternating on a set S_g of cardinality n_g, then $f(X_G)\in\Gamma$. Consequently there exists an integer M_g which bounds (from above) the cardinality of the g-alternating sets in any G-graded polynomial h which is not in Γ.*

We can now define $\mathrm{Ind}(\Gamma)$, the index of Γ. It will consist of a finite set of points (α,s) in the lattice $\Lambda_\Gamma\times D_\Gamma\cong(\mathbb{Z}^{\geqslant 0})^r\times(\mathbb{Z}^{\geqslant 0}\cup\infty)$ where $\alpha\in\Lambda_\Gamma$ and $s\in D_\Gamma$. We first determine the set $\mathrm{Ind}(\Gamma)_0$, namely the projection of $\mathrm{Ind}(\Gamma)$ into Λ_Γ and then for each point $\alpha\in\mathrm{Ind}(\Gamma)_0$ we determine $s(\alpha)\in D_\Gamma$ so that $(\alpha,s(\alpha))\in\mathrm{Ind}(\Gamma)$. In particular if (α,s_1) and (α,s_2) are both in $\mathrm{Ind}(\Gamma)$ then $s_1=s_2$. Before defining these sets we introduce a partial order on $(\mathbb{Z}^{\geqslant 0})^r\times(\mathbb{Z}^{\geqslant 0}\cup\infty)$ starting with a partial order on $(\mathbb{Z}^{\geqslant 0})^r$: If $\alpha=(\alpha_1,\dots,\alpha_r)$ and $\beta=(\beta_1,\dots,\beta_r)$ are elements of $(\mathbb{Z}^{\geqslant 0})^r$ we put $(\alpha_1,\dots,\alpha_r)\preccurlyeq(\beta_1,\dots,\beta_r)$ if and only if $\alpha_i\leqslant\beta_i$ for $i=1,\dots,r$.

Next, for (α,s) and (β,s') in $(\mathbb{Z}^{\geqslant 0})^r\times(\mathbb{Z}^{\geqslant 0}\cup\infty)$ put $(\alpha,s)\preccurlyeq(\beta,s')$ if and only if either

(1) $\alpha\prec\beta$ (i.e. $\alpha\preccurlyeq\beta$ and for some j, $\alpha_j<\beta_j$), or
(2) $\alpha=\beta$ and $s\leqslant s'$. (By definition $s<\infty$ for every $s\in\mathbb{Z}^{\geqslant 0}$.)

Definition 3.6. A point $\alpha=(\alpha_{g_1},\dots,\alpha_{g_r})$ is in $\mathrm{Ind}(\Gamma)_0$ if and only if for any integer ν there exists a multilinear G-graded polynomial outside Γ with ν alternating g-sets of cardinality α_g for every g in G.

Lemma 3.7. *The set* $\mathrm{Ind}(\Gamma)_0$ *is bounded (finite). Moreover if $\alpha=(\alpha_{g_1},\dots,\alpha_{g_r})\in\mathrm{Ind}(\Gamma)_0$ then any $\alpha'\preccurlyeq\alpha$ is also in* $\mathrm{Ind}(\Gamma)_0$.

Proof. The first statement is a consequence of the fact that $\Gamma\supseteq\mathrm{id}_G(A)$ where A is a finite-dimensional G-graded algebra. The second statement follows at once from the definition of $\mathrm{Ind}(\Gamma)_0$. □

The important points in $\mathrm{Ind}(\Gamma)_0$ are the extremal ones.

Definition 3.8.

(1) A point α in $\mathrm{Ind}(\Gamma)_0$ is *extremal* if for any point β in $\mathrm{Ind}(\Gamma)_0$, $\beta \succcurlyeq \alpha \Rightarrow \beta = \alpha$. We denote by $E_0(\Gamma)$ the set of all extremal points in $\mathrm{Ind}(\Gamma)_0$.
(2) For any point α in $E_0(\Gamma)$ and every integer ν consider the set $\Omega_{\alpha,\nu}$ of all ν-folds alternating polynomials in g-sets of cardinality α_g for every g in G, which are not in Γ. For such polynomials f we consider the number $s_\Gamma(\alpha, \nu, f)$ of alternating g-homogeneous sets (any $g \in G$) of disjoint variables, of cardinality $\alpha_g + 1$. We claim that the set of numbers $\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$ is bounded: If not, there exists a sequence of polynomials $f_1, f_2, \ldots$ in $\Omega_{\alpha,\nu}$ such that $s_i = s_\Gamma(\alpha, \nu, f_i)$ and $\lim_{i\to\infty} s_i = \infty$. Now, since the group G is finite, by the pigeonhole principle, there exist $g \in G$ and a subsequence $f_{i_1}, f_{i_2}, \ldots$ such that $\lim_{k\to\infty} s_{i_k,g} = \infty$ where $s_{i_k,g}$ is the number of alternating g-homogeneous sets of cardinality $\alpha_g + 1$. But this means that the point α' determined by $\alpha'_g = \alpha_g + 1$ and $\alpha'_h = \alpha_h$ for $h \neq g$ is in $\mathrm{Ind}(\Gamma)_0$ and we get a contradiction to the maximality of α. This proves the claim. Let $s_\Gamma(\alpha, \nu) = \max\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$. Note that the sequence $s_\Gamma(\alpha, \nu)$ is monotonically decreasing as a function of ν and hence there exists an integer $\mu = \mu(\Gamma, \alpha)$ for which the sequence stabilizes, that is for $\nu \geqslant \mu$ the sequence $s_\Gamma(\alpha, \nu)$ is constant. We let $s(\alpha) = \lim_{\nu\to\infty}(s_\Gamma(\alpha, \nu)) = s_\Gamma(\alpha, \mu)$. At this point the integer μ depends on α. However since the set $E_0(\Gamma)$ is finite we take μ to be the maximum of all μ's considered above.
(3) We define now the set $\mathrm{Ind}(\Gamma)$ as the set of points (α, s) in $(\mathbb{Z}^{\geqslant 0})^r \times (\mathbb{Z}^{\geqslant 0} \cup \infty)$ such that $\alpha \in \mathrm{Ind}(\Gamma)_0$ and $s = s_\Gamma(\alpha)$ if $\alpha \in E_0(\Gamma)$ (extremal) or $s = \infty$ otherwise.
(4) Given a G-graded T-ideal Γ which contains the graded identities of a finite-dimensional G-graded algebra A we let $\mathrm{Kemer}(\Gamma)$ (the Kemer set of Γ), to be set of points (α, s) in $\mathrm{Ind}(\Gamma)$ where α is extremal. We refer to the elements of $\mathrm{Kemer}(\Gamma)$ as the Kemer points of Γ.

Remark 3.9. Let $\Gamma_1 \supseteq \Gamma_2$ be T-ideals which contain $\mathrm{id}_G(A)$ where A is a finite-dimensional G-graded algebra. Then $\mathrm{Ind}(\Gamma_1) \subseteq \mathrm{Ind}(\Gamma_2)$. In particular for every Kemer point (α, s) in $\mathrm{Ind}(\Gamma_1)$ there is a Kemer point (β, s') in $\mathrm{Ind}(\Gamma_2)$ with $(\alpha, s) \preccurlyeq (\beta, s')$.

We are now ready to define Kemer polynomials for a G-graded T-ideal Γ.

Definition 3.10.

(1) Let (α, s) be a Kemer point of the T-ideal Γ. A G-graded polynomial f is said to be a *Kemer polynomial for the point* (α, s) if f is not in Γ and it has at least μ-folds of alternating g-sets of cardinality α_g (*small sets*) for every g in G and s homogeneous sets of disjoint variables Y_g (some g in G) of cardinality $\alpha_g + 1$ (*big sets*).
(2) A polynomial f is *Kemer* for the T-ideal Γ if it is Kemer for a Kemer point of Γ.
Note that a polynomial f cannot be Kemer simultaneously for different Kemer points of Γ.

4. The index of finite-dimensional G-graded algebras

As explained in Section 1 the Phoenix property of Kemer polynomials is essential for the proof of Theorem 1.1. Since we will need this notion not only with respect to Kemer polynomials, we give here a general definition.

Definition 4.1. (The Phoenix property) Let Γ be a T-ideal as above. Let P be any property which may be satisfied by G-graded polynomials (e.g. being Kemer). We say that P is "*Γ-Phoenix*" (or in short "*Phoenix*") if given a polynomial f having P which is not in Γ and any f' in $\langle f\rangle$ (the T-ideal generated by f) which is not in Γ as well, there exists a polynomial f'' in $\langle f'\rangle$ which is not in Γ and satisfies P. We say that P is "*strictly Γ-Phoenix*" if (with the above notation) f' itself satisfies P.

Let us pause for a moment and summarize what we have at this point. We are given a T-ideal Γ (the T-ideal of G-graded identities of a G-graded affine algebra W). We assume that W is *PI* and hence as shown in Section 2 there exists a G-graded finite-dimensional algebra A with $\Gamma \supseteq \mathrm{id}_G(A)$. To the T-ideals Γ and $\mathrm{id}_G(A)$ we attached the corresponding sets of Kemer points. By Remark 3.9 for every Kemer point (α, s) of Γ there is a Kemer point (β, s') of $\mathrm{id}_G(A)$ (or by abuse of language, a Kemer point of A) such that $(\alpha, s) \preceq (\beta, s')$. Our goal is to replace the algebra A by a finite-dimensional algebra A' with $\Gamma \supseteq \mathrm{id}_G(A')$ and such that

(1) The T-ideals Γ and $\mathrm{id}_G(A')$ have the same Kemer points.
(2) Every Kemer polynomial of A' is not in Γ (i.e. Γ and A' have the same Kemer polynomials).

This establishes the important connection between the combinatorics of the Kemer polynomials of Γ and the structure of G-graded finite-dimensional algebras. The "Phoenix" property for the Kemer polynomials of Γ will follow from that connection.

Let A be a finite-dimensional G-graded algebra over F. It is well known (see [17]) that the Jacobson radical $J(A)$ is G-graded and hence $\overline{A} = A/J(A)$ is semisimple and G-graded. Moreover by the Wedderburn–Malcev Principal Theorem for G-graded algebras (see [45]) there exists a semisimple G-graded subalgebra $\overline{A}$ of A such that $A = \overline{A} \oplus J(A)$ as G-graded vector spaces. In addition, the subalgebra $\overline{A}$ may be decomposed as a G-graded algebra into the direct product of (semisimple) G-simple algebras $\overline{A} \cong A_1 \times A_2 \times \cdots \times A_q$ (by definition A_i is G-simple if it has no non-trivial G-graded ideals).

Remark 4.2. This decomposition enables us to consider "semisimple" and "radical" substitutions. More precisely, since in order to check whether a given G-graded multilinear polynomial is an identity of A it is sufficient to evaluate the variables in any spanning set of homogeneous elements, we may take a basis consisting of homogeneous elements of $\overline{A}$ or of $J(A)$. We refer to such evaluations as semisimple or radical evaluations respectively. Moreover, the semisimple substitutions may be taken from the G-simple components. In what follows, whenever we evaluate G-graded polynomial on G-graded finite-dimensional algebras, we consider only evaluations of that kind.

In fact, in the proofs we will need a rather precise "control" of the evaluations in the G-simple components. This is provided by a structure theorem proved by Bahturin, Sehgal and Zaicev which will play a decisive role in the proofs of the main results of this paper.

Theorem 4.3. *(See [7].) Let $\widehat{A}$ be a G-simple algebra over F. Then there exists a subgroup H of G, a 2-cocycle $f : H \times H \to F^*$ where the action of H on F is trivial, an integer k and a k-tuple $(g_1 = 1, g_2, \ldots, g_k) \in G^{(k)}$ such that $\widehat{A}$ is G-graded isomorphic to $C = F^f H \otimes M_k(F)$ where $C_g = \mathrm{span}_F\{b_h \otimes E_{i,j}\colon\ g = g_i^{-1} h g_j\}$. Here $b_h \in F^f H$ is a representative of $h \in H$ and $E_{i,j} \in M_k(F)$ is the (i, j) elementary matrix.*

In particular the idempotents $1 \otimes E_{i,j}$ as well as the identity element of $\widehat{A}$ are homogeneous of degree $e \in G$.

Let $\overline{A} = \bigoplus_{g\in G} \overline{A}_g$ be the decomposition of $\overline{A}$ into its homogeneous components and let $d_{A,g} = \dim_F(\overline{A}_g)$, the dimension of $\overline{A}_g$ over F, for every g in G. Let n_A be the nilpotency index of $J(A)$. The following proposition establishes an easy connection between the parameters $d_{A,g_1}, \ldots, d_{A,g_r}, n_A$ and the Kemer points of A. We denote by $G - \mathrm{Par}(A)$ the $(r+1)$-tuple $(d_{A,g_1}, \ldots, d_{A,g_r}, n_A - 1)$ in $(\mathbb{Z}^{\geqslant 0})^r \times (\mathbb{Z}^{\geqslant 0})$.

Proposition 4.4. *If $(\alpha, s) = (\alpha_{g_1}, \ldots, \alpha_{g_r}, s)$ is a Kemer point of A then $(\alpha, s) \preceq G - \mathrm{Par}(A)$.*

Proof. Assume this is false. It follows that either $\alpha_g > \dim_F(\overline{A}_g)$ for some $g \in G$ or else $\alpha_g = \dim_F(\overline{A}_g)$ for every $g \in G$ and $s > n_A - 1$. We show that both are impossible. First recall that (α, s) being a Kemer point of A implies the existence of polynomials f which are non-identities of A with arbitrary many alternating g-sets of cardinality α_g. Hence, if $\alpha_g > \dim_F(\overline{A}_g)$ for some $g \in G$, it follows that in each such alternating set there must be at least one radical substitution in any non-zero evaluation of f. This implies that we cannot have more than $n_A - 1$ g-alternating sets of cardinality α_g contradicting our previous statement. Next, suppose that $\alpha_g = \dim_F(\overline{A}_g)$ for all $g \in G$ and $s > n_A - 1$. This means that we have s alternating g-sets for some g's in G, of cardinality $\alpha_g + 1 = \dim_F(\overline{A}_g) + 1$. Again this means that f will vanish if we evaluate any of these sets by semisimple elements. It follows that in each one of these s sets at least one of the evaluations is radical. Since $s > n_A - 1$, the polynomial f vanishes on such evaluations as well and hence f is a G-graded identity of A. Contradiction. □

In the next examples we show that the Kemer points of A may be quite far from $G - \mathrm{Par}(A)$.

Example 4.5. Let A be G-simple and let $A(n) = A \times A \times \cdots \times A$ (n-times). Clearly, $\mathrm{id}_G(A(n)) = \mathrm{id}_G(A)$ and hence A and $A(n)$ have the same Kemer points. On the other hand $G - \mathrm{Par}(A(n))$ increases with n.

The next construction (see [23, Example 4.50]) shows that also the nilpotency index may increase indefinitely whereas that Kemer points remain unchanged.

Let B be any finite-dimensional G-grade algebra and let $B' = \overline{B} * \{x_{g_1}, \ldots, x_{g_n}\}$ be the algebra of G-graded polynomials in the graded variables $\{x_{g_i}\}_{i=1}^n$ with coefficients in $\overline{B}$, the semisimple component of B. The number of g-variables that we take is at least the dimension of the g-component of $J(B)$. Let I_1 be the ideal of B generated by all evaluations of polynomials of $\mathrm{id}_G(B)$ on B' and let I_2 be the ideal generated by the variables $\{x_{g_i}\}_{i=1}^n$. Consider the algebra $\widehat{B}_u = B/(I_1 + I_2^u)$.

Proposition 4.6.

(1) *$\mathrm{id}_G(\widehat{B}_u) = \mathrm{id}_G(B)$ whenever $u \geqslant n_B$ (n_B denotes the nilpotency index of B). In particular $\widehat{B}_u$ and B have the same Kemer points.*
(2) *$\widehat{B}_u$ is finite-dimensional.*
(3) *The nilpotency index of $\widehat{B}_u$ is u.*

Proof. Note that by the definition of $\widehat{B}_u$, $\mathrm{id}_G(\widehat{B}_u) \supseteq \mathrm{id}_G(B)$. On the other hand there is a G-graded surjection $\mathrm{id}_G(\widehat{B}_u) \to \mathrm{id}_G(B)$ which maps the variables $\{x_{g_i}\}_{i=1}^n$ into graded elements of $J(B)$ where $\overline{B}$ is mapped isomorphically. This shows (1). To see (2) observe that any element of $\widehat{B}_u$ is represented by a sum of monomials of the form $b_1 z_1 b_2 z_2 \cdots b_j z_j b_{j+1}$ where $j < u$, $b_i \in \overline{B}$ and $z_i \in \{x_{g_i}\}_{i=1}^n$. Clearly the subspace spanned by monomials for a given configuration of the z_i's (and arbitrary b_i's) has finite dimension. On the other hand the number of different configurations is finite and so the result follows. The 3rd statement follows from the fact that the product of less than $u-1$ variables is non-zero in $\widehat{B}_u$. □

In view of the examples above, in order to establish a precise relation between Kemer points of a finite-dimensional G-graded algebra A and its structure we need to find appropriate finite-dimensional algebras which will serve as *minimal models* for a given Kemer point. We start with the decomposition of a G-graded finite-dimensional algebra into the product of subdirectly irreducible components.

Definition 4.7. A G-graded finite-dimensional algebra A is said to be *subdirectly irreducible*, if there are no non-trivial G-graded ideals I and J of A, such that $I \cap J = \{0\}$.

Lemma 4.8. *Let A be a finite-dimensional G-graded algebra over F. Then A is G-graded PI-equivalent to a direct product $C_1 \times \cdots \times C_n$ of finite-dimensional G-graded algebras, each subdirectly irreducible. Furthermore for every $i = 1, \ldots, n$, $\dim_F(C_i) \leqslant \dim_F(A)$ and the number of G-simple components in C_i is bounded by the number of such components in A.*

Proof. The proof is identical to the proof in the ungraded case. If A is not subdirectly irreducible we can find non-trivial G-graded ideals I and J such that $I \cap J = \{0\}$. Note that $A/I \times A/J$ is *PI*-equivalent to A. Since $\dim_F(A/I)$ and $\dim_F(A/J)$ are strictly smaller than $\dim_F(A)$ the result follows by induction. □

The next condition is

Definition 4.9. We say that a finite-dimensional G-graded algebra A is *full* with respect to a G-graded multilinear polynomial f, if there exists a non-vanishing evaluation such that every G-simple component is represented (among the semisimple substitutions). A finite-dimensional G-graded algebra A is said to be full if it is full with respect to some G-graded polynomial f.

We wish to show that any finite-dimensional algebra may be decomposed (up to *PI*-equivalence) into the direct product of full algebras. Algebras without an identity element are treated separately.

Lemma 4.10. *Let A be a G-graded, subdirectly irreducible which is not full.*

(1) *If A has an identity element then it is PI-equivalent to a direct product of finite-dimensional algebras, each having fewer G-simple components.*
(2) *If A has no identity element then it is PI-equivalent to a direct product of finite-dimensional algebras, each having either fewer G-simple components than A or else it has an identity element and the same number of G-simple components as A.*

The lemma above together with Lemma 4.8 yields:

Corollary 4.11. *Every finite-dimensional G-graded algebra A is PI-equivalent to a direct product of full, subdirectly irreducible finite-dimensional algebras.*

Proof of Lemma 4.10. The proof is similar to the proof in [23]. Assume first that A has an identity element. Consider the decompositions mentioned above $A \cong \overline{A} \oplus J$ and $\overline{A} \cong A_1 \times A_2 \times \cdots \times A_q$ (A_i are G-simple algebras). Let e_i denote the identity element of A_i and consider the decomposition $A \cong \bigoplus_{i,j=1}^{q} e_i A e_j$. By assumption we have that $e_{i_1} A e_{i_2} \cdots e_{i_q-1} A e_{i_q} = e_{i_1} J e_{i_2} \cdots e_{i_q-1} J e_{i_q} = 0$ whenever $i_1, \dots, i_q$ are distinct.

Consider the commutative algebra $H = F[\lambda_1, \dots, \lambda_q]/I$ where I is the (ungraded) ideal generated by $\lambda_i^2 - \lambda_i$ and $\lambda_1 \cdots \lambda_q$. Then, if we write $\widetilde{e}_i$ for the image of λ_i, we have $\widetilde{e}_i^2 = \widetilde{e}_i$ and $\widetilde{e}_1 \cdots \widetilde{e}_q = 0$. Consider the algebra $A \otimes_F H$, G-graded via A. Let $\widetilde{A}$ be the G-graded subalgebra generated by all $e_i A e_j \otimes \widetilde{e}_i \widetilde{e}_j$ for every $1 \leqslant i, j \leqslant q$. We claim that the graded algebras A and $\widetilde{A}$ are *PI*-equivalent as G-graded algebras: Clearly $\mathrm{id}_G(A) \subseteq \mathrm{id}_G(A \otimes_F H) \subseteq \mathrm{id}_G(\widetilde{A})$, so it suffices to prove that any graded non-identity f of A is also a non-identity of $\widetilde{A}$. Clearly, we may assume that f is multilinear and strongly homogeneous. Note: $\widetilde{A}$ is graded by the number of distinct $\widetilde{e}_i$ appearing in the tensor product of an element. In evaluating f on A it suffices to consider specializations of the form $x_g \mapsto v_g$ where $v_g \in e_{i_k} A e_{i_{k+1}}$. In order to have $v_{g_1} \cdots v_{g_n} \neq 0$, the set of indices i_k appearing must contain at most $q-1$ distinct elements, so $e_{i_1} \cdots e_{i_{n+1}} \neq 0$. Then $f(v_{g_1} \otimes \widetilde{e}_{i_1}, \dots, v_{g_n} \otimes \widetilde{e}_{i_n}) = f(v_{g_1}, \dots, v_{g_n}) \otimes \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_n} \neq 0$ so f is not in $\mathrm{id}_G(\widetilde{A})$. We conclude that $\widetilde{A} \sim_{PI} A$, as claimed. Now we claim that $\widetilde{A}$ can be decomposed into direct product of G-graded algebras with fewer G-simple components. To see this let $I_j = \langle e_j \otimes \widetilde{e}_j \rangle \lhd \widetilde{A}$. Note that by Theorem 4.3, the element e_j is homogeneous and hence I_j is G-graded. We see that

$$\bigcap_{j=1}^{q} I_j = (1 \otimes \widetilde{e}_1 \cdots 1 \otimes \widetilde{e}_q)^{-1} \left(\bigcap_{j=1}^{q} I_j \right) = (1 \otimes \widetilde{e}_1 \cdots \widetilde{e}_q) \left(\bigcap_{j=1}^{q} I_j \right) = \{0\}$$

so $\widetilde{A}$ is subdirectly reducible to the direct product of $\widetilde{A}/I_j$. Furthermore, each component $\widetilde{A}/I_j$ has less than q G-simple components since we eliminated the idempotent corresponding to the j-th G-simple component. This completes the proof of the first part of the lemma.

Consider now the case where the algebra A has no identity element. There we have $A \cong \bigoplus_{i,j=0}^{q} e_i A e_j$ where $e_0 = 1 - (e_1 + \cdots + e_q)$ (here the element 1 and hence e_0 are given degree $e \in G$). We proceed as above but now with $q+1$ idempotents, variables, etc. We see as above that $\widetilde{A}/I_j$ will have less than q G-simple components if $1 \leqslant j \leqslant q$ whereas $\widetilde{A}/I_0$ will have an identity element and q G-simple components. This completes the proof of the lemma. □

Remark 4.12. Note that in the decomposition above the nilpotency index of the components in the direct product is bounded by the nilpotency index of A.

Now we come to the definition of *PI*-minimal.

Definition 4.13. We say that a finite-dimensional G-graded algebra A is *PI*-minimal if $G - \mathrm{Par}_A$ is minimal (with respect to the partial order defined above) among all finite-dimensional G-graded algebras which are *PI*-equivalent to A.

Definition 4.14. A finite-dimensional G-graded algebra A is said to be *PI-basic* (or just *basic*) if it is PI-minimal, full and subdirectly irreducible.

Proposition 4.15. *Every finite-dimensional G-graded algebra A is PI-equivalent to the direct product of a finite number of G-graded PI-basic algebras.*

In the next two sections we show that any basic A algebra has a Kemer set which consists of a unique point $(\alpha, s(\alpha))$. Moreover, $(\alpha, s(\alpha)) = G - \mathrm{Par}_A$.

5. Kemer's Lemma 1

The task in this section is to show that if A is subdirectly irreducible and full, then there is a point $\alpha \in E_0(A)$ with $\alpha = (d_{A,g_1}, \dots, d_{A,g_r})$. In particular, $E_0(A)$ consists of a unique point.

Proposition 5.1. *Let A be a finite-dimensional algebra, G-graded, full and subdirectly irreducible. Then there is an extremal point α in $E_0(A)$ with $\alpha_g = \dim_F(\overline{A}_g)$ for every g in G. In particular, the extremal point is unique.*

Proof. Note that the uniqueness follows from Lemma 4.4 since $\alpha \preceq (\dim_F(\overline{A}_{g_1}), \dots, \dim_F(\overline{A}_{g_r}))$ for every extremal point α.

For the proof we need to show that for an arbitrary large integer ν there exists a non-identity f such that for every $g \in G$ there are ν folds of alternating g-sets of cardinality $\dim_F(\overline{A}_g)$.

Since the algebra A is full, there is a multilinear, G-graded polynomial $f(x_{1,g_{i_1}}, \dots, x_{q,g_{i_q}}, \overrightarrow{y})$, which does not vanish upon an evaluation of the form $x_{j,g_{i_j}} = \bar{x}_{j,g_{i_j}} \in \overline{A}_{j,g_{i_j}}$, $j = 1, \dots, q$ and the variables of $\overrightarrow{y}$ get homogeneous values in A. The idea is to produce polynomials $\widehat{f}$'s in the T-ideal generated by f which will remain non-identities of A and that will reach eventually the desired form. The way one checks that the polynomials $\widehat{f}$'s are non-identities is by presenting suitable evaluations on which they do not vanish. Let us reformulate what we need in the following lemma.

Lemma 5.2 *(Kemer's Lemma 1 for G-graded algebras). Let A be a finite-dimensional algebra, G-graded, subdirectly irreducible and full with respect to a polynomial f. Then for any integer ν there exists a non-identity f' (of A) in the T-ideal generated by f with ν-folds of alternating g-sets of cardinality $\dim_F(\overline{A}_g)$ for every g in G.*

Proof. Let f_0 be the polynomial obtained from f by multiplying (on the left say) each one of the variables $x_{1,g_{i_1}}, \dots, x_{q,g_{i_q}}$ by e-homogeneous variables $z_{1,e}, \dots, z_{q,e}$ respectively. Note that the polynomial obtained is a G-graded non-identity since the variables $z_{i,e}$'s may be evaluated by the (degree e) elements $1_{\overline{A}_i}$'s where

$$1_{\overline{A}_i} = 1 \otimes E^i_{1,1} + \dots + 1 \otimes E^i_{k_i,k_i}.$$

(Here we use the notation of Theorem 4.3, $\overline{A}_i = F^{c_i} H_i \otimes M_{k_i}(F)$.)

Applying linearity there exists a non-zero evaluation where the variables $z_{1,e}, \dots, z_{q,e}$ take values of the form $1 \otimes E^1_{j_1,j_1}, \dots, 1 \otimes E^q_{j_q,j_q}$ where $1 \leqslant j_i \leqslant k_i$ for $i = 1, \dots, q$.

Our aim is to replace each one of the variables $z_{1,e},\dots,z_{q,e}$ by G-graded polynomials $Z_1,\dots,Z_q$ such that:

(1) For every $i=1,\dots,q$ and for every $g\in G$, the polynomial Z_i is alternating in ν-folds of g-sets of cardinality $\dim_F(\overline{A}_i)_g$.
(2) For every $i=1,\dots,q$ the polynomial Z_i assumes the value $1\otimes E^i_{j_i,j_i}$.

Once this is accomplished, we complete the construction by alternating the g-sets which come from different Z_i's. Clearly, the polynomial f' obtained

(1) is a non-identity since any non-trivial alternation of the evaluated variables (as described above) vanishes,
(2) for every $g\in G$, f' has ν-folds of alternating g-sets of cardinality $\dim_F(\overline{A}_g)$.

We now show how to construct the G-graded polynomials Z_i.

In order to simplify the notation we put $\widehat{A}=\overline{A}_i$ $(=F^cH\otimes M_k(F))$ and $\widehat{A}=\bigoplus_{g\in G}\widehat{A}_g$ where $\overline{A}_i$ is the i-th G-simple component.

Fix a product of the k^2 different elementary matrices $E_{i,j}$ of $M_k(F)$ with value $E_{1,1}$ (it is not difficult to show the such a product exists). For each $h\in H$ we consider the basis elements of $\widehat{A}$ of the form $b_h\otimes E_{i,j}$ where $1\leqslant i,j\leqslant k$. If we multiply these elements (in view of the $E_{i,j}$'s) in the same order as above we obtain $b_h^{k^2}\otimes E_{1,1}$. Observe that since b_h is invertible in F^fH, the element $b_h^{k^2}\otimes E_{1,1}$ is not zero. Repeating this process for every $h\in H$ and multiplying all together we obtain a non-zero product of all basis elements $b_h\otimes E_{i,j}$ where $1\leqslant i,j\leqslant k$ and $h\in H$.

Note that we obtained an homogeneous element of the form $\lambda b_h\otimes E_{1,1}$ where $\lambda\in F^*$ and $h\in H$. Finally we may multiply the entire product by $(\lambda b_h)^{-1}\otimes E_{1,1}$ and get $1\otimes E_{1,1}$. Consider now the graded monomial consisting of $\operatorname{ord}(H)\cdot k^2+1$ graded variables which may be evaluated by the product above. We extend this monomial by inserting $\operatorname{ord}(H)\cdot k^2+1$ e-variables $y_{i,e}$ bordering each one of the basis elements (there is no need to border the extra h^{-1}-variable). Clearly, the variables $y_{i,e}$ may be substituted by elements of the form $1\otimes E_{i,i}\in\widehat{A}_e$.

Remark 5.3. Note that for any (i,j) the number of basis elements $b_h\otimes E_{i,j}$ which are bordered by the pair $(1\otimes E_{i,i},1\otimes E_{j,j})$ is precisely $\operatorname{ord}(H)$. The key observation here is that basis elements which are bordered by the same pair belong to *different* homogeneous components of $\widehat{A}$ (and hence there is no need to alternate the corresponding variables).

Now, we can complete the proof of Lemma 5.2. By Remark 5.3, if we alternate the g-variables that correspond (in the evaluation above) to the g-basis of any G-simple component, we obtain a non-identity (indeed, the evaluations which correspond to non-trivial permutations will vanish since the borderings of these g-elements are different). Furthermore, alternating g-variables which correspond to g-bases of different G-simple components again yields a non-identity since (again) the evaluations which correspond to non-trivial permutations will vanish (here we are multiplying two central idempotents of different G-simple components). This completes the proof of Lemma 5.2 and of Proposition 5.1. $\square$

6. Kemer's Lemma 2

In this section we prove the G-graded versions of Kemer's Lemma 2. Before stating the precise statement we need an additional reduction which enables us to control the number of radical evaluations in certain non-identities.

Let f be a multilinear, graded polynomial which is not in $\mathrm{id}_G(A)$. Clearly, any non-zero evaluation cannot have more than $n_A - 1$ radical evaluations.

Lemma 6.1. *Let A be a finite-dimensional algebra, G-graded. Let $(\alpha, s_A(\alpha))$ be a Kemer point of A. Then $s_A(\alpha) \leqslant n_A - 1$.*

Proof. By the definition of the parameter $s = s_A(\alpha)$ we know that for arbitrary large ν there exist multilinear graded polynomials, not in $\mathrm{id}_G(A)$, such that have ν-folds of alternating g-sets (small) of cardinality $d_g = \dim_F(\overline{A}_g)$ and $s_A(\alpha)$ (big) sets of cardinality $d_g + 1$. It follows that an alternating g-set of cardinality $d_g + 1$ in a non-identity polynomial must have at least one radical evaluation. Consequently we cannot have more than $n_A - 1$ of such alternating sets, proving the lemma. □

The next definition and proposition are taken from [23].

Definition 6.2. Let f be a multilinear G-graded polynomial which is not in $\mathrm{id}_G(A)$. We say that f has property K if f vanishes on every evaluation with less than $n_A - 1$ radical substitutions.

We say that a finite-dimensional G-graded algebra A has property K if it satisfies the property with respect to some non-identity multilinear polynomial.

Proposition 6.3. *Let A be a PI-minimal G-graded, finite-dimensional F-algebra. Then it has property K.*

Proof. Assume property K always fails. This means that any multilinear polynomial which vanishes on less than $n_A - 1$ radical evaluations is in $\mathrm{id}_G(A)$. Recall (from Proposition 4.6) the algebra $\widehat{A}_u = A'/(I_1 + I_2^u)$ where $A' = \overline{A} * F\{x_{1,g_1}, \ldots, x_{\nu,g_\nu}\}$. We claim that for $u = n_A - 1$, $\widehat{A}_u$ is PI-equivalent to A. Then noting that the nilpotency index of $\widehat{A}_{n_A-1}$ is $n_A - 1$ we get a contradiction to the minimality of A. Clearly, by construction $\mathrm{id}_G(A) \subseteq \mathrm{id}_G(\widehat{A}_{n_A-1})$. For the converse take a polynomial f which is not in $\mathrm{id}_G(A)$. Then by assumption, there is a non-zero evaluation on A with less than $n_A - 1$ radical substitutions. Now, if we replace these radical values by $\{x_{i,g}\}$'s in A' we get a polynomial (in $\{x_{i,g}\}$'s) which is not in $I_1 + I_2^{n_A-1}$ and hence f is not in $\mathrm{id}_G(\widehat{A}_{n_A-1})$. This completes the proof of the proposition. □

Let A be a basic algebra (i.e. minimal, full and subdirectly irreducible). Let $G - \mathrm{Par}_A = (d_{g_1}, \ldots, d_{g_r}; n_A - 1)$ where d_g is the dimension of the g-homogeneous components of the semisimple part of A and n_A the nilpotency index of $J(A)$. By Proposition 6.3 the algebra satisfies property K with respect to a non-identity polynomial f, that is f vanishes on any evaluation whenever we have less than $n_A - 1$ radical substitutions. Furthermore, there is possibly a different non-identity polynomial h with respect to which A is full, that is h has a non-zero evaluation which "visits" each one of the G-simple components of $\overline{A}$. In order to proceed we need both properties to be satisfied by the same polynomial.

We start with two preliminary lemmas which show that these two properties, namely property K and the property of being full are "preserved" in a T-ideal.

Lemma 6.4. *Let A be a G-graded finite-dimensional algebra over F.*

(1) *Let f be a G-graded non-identity polynomial, strongly homogeneous which is μ-fold alternating on g-sets of cardinality $d_g = \dim_F(\overline{A}_g)$ for every g in G (in particular A is full with respect to f). If $f' \in \langle f \rangle$ is a non-identity in the T-ideal generated by f, then there exists a non-identity $f'' \in \langle f' \rangle$ which is μ-fold alternating on g-sets of cardinality d_g for every g in G. In other words, the property of being μ-fold alternating on g-sets of cardinality d_g for every g in G is A-Phoenix.*
(2) *Property K is strictly A-Phoenix.*

Proof. Let f be a G-graded non-identity polynomial, strongly homogeneous which is μ-fold alternating on g-sets of cardinality d_g for every g in G and let f' be a non-identity in $\langle f \rangle$. In view of Lemma 5.2, it is sufficient to show that A is full with respect to f'. Note that by the definition of μ, for each g in G, in at least one alternating g-set the evaluations of the corresponding variables must consist of semisimple elements of A in any non-zero evaluation of the polynomial. The result is clear if f' is in the ideal (rather than the T-ideal) generated by f. We assume therefore that f' is obtained from f by substituting variables x_g's by monomials Z_g's. Clearly, if one of the evaluations in any of the variables of Z_g is radical, then the value of Z_g is radical. Hence in any non-zero evaluation of f' there is an alternating g-set Δ_g of cardinality d_g in f for every $g \in G$, such that the variables in monomials of f' (which correspond to the variables in Δ_g) assume only semisimple values. Furthermore, each G-simple component must be represented in these evaluations for otherwise we would have a G-simple component not represented in the evaluations of the Δ_g's and this is impossible. We have shown that A is full with respect to f'. Applying Lemma 5.2 we obtain a polynomial $f'' \in \langle f' \rangle \subseteq \langle f \rangle$ with the desired property. This proves the first part of the lemma.

For the second part note that if f' is a non-identity in the T-ideal generated by f then if f' has less than $n_A - 1$ radical evaluations then the same is true for f and hence vanishes. In other word, f' satisfies property K. □

Remark 6.5. Note that we are not claiming that the property "full" is Phoenix.

Now we combine these two properties.

Lemma 6.6. *Let A be a finite-dimensional algebra, G-graded which is full, subdirectly irreducible and satisfying property K. Let f be a non-identity with μ-folds of alternating g-sets of cardinality $d_g = \dim_F(\overline{A}_g)$ for every g in G and let h be a polynomial with respect to which A has property K. Then there is a non-identity in $\langle f \rangle \cap \langle h \rangle$. Consequently there exists a non-identity polynomial $\widehat{f}$ which has μ-folds of alternating g-sets of cardinality d_g for every g in G and with respect to which A has property K.*

Proof. Suppose this is false, that is the intersection is contained in $\mathrm{id}_G(A)$. Consider the ideals $\bar{I}$ and $\bar{J}$ generated by all evaluations on A of the polynomials in the T-ideals $I = \langle f \rangle$ and $J = \langle h \rangle$ respectively. Since the ideals I and J are not contained in $\mathrm{id}_G(A)$, the ideals $\bar{I}$ and $\bar{J}$ are non-zero. On the other hand, by construction, their intersection is zero and we get a contradiction to the subdirectly irreducibility of A. Take a non-identity $f' \in I \cap J$. By the first part of Lemma 6.4 there is a non-identity $\widehat{f} \in \langle f' \rangle \subseteq I \cap J$ which has μ-folds of alternating g-sets of cardinality d_g for every g in G. By the second part of the lemma $\widehat{f}$ has property K. □

We can now state and prove Kemer's Lemma 2 for G-graded algebras.

Lemma 6.7 *(Kemer's Lemma 2 for G-graded algebras). Let A be a finite-dimensional G-graded algebra. Assume A is basic. Let $(d_{g_1}, \dots, d_{g_r})$ be the dimensions of the g_i-homogeneous components of $\overline{A}$ (the semisimple part of A) and n_A be the nilpotency index of $J(A)$. Then for any integer ν there exists a G-graded, multilinear, non-identity polynomial f such that for every g in G, it has ν-folds of g-sets of alternating variables of cardinality d_g and a total of $n_A - 1$ sets of variables which are g-homogeneous and of cardinality $d_g + 1$ for some g in G.*

Note 6.8. Any non-zero evaluation of such f must consists only of semisimple evaluations in the ν-folds and each one of the big sets (namely the sets of cardinality $d_g + 1$) must have exactly one radical evaluation.

Proof. By the preceding lemma we take a multilinear (strongly homogeneous) non-identity polynomial f, with respect to which A is full and has property K. Let us fix a non-zero evaluation $x_g \mapsto \widehat{x}_g$. We will consider four cases. These correspond to whether A has or does not have an identity element and whether q (the number of G-simple components) > 1 or $q = 1$.

Case $(1, 1)$ (A has an identity element and $q > 1$).

Fix a non-zero evaluation of f with respect to which A is full (i.e. "visits" in every G-simple component) and has precisely $n_A - 1$ radical substitutions. Moreover any evaluation with fewer radical substitutions vanishes. We choose a monomial X in f which does not vanish upon the above evaluation.

Notice that in the monomial X, the variables which get semisimple evaluations from different G-simple component must be separated by variables with radical values. Let us denote by $w_{1,g_{i_1}}, \dots, w_{n_a-1,g_{i_{n_A-1}}}$ the variables which get radical values and by $\widehat{w}_{1,g_{i_1}}, \dots, \widehat{w}_{n_a-1,g_{i_{n_A-1}}}$ their values (g_{i_j} is the G-degree of $w_{j,g_{i_j}}$).

By Theorem 4.3 (and linearity), each $\widehat{w}_{i,g_i}$ may be bordered (i.e. multiplied from left and right) by elements of the form $1 \otimes E^i_{k_i,k_i} \in A_i$ and still giving a non-zero value. We refer to any two such elements of the form $1 \otimes E^i_{k_i,k_i}$ which border a radical evaluation as *partners*.

Claim 6.9. *The elements $1 \otimes E^i_{k_i,k_i}$ which appear in the borderings above, represent all the G-simple components of $\overline{A}$.*

Indeed, suppose that the G-component A_1 (say) is not represented among the $1 \otimes E^i_{k_i,k_i}$'s. Since our original evaluation was full there is a variable which is evaluated by an element u_g of A_1. "Moving" along the monomial X to the left or right of u_g we will hit a bordering value of the form $1 \otimes E^i_{k_i,k_i}$ before we hit any radical evaluation. But this is possible only if both u_g and $1 \otimes E^i_{k_i,k_i}$ belong to the same G-simple component. This proves the claim.

But we need more: Consider the radical evaluations which are bordered by pairs of elements $1 \otimes E^i_{k_i,k_i}, 1 \otimes E^j_{k'_j,k'_j}$ that belong to G-simple components A_i and A_j where $i \neq j$.

Claim 6.10. *Every G-simple component is represented by one of the elements in these pairs.*

Again, assume that A_1 is not represented among these pairs. By the preceding claim A_1 is represented, so it must be represented by both partners in each pair it appears. Take such a pair: $1 \otimes E^1_{k_1,k_1}$, $1 \otimes E^1_{k_1',k_1'}$. Moving along the monomial X to the left of $1 \otimes E^1_{k_1,k_1}$ or to the right of $1 \otimes E^1_{k_1',k_1'}$, we will hit a value in a different G-simple component. But before that we must hit a radical evaluation which is bordered by a pair where one of the partners is from A_1 and the other from a different G-simple component. This contradicts our assumption and hence the claim is proved.

For $t = 1, \ldots, q$ we choose a variable $w_{j_t, g_{i_{j_t}}}$ whose radical value $\widehat{w}_{j_t, g_{i_{j_t}}}$ is bordered by partners which

(1) belong to different G-simple components,
(2) one of them is an idempotent in the t-th G-simple component.

We replace now the variable $w_{j_t, g_{i_{j_t}}}$ by the product $y_{t,e} w_{j_t, g_{i_{j_t}}}$ or $w_{j_t, g_{i_{j_t}}} y_{t,e}$ (according to the position of the bordering) were $y_{1,e}, \ldots, y_{q,e}$ are e-variables. Clearly we obtained a non-identity.

Applying Lemma 5.2 we can insert in the $y_{j,e}$'s suitable graded polynomials and obtain a G-graded polynomial with ν-folds of (small) g-sets of alternating variables where each g-set is of cardinality $\dim(\overline{A}_g)$.

Consider the variables with radical evaluations which are bordered by e-variables with evaluations from different G-simple components (these include the variables which are bordered by the $y_{j,e}$). Let z_g be such a variable (assume it is homogeneous of degree g). We attach it to a (small) g-alternating set. We claim that if we alternate this set (of cardinality $d_g + 1$) we obtain a non-identity. Indeed, all g-variables in the small set are bordered by e-variables which are evaluated with elements from the *same* G-simple component whereas the radical element is bordered with elements of *different* G-simple components. Consequently any non-trivial permutation of the evaluated monomials vanishes. At this point we have constructed the desired number of small sets and *some* of the big sets. We still need to attach the radical variables which are bordered by e-variables from the same G-simple component. We attach them as well to (small) g-sets. We claim also here that if we alternate this set (of cardinality $d_g + 1$) we obtain a non-identity. Indeed, any non-trivial permutation represents an evaluation with fewer radical evaluations in the original polynomial which must vanish by property K. This completes the proof where q, the number of G-simple components, is > 1.

Case $(1,2)$ (A has an identity element and $q = 1$). We start with a non-identity f which satisfies property K. Clearly we may multiply f by a variable x_e and get a non-identity (since x_e may be evaluated by 1). Again by Lemma 5.2 we may replace x_e by a polynomial h with ν-folds of g-sets of alternating variables of cardinality d_g. Consider the polynomial hf. We attach the radical variables of f to some of the small sets in h. Any non-trivial permutation vanishes because f satisfies property K. This completes the proof of the Lemma 6.7 in case A has an identity element.

Case $(2,1)$. Suppose now A has no identity element and $q > 1$. The proof in this case is basically the same as in the case where A has an identity element. Let $e_0 = 1 - 1_{A_1} - 1_{A_2} - \cdots - 1_{A_q}$ and include e_0 to the set of elements which border the radical values $\widehat{w}_{j, g_{i_j}}$. A similar argument shows that also here every G-simple component $(A_1, \ldots, A_q)$ is represented in one of the bordering pairs where the partners are *different* (the point is that one of the partners (among these pairs) may be e_0). Now we complete the proof exactly as in case $(1,1)$.

Case $(2,2)$. In order to complete the proof of the lemma we consider the case where A has no identity element and $q=1$. The argument in this case is somewhat different. For simplicity we denote by $e_1 = 1_{A_1}$ and $e_0 = 1 - e_1$. Let $f(x_{1,g_{i_1}},\dots,x_{n,g_{n_1}})$ be a non-identity of A which satisfies property K and let $f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}})$ be a non-zero evaluation. If $e_1 f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}}) \neq 0$ (or $f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}})e_1 \neq 0$) we proceed as in case $(1,2)$. To treat the remaining case we may assume that:

(1) f is a non-identity and satisfies property K.
(2) A is full with respect to f.
(3) $e_0 f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}})e_0 \neq 0$.

First note that if one of the radical values (say $\widehat{w}_g$) in $f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}})$ allows a bordering by the pair (e_0,e_1) (and remains non-zero), then replacing w_g by $w_g y_e$ where y_e is an e-variable, yields a non-identity. Invoking Lemma 5.2 we may replace the variable y_e by a G-graded polynomial h with ν-folds of alternating (small) g-sets of cardinality $\dim_F(\overline{A}_g) = \dim_F((A_1)_g)$ for every g in G. Then we attach the radical variable w_g to a suitable small set (same G-degree). Clearly, the value of any alternation of this (big) set is zero since the borderings are different. Finally we attach the remaining radical variables to suitable small sets in h. Again any alternation vanishes because of property K. This settles this case. Obviously, the same holds if the bordering pair above is (e_1,e_0). The outcome is that we may assume that all radical values may be bordered by either (e_0,e_0) or (e_1,e_1).

Claim 6.11. *Under the above assumption, all pairs that border radical values are equal.*

Indeed, if we have of both kinds, we must have a radical value which is bordered by a mixed pair since the semisimple variable can be bordered only the pair (e_1,e_1).

Now, assume all the bordering pairs of the radical values are (e_1,e_1). Since also the semisimple values can be bordered (only) by that pair it follows that the entire value of the polynomial, namely $f(\widehat{x}_{1,g_{i_1}},\dots,\widehat{x}_{n,g_{i_n}})$, may be multiplied by (e_1,e_1) but this case was already taken care of.

The last case to consider, is the case where the all bordering pairs of the radical values are (e_0,e_0). Here we use the fact that the polynomial is full (rather than satisfying property K as in previous cases) and replace one of the semisimple variables (say x_g) by $x_g y_e$. Then as above we replace y_e by G-graded polynomial h with ν-folds of alternating (small) g-sets of cardinality $\dim_F(\overline{A}_g) = \dim_F((A_1)_g)$ for every g in G. The point in this case is that we may attach all radical variables to suitable small sets from h. Clearly, since the borderings are different ((e_0,e_0) for the radical values and (e_1,e_1) for the semisimple ones) any non-trivial alternation will vanish. This completes the proof of the lemma. $\square$

Corollary 6.12. *If A is basic then its Kemer set consists of precisely one point* $(\alpha, s(\alpha)) = (\alpha_{g_1},\dots,\alpha_{g_r}; s(\alpha)) = (d_{g_1},\dots,d_{g_r}; n_A - 1)$.

Corollary 6.13.

(1) *Let A be a basic algebra and let f be a Kemer polynomial of A i.e. a Kemer polynomial of its unique Kemer point $(\alpha, s(\alpha))$. Then it satisfies the A-Phoenix property.*
(2) *More generally: let A be a finite-dimensional algebra A and let f be a Kemer polynomial of a Kemer point of A. Then it satisfies the A-Phoenix property.*

Proof. Clearly if f is Kemer then A is full and satisfies property K with respect to f. The first part of the corollary now follows from Lemmas 6.4, 6.6 and 6.7.

The second part follows at once from the first. $\square$

7. More tools

In this section we present several concepts and results which will be essential for the proof of the main theorems. These concepts are borrowed from the classical *PI*-theory.

7.1. Finite generation of the relatively free algebra

It is well known that if $\mathcal{W}$ is a relatively free algebra over a field of characteristic zero which satisfies the Capelli identity c_n, then it has basic rank $< n$ (i.e. mod c_n, any identity of $\mathcal{W}$ is equivalent to an identity with less than n variables). Indeed, any non-zero multilinear polynomial with m variables, generating an irreducible S_m-module corresponds to a Young tableau with strictly less than n-rows and hence is equivalent (via linearization) to a homogeneous polynomial with less than n-variables (see [24, Section 1]). The same holds for G-graded polynomials, i.e. a polynomial with m_i g_i-variables, $i = 1, \dots, r$. Here one considers the action of the group $S_{m_1} \times \dots \times S_{m_r}$ on the set of multilinear polynomial with $m_1 + \dots + m_r = m$ variables (each symmetric group acts on the corresponding variables) and shows that such a polynomial is equivalent to an homogeneous polynomial with less than n variables of each type (i.e. $< rn$). This gives:

Corollary 7.1. *Let W be a G-graded affine algebra which is (ungraded) PI. Then there exists a relatively free affine G-graded algebra $\mathcal{W}_{affine}$ with $\mathrm{id}_G(W) = \mathrm{id}_G(\mathcal{W}_{affine})$.*

Corollary 7.2. *All G-graded Kemer polynomials of W are obtained (via linearization) from Kemer polynomials with a bounded number of variables.*

Remark 7.3. We could obtain the corollaries above from Berele and Bergen result (see [11, Lemma 1]).

7.2. The G-graded generic algebra

We start with an alternative description of the relatively free algebra $\mathcal{A} = F\langle X_G = \bigcup X_g\rangle / \mathrm{id}_G(A)$ of a finite-dimensional algebra A. Note that by the virtue of Corollary 7.1 we may (and will) assume that the set of g-variables is finite, say m_g, for every $g \in G$.

Let $\{b_{1,g}, b_{2,g}, \dots, b_{t_g,g}\}$ be a basis of the g-component of A and let $\Lambda_G = \{\lambda_{i,j,g} \mid i = 1, \dots, m,\ j = 1, \dots, t_g,\ g \in G\}$ be a set of commuting variables which centralize the elements of A. For any g in G, consider the elements

$$y_{i,g} = \sum_j b_{j,g}\lambda_{i,j,g}$$

for $i = 1, \dots, m_g$ and consider the subalgebra $\widetilde{\mathcal{A}}$ they generate in the polynomial algebra $A[\Lambda_G]$. The proof of the following lemma is identical to the proof in [23], Section 3.3.1 and is omitted.

Lemma 7.4. *The map $\pi : \mathcal{A} \to \widetilde{\mathcal{A}}$ defined by $\pi(x_{i,g}) = y_{i,g}$ is a G-graded isomorphism.*

In particular, the relatively free algebra of a finite-dimensional algebra A is representable i.e. it can be G-graded embedded in a finite-dimensional algebra. The next claim is well known.

Claim 7.5. *Any G-graded finite-dimensional algebra A over a field K can be G-embedded in a G-graded matrix algebra.*

Proof. Let $n = \dim_K(A)$ and let $M = \mathrm{End}_K(A) \cong M_n(K)$ be the algebra of all endomorphisms of A. We may introduce a G-grading on $\mathrm{End}_K(A)$ by setting $M_g = \{\varphi \in M$ such that $\varphi(A_h) \subseteq A_{gh}\}$. Let us show that any endomorphism of A can be written as a sum of homogeneous elements φ_g. Indeed, if φ is in M and $h \in G$ we define φ_g on A_h by $\varphi_g = P_{gh} \circ \varphi$ where P_{gh} is the projection of A onto A_{gh}. Taking $a_h \in A_h$ we have $\bigoplus_g \varphi_g(a_h) = \bigoplus_g P_{gh} \circ \varphi(a_h) = \varphi(a_h)$. Since this is for every h in G the result follows. □

7.3. Shirshov (essential) base

For the reader convenience we recall the definition from classical *PI*-theory (i.e. ungraded).

Definition 7.6. Let W be an affine *PI*-algebra over F. Let $\{a_1, \dots, a_s\}$ be a set of generators of W. Let m be a positive integer and let Y be the set of all words in $\{a_1, \dots, a_s\}$ of length $\leqslant m$. We say that W has *Shirshov* base of length m and of height h if elements of the form $y_{i_1}^{k_1} \cdots y_{i_l}^{k_l}$ where $y_{i_i} \in Y$ and $l \leqslant h$, span W as a vector space over F.

Theorem 7.7. *If W is an affine PI-algebra, then it has a Shirshov base for some m and h. More precisely, suppose W is generated by a set of elements of cardinality s and suppose it has PI-degree m (i.e. there exists an identity of degree m and m is minimal) then W has a Shirshov base of length m and of height h where $h = h(m, s)$.*

In fact we will need a weaker condition (see [10]).

Definition 7.8. Let W be an affine *PI*-algebra. We say that a set Y as above is an *essential Shirshov* base of W (of length m and of height h) if there exists a finite set $D(W)$ such that the elements of the form $d_{i_1} y_{i_1}^{k_1} d_{i_2} \cdots d_{i_l} y_{i_l}^{k_l} d_{i_{l+1}}$ where $d_{i_j} \in D(W)$, $y_{i_j} \in Y$ and $l \leqslant h$ span W.

An essential Shirshov's base gives

Theorem 7.9. *Let C be a commutative ring and let $W = C\langle\{a_1, \dots, a_s\}\rangle$ be an affine algebra over C. If W has an essential Shirshov base (in particular, if W has a Shirshov base) whose elements are integral over C, then it is a finite module over C.*

Returning to G graded algebras we have

Proposition 7.10. *Let W be an affine, PI, G-graded algebra. Then it has an essential G-graded Shirshov base of elements of W_e.*

Proof. W is affine so it is generated by a finite set of elements $\{a_1, \dots, a_s\}$ which can be assumed to be homogeneous. We form the set Y of words in the a's, of length $\leqslant m$ (say), so that Y provides a Shirshov base of W. Now each element y of Y corresponds to an homogeneous component, say g. Hence, raised to the order of g in G it represents an element of W_e. Let Y_e be the subset of W_e consisting of elements $y^{\operatorname{ord}(g)}$ where $y \in Y$ of degree g and let $D(W)$ be the set consisting of all elements of the form $(1, y, y^2, \dots, y^{\operatorname{ord}(g)-1})$. Clearly, Y_e is an essential Shirshov base of W. □

7.4. The trace ring

Let A be a finite-dimensional G-graded algebra over F and let $\mathcal{A}$ be the corresponding relatively free algebra. By Lemma 7.4, $\mathcal{A}$ is representable, i.e. can be embedded (as a G-graded algebra) into a matrix algebra M over a suitable field K. For every element $x_e \in \mathcal{A}_e$ (viewed in M) we consider its trace $\operatorname{Tr}(x_e) \in K$. We denote by $R_e = F[\{\operatorname{Tr}(x_{s,e})\}]$ the F-algebra generated by the trace elements of $\mathcal{A}_e$. Note that R_e centralizes $\mathcal{A}$ and hence we may consider the extension $\mathcal{A}_{R_e} = R_e \otimes_F \mathcal{A}$. We refer to $\mathcal{A}_{R_e}$ as the extension of $\mathcal{A}$ by traces (of $\mathcal{A}_e$). In particular we may consider $(\mathcal{A}_e)_{R_e}$, namely the extension of $\mathcal{A}_e$ by traces.

Remark 7.11. $\mathcal{A}_e = 0$ if and only if $A_e = 0$. In that case $\mathcal{A}_{R_e} = \mathcal{A}$.

Lemma 7.12. *The algebras $\mathcal{A}_{R_e}$ and $(\mathcal{A}_e)_{R_e}$ are finite modules over R_e.*

Proof. By the Cayley–Hamilton theorem any element of $(\mathcal{A}_e)_{R_e}$ is integral over R_e and hence it has a Shirshov base consisting of elements which are integral over R_e. It follows by Theorem 7.9 that $(\mathcal{A}_e)_{R_e}$ is a finite module over R_e. Now, as noted above, since G is finite, $\mathcal{A}_{R_e}$ has an essential Shirshov base ($\subset (\mathcal{A}_e)_{R_e}$) whose elements are integral over R_e. Applying Theorem 7.9 the result follows. □

8. Kemer polynomials, emulation of traces and representability

Let A be a basic G-graded algebra and let $\mathcal{A}$ be the corresponding relatively free algebra.

Lemma 8.1. *Let I be a G-graded T-ideal of $\mathcal{A}$ which is closed under multiplication by traces. Then $\mathcal{A}/I$ is representable. (This is abuse of language: we should have said I is the G-graded ideal of $\mathcal{A}$ generated by the evaluations (on $\mathcal{A}$) of a G-graded T-ideal.)*

Proof. Indeed, by Lemma 7.12, $\mathcal{A}_{R_e}$ is a finite module over R_e. Hence $\mathcal{A}_{R_e}/I_{R_e}$ is a finite module as well. Our assumption on I says $I_{R_e} = I$ and hence $\mathcal{A}_{R_e}/I$ is a finite module over R_e. Now, R_e is a commutative Noetherian ring and hence applying [9] we have that $\mathcal{A}_{R_e}/I$ is representable. Since $\mathcal{A}/I \subseteq \mathcal{A}_{R_e}/I$, $\mathcal{A}/I$ is representable as well. □

A key property of Kemer polynomials is *emulation of traces*. This implies that T-ideals generated by Kemer polynomials are closed under multiplication by traces. Here is the precise statement.

Proposition 8.2. *Let $A = A_1 \times \cdots \times A_u$ where the A_i's are basic, G-graded algebras. Here A is G-graded in the obvious way. Assume the algebras A_i have the same Kemer point. Let $\mathcal{A}_i$ be the relatively free algebra of A_i and set $\widehat{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_u$. Let S be the T-ideal generated by a set of Kemer polynomials of some of the A_i's and let $S_{\widehat{A}}$ be the ideal of $\widehat{A}$ generated by all evaluation of S on $\widehat{A}$. Then $S_{\widehat{A}}$ is closed under multiplication by traces (of elements of $\widehat{A}_e$).*

Proof. Let $(\alpha_{g_1}, \ldots, \alpha_{g_r}, n_A - 1)$ be the Kemer point which corresponds to A. Recall that if $\alpha_{g_1} = 0$ (i.e. $\overline{A}_e$, the semisimple part of the e-component of A, is zero) then $R_e = F$ and the proposition is clear. We assume therefore that $\overline{A}_e \neq 0$. Let z_e be in $\mathcal{A}_e$ and f in S. We need to show that $\mathrm{Tr}(z_e) f$ evaluated on $\mathcal{A}$ is in $S_{\widehat{A}}$. Clearly, we may assume that f is a Kemer polynomial of A_1. To simplify the notation we put $d = \alpha_{g_1}$ and write $f = f(x_{e,1}, \ldots, x_{e,d}, \overrightarrow{y})$ where the variables $x_{e,i}$'s alternate. Let us recall the following important result on alternating (ungraded) polynomials from [23]:

Theorem 8.3. *(See [23, Theorem J].) Suppose $B \subseteq M_n(K)$ is an algebra over K, and let V be a t-dimensional K-subspace of $M_n(K)$ with a base $a_1, \ldots, a_t$ consisting of elements of B. Let $f(x_1, \ldots, x_t; \overrightarrow{y})$ be an alternating polynomial in the x's. If T is a C-linear map ($C = Z(B)$) $T : V \to V$, then*

$$\mathrm{Tr}(T) f(a_1, \ldots, a_t; \overrightarrow{b}) = \sum_{k=1}^{t} f(a_1, \ldots, a_{k-1}, T a_k, a_{k+1}, \ldots, a_t; \overrightarrow{b}).$$

First note that the same result (with the same proof) holds for a G-graded polynomial f where the x's are e-variables and the space V is contained in the e-component of B. For our purposes we consider $\mathcal{A}_1$ to be the relatively free algebra of A_1. Extending scalars to K we have $B = K \otimes \mathcal{A}_1$. Then we take V to be the e-component of the semisimple part of B. The key observation here is that since f is a Kemer polynomial, on any non-zero evaluation, the variables $x_{e,1}, \ldots, x_{e,d}$ may assume only values which form a basis of V and hence the result follows from the G-graded version of Theorem 8.3. This completes the proof of Theorem 8.2. $\square$

Combining Lemma 8.1 and Proposition 8.2 we obtain the following corollary (the notation is as in Proposition 8.2).

Corollary 8.4. *$\widehat{A}/S_{\widehat{A}}$ is representable.*

9. Γ-Phoenix property

Let A be a finite-dimensional G-graded algebra. Recall that by Proposition 4.15, A is *PI*-equivalent to a direct product of basic algebras.

Let $A \sim A_1 \times \cdots \times A_s$ where the A_i's are basic. For each A_i we consider its Kemer point $(\alpha_{A_i}, s(\alpha_{A_i})) = ((d_{A_i,g_1}, d_{A_i,g_2}, \ldots, d_{A_i,g_r}), n_{A_i} - 1)$. Let $(\alpha_{A_1}, s(\alpha_{A_1})), \ldots, (\alpha_{A_t}, s(\alpha_{A_t}))$ be the Kemer points which are maximal among the Kemer points $(\alpha_{A_1}, s(\alpha_{A_1})), \ldots, (\alpha_{A_s}, s(\alpha_{A_s}))$ (after renumbering if necessary).

Proposition 9.1. $\mathrm{Kemer}(A) = \bigcup_{1 \leqslant i \leqslant t} \mathrm{Kemer}(A_i)$. *Furthermore, a polynomial f is Kemer of A if and only if is Kemer of one of the A_i, $i = 1, \ldots, t$.*

Proof. Since for any $i = 1, \dots, t$, $\mathrm{id}_G(A_i) \supseteq \mathrm{id}_G(A)$, there exists a Kemer point $(\alpha, s(\alpha))$ of A with $(\alpha_{A_i}, s(\alpha_{A_i})) \preceq (\alpha, s(\alpha))$. On the other hand if $(\beta, s(\beta))$ is a Kemer point of A and f is a Kemer polynomial of A which corresponds to $(\beta, s(\beta))$, f is not an identity of A and hence not an identity of A_j for some $j = 1, \dots, s$. It follows that $(\beta, s(\beta)) \preceq (\alpha_{A_j}, s(\alpha_{A_j}))$. Thus we have two finite subsets of points, namely $\mathrm{Kemer}(A)$ and $\bigcup_{1 \leqslant i \leqslant s} \mathrm{Kemer}(A_i)$ in a partially ordered set in $(\mathbb{Z}^{\geqslant 0})^r \times (\mathbb{Z}^{\geqslant 0} \cup \infty)$ such that for any point $(u, s(u))$ in any subset there is a point $(v, s(v))$ in the other subset with $(u, s(u)) \preceq (v, s(v))$. Since $\mathrm{Kemer}(A)$ and $\bigcup_{1 \leqslant i \leqslant t} \mathrm{Kemer}(A_i)$ are maximal, they must coincide. In particular, note that the polynomial f above must be a non-identity (and hence Kemer) of A_j for some $j = 1, \dots, t$. It remains to show that a Kemer polynomial of A_j for $j = 1, \dots, t$ is a Kemer polynomial of A, but this is clear. □

Thus our T-ideal Γ (the T-ideal of identities of a G-graded affine algebra) contains $\mathrm{id}(A) = \mathrm{id}(A_1 \times \cdots \times A_s)$ where A_i are basic algebras.

As noted in Remark 3.9, $\mathrm{Ind}(\Gamma) \subseteq \mathrm{Ind}(A)$ and if α is a point in $E_0(\Gamma) \cap E_0(A)$ (i.e. is extremal for both ideals) then $s_\Gamma(\alpha) \leqslant s_A(\alpha)$.

Our aim now (roughly speaking) is to replace the finite-dimensional algebra A with a finite-dimensional algebra A' with $\Gamma \supseteq \mathrm{id}_G(A')$ but *PI* "closer" to Γ.

Here is the precise statement (see [23] for the ungraded version).

Proposition 9.2. *Let Γ and A be as above. Then there exists a G-graded finite-dimensional algebra A' with the following properties:*

(1) $\Gamma \supseteq \mathrm{id}_G(A')$.
(2) *The Kemer points of Γ coincide with the Kemer points of A'.*
(3) *Any Kemer polynomial of A' (i.e. a Kemer polynomial which corresponds to a Kemer point of A') is not in Γ (i.e. Γ and A' have the same Kemer polynomials).*

Remark 9.3. The proof is similar but not identical to the proof of Proposition 4.61 in [23]. For the reader convenience we give a complete proof here.

Proof. Let $(\alpha, s(\alpha))$ be a Kemer point of A (i.e. it corresponds to some of the basic components of A). After renumbering the components we can assume that $(\alpha, s(\alpha))$ is the Kemer point of $A_1, \dots, A_u$ and not of $A_{u+1}, \dots, A_s$. Suppose that $(\alpha, s(\alpha))$ is not a Kemer point of Γ. Note that since $\Gamma \supseteq \mathrm{id}_G(A)$, there is no Kemer point $(\delta, s(\delta))$ of Γ with $(\delta, s(\delta)) \succeq (\alpha, s(\alpha))$ and hence any Kemer polynomial of A which corresponds to the point $(\alpha, s(\alpha))$ is in Γ. Now for $i = 1, \dots, u$, let $\mathcal{A}_i$ be the relatively free algebra of A_i. For the same indices let S_i be the T-ideal generated by all Kemer polynomials of A_i and let $S_{\mathcal{A}_i}$ be the ideal of $\mathcal{A}_i$ generated by the evaluations of S_i on $\mathcal{A}_i$. By Corollary 8.4 we have that $\mathcal{A}_i / S_{\mathcal{A}_i}$ is representable.

Claim 9.4. *For any $i = 1, \dots, u$, if $(\beta, s(\beta))$ is any Kemer point of $\mathcal{A}_i / S_{\mathcal{A}_i}$, then $(\beta, s(\beta)) \prec (\alpha_{A_i}, s(\alpha_{A_i}))$.*

(In this claim one may ignore our assumption above that $(\alpha_{A_i}, s(\alpha_{A_i})) = (\alpha, s(\alpha))$ for $i = 1, \dots, u$.)

Assume the claim is false. This means that $\mathcal{A}_i / S_{\mathcal{A}_i}$ has a Kemer point $(\beta, s(\beta))$ for which $(\beta, s(\beta))$ and $(\alpha_{A_i}, s(\alpha_{A_i}))$ are either not comparable or $(\beta, s(\beta)) \succeq (\alpha_{A_i}, s(\alpha_{A_i}))$.

If $(\beta, s(\beta))$ and $(\alpha_{A_i}, s(\alpha_{A_i}))$ are not comparable then there is an element g in G with $\beta_g > \alpha_{A_i,g}$. But this contradicts $\mathrm{id}_G(A_i) \subset \mathrm{id}_G(\mathcal{A}_i/S_{\mathcal{A}_i})$ for if f is a Kemer polynomial for the Kemer point $(\beta, s(\beta))$ of $\mathcal{A}_i/S_{\mathcal{A}_i}$, it must vanish on A_i and hence is in $\mathrm{id}_G(A_i)$. The same argument yields a contradiction in case $(\beta, s(\beta)) \succ (\alpha_{A_i}, s(\alpha_{A_i}))$.

Assume now $(\beta, s(\beta)) = (\alpha_{A_i}, s(\alpha_{A_i}))$ and let f be a Kemer polynomial of the Kemer point $(\beta, s(\beta))$ of $\mathcal{A}_i/S_{\mathcal{A}_i}$. The polynomial f is not in $\mathrm{id}_G(\mathcal{A}_i/S_{\mathcal{A}_i})$ and hence is not in $\mathrm{id}_G(A_i)$. Hence f is a Kemer polynomial of A_i and therefore, by construction, it is in $\mathrm{id}_G(\mathcal{A}_i/S_{\mathcal{A}_i})$. This is a contradiction and the claim is proved.

We replace now each algebra in $A_1 \times \cdots \times A_u$ by $\mathcal{A}_i/S_{\mathcal{A}_i}$ (in the product $A = A_1 \times \cdots \times A_s$). Clearly, the set of Kemer points of the algebra $\mathcal{A}_1/S_{\mathcal{A}_1} \times \cdots \times \mathcal{A}_u/S_{\mathcal{A}_u} \times A_{u+1} \times \cdots \times A_s$ is strictly contained in the set of Kemer points of $A_1 \times \cdots \times A_u \times A_{u+1} \times \cdots \times A_s$ so parts 1 and 2 of the proposition will follow by induction if we show that $\mathrm{id}_G(\mathcal{A}_1/S_{\mathcal{A}_1} \times \cdots \times \mathcal{A}_u/S_{\mathcal{A}_u} \times A_{u+1} \times \cdots \times A_s) \subseteq \Gamma$. To see this note that $\mathcal{A}_1/S_{\mathcal{A}_1} \times \cdots \times \mathcal{A}_u/S_{\mathcal{A}_u} = B/S_B$ where $B = \mathcal{A}_1 \times \cdots \times \mathcal{A}_u$. Here S_B is the ideal of B generated by all evaluations of S on B and S is the T-ideal generated by all polynomial which are Kemer with respect to A_i for some $i = 1, \ldots, u$.

Let $z \in \mathrm{id}_G(\mathcal{A}_1/S_{\mathcal{A}_1} \times \cdots \times \mathcal{A}_u/S_{\mathcal{A}_u} \times A_{u+1} \times \cdots \times A_s) = \langle \mathrm{id}_G(\mathcal{A}_1 \times \cdots \times \mathcal{A}_u) + S\rangle_T \cap \mathrm{id}_G(A_{u+1}) \cap \cdots \cap \mathrm{id}_G(A_s)$ and write $z = h + f$ where $h \in \mathrm{id}_G(\mathcal{A}_1 \times \cdots \times \mathcal{A}_u)$ and $f \in S$. Clearly, we may assume that f is a Kemer polynomial of A_i for some $1 \leqslant i \leqslant u$. Now since the Kemer point of A_i, $i = 1, \ldots, u$, is maximal among the Kemer points of A, f and hence $h = z - f$ are in $\mathrm{id}_G(A_{u+1} \times \cdots \times A_s)$. It follows that $h \in \mathrm{id}_G(\mathcal{A}_1 \times \cdots \times \mathcal{A}_u) \cap \mathrm{id}_G(A_{u+1} \times \cdots \times A_s) = \mathrm{id}_G(A_1 \times \cdots \times A_s) \subseteq \Gamma$. But $S \subseteq \Gamma$ (since the point $(\alpha, s(\alpha))$ is not a Kemer point of Γ) and hence $z = h + f \in \Gamma$ as desired.

Now for the proof of (3) we may assume that Γ and A have the same Kemer points. Let $(\alpha, s(\alpha))$ be such a Kemer point and assume that some Kemer polynomials which correspond to $(\alpha, s(\alpha))$ are in Γ. After renumbering the basic components of A we may assume that $(\alpha, s(\alpha))$ is the Kemer point of A_i, $i = 1, \ldots, u$. We repeat the argument above but now instead of taking the set of all Kemer polynomials of the point $(\alpha, s(\alpha))$ we take only the set of Kemer polynomial of $(\alpha, s(\alpha))$ which are contained in Γ. This completes the proof of the proposition. □

Remark 9.5. The proof of Proposition 9.2 can be sketched as follows: We start with $\Gamma \supseteq \mathrm{id}_G(A)$ where A is finite-dimensional and G-graded. Let $\mathcal{A}$ be the relatively free algebra of A. By Lemma 7.4 $\mathcal{A}$ is representable and hence we may consider the trace values of elements of $\mathcal{A}_e$. Let $\Gamma_0 \subseteq \Gamma$ be the maximal T-ideal contained in Γ which is closed under multiplication by traces. By Proposition 8.2, Γ_0 *contains the T-ideal generated by Kemer polynomials of A which are contained in Γ*. Now, it follows from [9], that $\mathcal{A}/I_{\Gamma_0}$ is representable, where I_{Γ_0} is the ideal generated by all evaluations of Γ_0 on $\mathcal{A}$. Hence $\mathcal{A}/I_{\Gamma_0}$ is *PI*-equivalent to some finite-dimensional algebra A' with $\mathrm{id}_G(A') \subseteq \Gamma$. Finally, one sees that either the Kemer points of A' are smaller comparing to those of A or else the intersection of Γ with Kemer polynomials of A' is zero (it's preimage in $\mathcal{A}$ must be in Γ_0).

Corollary 9.6 *(Γ-Phoenix property). Let Γ be a T-ideal as above and let f be a Kemer polynomial of Γ. Then it satisfies the Γ-Phoenix property.*

Proof. By Proposition 9.2 f is a Kemer polynomial of a Kemer point of a finite-dimensional algebra A. By Corollary 6.13 for every polynomial $f' \in \langle f\rangle$ there is a polynomial $f'' \in \langle f'\rangle$ which is Kemer for A. Applying once again Proposition 9.2 the result follows. □

10. Zubrilin–Razmyslov traces and representable spaces

As explained in Section 1, the proof of representability of G-graded affine algebras (Theorem 1.1) has two main ingredients. One is the *Phoenix property* of Kemer polynomials (which is the final statement of the last section) and the other one (which is our goal in this section) is the construction of a *representable* algebra which we denoted there by $B_{(\alpha,s)}$.

Choose a Kemer point $(\alpha, s(\alpha))$ of Γ and let $S_{(\alpha,s(\alpha))}$ be the T-ideal generated by all Kemer polynomial which correspond to the point $(\alpha, s(\alpha))$ with at least μ-folds of small sets. Note that by Remark 7.2 we may assume that the total number of variables in these polynomials is bounded. Let $\mathcal{W}_\Gamma$ be the relatively free algebra of Γ. In what follows it will be important to assume (as we may by Corollary 7.1) that $\mathcal{W}_\Gamma$ is affine. Since we will not need to refer explicitly to the variables in the construction of $\mathcal{W}_\Gamma$ we keep the notation X_G of G-graded variables for a different purpose. Let $X_G = \bigcup X_g$ be a set of G-graded variables where X_g has cardinality $\mu\alpha_g + s(\alpha)(\alpha_g + 1)$ (i.e. enough g-variables to support Kemer polynomials with μ small sets and possibly $s(\alpha)$ big sets which are g-homogeneous). Let $\mathcal{W}'_\Gamma = \mathcal{W}_\Gamma * F\{X_G\}$ (G-graded) and $\mathcal{U}_\Gamma = \mathcal{W}'_\Gamma / I_1$ where I_1 is the ideal generated by all evaluations of Γ on $\mathcal{W}'_\Gamma$. Note that the algebra $\mathcal{U}_\Gamma$ is G-graded isomorphic to the relatively free algebra of Γ and hence $\mathrm{id}_G(\mathcal{U}_\Gamma) = \Gamma$.

Consider all possible evaluations in $\mathcal{W}'_\Gamma$ of the Kemer polynomials in $S_{(\alpha,s(\alpha))}$ in such a way that *precisely* μ folds of small sets and all big sets (and no other variables) are evaluated on different variables of X_G. Denote by S_0 the space generated by these evaluations. Note that every non-zero polynomial in S_0 has an evaluation of that kind which is non-zero in $\mathcal{U}_\Gamma$. In other words $S_0 \cap I_1 = 0$.

Our aim is to construct a representable algebra $B_{(\alpha,s(\alpha))}$ and a G-graded epimorphism $\varphi : \mathcal{U}_\Gamma \to B_{(\alpha,s(\alpha))}$ (in particular $\Gamma \subseteq \mathrm{id}_G(B_{(\alpha,s(\alpha))})$), such that φ maps the space S_0 isomorphically into $B_{(\alpha,s(\alpha))}$. Let us introduce the following general terminology.

Definition 10.1. Let W be a G-graded algebra over a field F. Let S be an F-subspace of W. We say that S is a *representable space* of W if there exists a G-graded representable algebra B and a G-graded epimorphism

$$\phi : W \to B$$

such that ϕ maps S isomorphically into B.

We can now state the main result of this section.

Theorem 10.2 *(Representable space). With the above notation, there exists a representable algebra $B_{(\alpha,s(\alpha))}$ and a G-graded surjective homomorphism $\varphi : \mathcal{U}_\Gamma \to B_{(\alpha,s(\alpha))}$ (hence $\Gamma \subseteq \mathrm{id}_G(B_{(\alpha,s(\alpha))})$) and such that φ maps the space S_0 isomorphically into $B_{(\alpha,s(\alpha))}$. In particular the space S_0 is representable.*

It is appropriate to view the theorem above as a “partial success”: Our final goal is to show that the algebra $\mathcal{U}_\Gamma$ is representable but here we “only” prove that the subspace S_0 (spanned by Kemer polynomials) is representable. In order to complete the proof of Theorem 1.1 we must invoke the Phoenix property of Kemer polynomials. The reader may want to “jump” to Section 11 and see how to finalize the proof of Theorem 1.1 using the Phoenix property of Kemer polynomials and the representability of the space S_0.

The construction of $B_{(\alpha,s(\alpha))}$ is based on two key lemmas. One is the "Zubrilin–Razmyslov identity" and the second is a lemma named as the "interpretation lemma". We start with the "Zubrilin–Razmyslov identity" (see [23] for the ungraded case).

Let $\{x_{1,e},\dots,x_{n,e},x_{n+1,e}\}$ be a set of *e-variables*, Y_G a set of arbitrary G-graded variables and $z=z_e$ an additional *e-variable*. For a given G-graded polynomial $f(x_{1,e},\dots,x_{n,e},x_{n+1,e};Y_G)$, multilinear in the x's, we define $u_j^z(f)$ to be the homogeneous component of degree j in z in the polynomial $f((z+1)x_{1,e},\dots,(z+1)x_{n,e},x_{n+1,e};Y_G)$. In other words $u_j^z(f)$ is the sum of all polynomials obtained by replacing $x_{i,e}$ by $zx_{i,e}$ in j positions from $\{x_{1,e},\dots,x_{n,e}\}$. Clearly, if f alternates in the variables $\{x_{1,e},\dots,x_{n,e}\}$ then $u_j^z(f)$ alternates in these variables as well. Note that for any $1\leqslant i,j\leqslant n$, the operators u_i^z and u_i^z commute.

Let A be any G-graded algebra over F. Let f be as above and assume it alternates in $\{x_{1,e},\dots,x_{n,e}\}$. Consider the polynomial

$$\begin{aligned}\widetilde{f}(x_{1,e},\dots,x_{n,e},x_{n+1,e};Y_G)={}&f(x_{1,e},\dots,x_{n,e},x_{n+1,e};Y_G)\\&-\sum_{k=1}^{n}f(x_{1,e},\dots,x_{k-1,e},x_{n+1,e},x_{k+1,e},\dots,x_{n,e},x_{k,e};Y_G).\end{aligned}$$

Note that $\widetilde{f}(x_{1,e},\dots,x_{n,e},x_{n+1,e};Y_G)$ alternates in the variables $\{x_{1,e},\dots,x_{n+1,e}\}$. The proof of the following proposition is identical to the proof of Proposition 2.44 in [23] and hence is omitted.

Proposition 10.3 *(Zubrilin–Razmyslov identity). With the above notation: if*

$$\widetilde{f}(x_{1,e},\dots,x_{n,e},x_{n+1,e};Y_G)$$

is a G-graded identity of A then also is

$$\sum_{j=0}^{n}(-1)^j u_j^z f\bigl(x_{1,e},\dots,x_{n,e},z^{n-j}x_{n+1,e};Y_G\bigr).$$

Lemma 10.4 *(Interpretation lemma). Let A be a G-graded algebra over a field F and I a G-graded ideal of A. Let $\Lambda=F[\theta_1,\dots,\theta_n]$ be a commutative, finitely generated F-algebra. Suppose Λ acts on I as linear operators and the action commutes with the multiplication in A (we view the elements of Λ as homogeneous of degree $e\in G$). Consider the extension of A by commuting e-variables $\{\lambda_1,\dots,\lambda_n\}$ and let K be the G-graded ideal of $A[\lambda_1,\dots,\lambda_n]$ generated by the elements $(\lambda_i x-\theta_i x)$, $i=1,\dots,n$ and $x\in I$. Then the natural map $A\to A'=A[\lambda_1,\dots,\lambda_n]/K$ is an embedding.*

Proof. We prove the lemma by giving an explicit description of A'. Let V be a complement of I in A (as an F-vector space). Since I is G-graded we may assume that V is spanned by homogeneous elements. Let $F[\lambda_1,\dots,\lambda_n]\otimes_F V$ be the extension V by the λ_i's and consider the subspace $C=F[\lambda_1,\dots,\lambda_n]\otimes_F V+I$ of $A[\lambda_1,\dots,\lambda_n]$. We introduce an action of $F[\lambda_1,\dots,\lambda_n]$ on C as follows: The action on $F[\lambda_1,\dots,\lambda_n]\otimes_F V$ is the obvious one where the action on I is given by $\lambda_i x=\theta_i x$ for every $x\in I$. Next we introduce a multiplication on C: Let $\mu: F[\lambda_1,\dots,\lambda_n]\to F[\theta_1,\dots,\theta_n]$ be the algebra map defined by $\lambda_i\mapsto\theta_i$. Take v_1 and v_2 in

V and let their multiplication in A be given by $v_1v_2 = v_3 + a$ where $v_3 \in V$ and $a \in I$. Then we define $(r_1 \otimes v_1)(r_2 \otimes v_2) = r_1r_2 \otimes v_3 + \mu(r_1r_2)a$. The product of $r \otimes v$ and an element of I is defined in the same way (using the map μ). Now it is clear that the algebras C and A' are isomorphic and that A is embedded in C. $\square$

Remark 10.5. A similar statement can be proved for algebras over an arbitrary commutative, Noetherian ring R. Instead of the space V one can consider the coset representatives of I in A and be more "careful" with the addition operation.

We can turn now to the construction of $B_{(\alpha,s(\alpha))}$.

Consider the ideal I_2 of $\mathcal{W}'_\Gamma$ generated by all elements of the form $x_g z x_g$, $z \in \mathcal{W}'_\Gamma$ and $x_g \in X_g$. Clearly the natural map $\mathcal{W}'_\Gamma \to \mathcal{W}'_\Gamma / I_2$ maps the space S_0 isomorphically. To simplify the notation we denote the image of S_0 in $\mathcal{W}'_\Gamma / I_2$ again by S_0. Note that the ideal of $\mathcal{W}'_\Gamma / I_2$ generated by the elements of X_G is nilpotent.

In order to construct the algebra $B_{(\alpha,s(\alpha))}$ we construct a sequence of algebras $B^{(r)}$, $r = 0, \ldots, t$, where $B^{(0)} = \mathcal{W}'_\Gamma / I_2$, $B^{(t)} = B_{(\alpha,s(\alpha))}$, and $B^{(r+1)}$ is obtained from $B^{(r)}$ by first extending its centroid with a certain finite set of indeterminates $\lambda_{i,1}, \ldots, \lambda_{i,n}$ (n is the cardinality of an e-small set) and then by modding out from $B^{(r)}[\lambda_{i,1}, \ldots, \lambda_{i,n}]$ a suitable ideal which we denote by J_{a_r}. Our main tasks will be:

(1) to show that $B_{(\alpha,s(\alpha))}$ is a finite module over its centroid (and hence representable by [9]),
(2) to show that the subspace of $B^{(r)}$ spanned by the image of S_0 is mapped isomorphically into $B^{(r+1)}$.

We choose an essential Shirshov base $\{a_0, \ldots, a_{t-1}\}$ of $\mathcal{W}'_\Gamma$. As shown in Proposition 7.10, these elements can be taken from $(\mathcal{W}'_\Gamma)_e$. Moreover, since the ideal generated X_G is nilpotent we can assume the a_i's are X_G-free. Clearly, the (images of) elements $\{a_0, \ldots, a_{t-1}\}$ form an essential Shirshov base of $B^{(0)} = \mathcal{W}'_\Gamma / I_2$. Moreover since the construction of $B^{(j+1)}$ consists of extending the centroid of $B^{(j)}$ and modding out by a certain ideal, the image of $\{a_0, \ldots, a_{t-1}\}$ in $B^{(j)}$ is an essential Shirshov base for $B^{(j)}$, $j = 0, \ldots, t$. We are now ready to define $B^{(j+1)}$.

Let $B^{(j+1)} = B^{(j)}[\lambda_{j,1}, \ldots, \lambda_{j,n}]/J_{a_j}$ where J_{a_j} is the ideal generated by the expression

$$a_j\big(a_j^n + \lambda_{j,1}a_j^{n-1} + \lambda_{j,2}a_j^{n-2} + \cdots + \lambda_{j,n}\big) = a_j^{n+1} + \lambda_{j,1}a_j^n + \lambda_{j,2}a_j^{n-1} + \cdots + \lambda_{j,n}a_j.$$

From the definition of J_{a_j} it follows that the image of a_j in $B^{(t)}$ is integral over the centroid. In other words $B^{(t)}$ has an essential Shirshov base consisting of integral elements and so it is a finite module over its centroid. This proves (1).

For the proof of (2) we need to show that $\bar{S}_0$, the image of S_0 in $B^{(r)}[\lambda_{r,1}, \ldots, \lambda_{r,n}]$, intersects trivially J_{a_r}. This is an immediate consequence of the lemma below. We insist in rephrasing it as a separate lemma in order to emphasize that its proof is independent of the inductive process presented above.

Let W be a *PI*, G-graded affine algebra over a field K. Let $(\alpha, s(\alpha))$ be a Kemer point of W. Fix a configuration of big sets according to the Kemer point $(\alpha, s(\alpha))$, that is we fix an $s(\alpha)$-tuple $u = (g_1, \ldots, g_{s(\alpha)})$ in $G^{s(\alpha)}$. Let $X_{(\alpha,s(\alpha),u)}$ be a set of graded variables with μ small sets of g-variables of cardinality α_g, for every $g \in G$ and big sets Λ_{g_i} of g_i-variables, of cardinality $\alpha_{g_i} + 1$ for $i = 1, \dots, s(\alpha)$. Thus the total number of variables in $X_{(\alpha,s(\alpha),u)}$ is given by

$$\mu \sum_{g \in G} \alpha_g + \sum_{i=1}^{s(\alpha)} (\alpha_{g_i} + 1).$$

(Note that here we don't require that W has a Kemer polynomial with such configuration.)

Consider the algebra $\widehat{W} = W * \{X_{(\alpha,s(\alpha),u)}\}/(I_1 + I_2)$ where I_1 is the ideal of $W * \{X_{(\alpha,s(\alpha),u)}\}$ generated by all evaluations of $\mathrm{id}_G(W)$ on $W * \{X_{(\alpha,s(\alpha),u)}\}$ and I_2 the ideal of $W * \{X_{(\alpha,s(\alpha),u)}\}$ generated by elements of the form $x_g w' x_g$ where $w' \in W * \{X_{(\alpha,s(\alpha),u)}\}$. Consider the scalar extension $\widehat{W}[\lambda_1, \dots, \lambda_n]$ of $\widehat{W}$ where λ_i are indeterminates and $n = \alpha_e$. Given an element $b \in W$, let J_b be the ideal of $W * \{X_{(\alpha,s(\alpha))}\}/(I_1 + I_2)$ generated by the expression

$$b\bigl(b^n + \lambda_1 b^{n-1} + \lambda_2 b^{n-2} + \cdots + \lambda_n\bigr) = b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b.$$

Lemma 10.6. *Let S be the subspace of $\widehat{W}$ spanned by all polynomials in the graded variables of $X_{(\alpha,s(\alpha),u)}$ which alternate on small and big sets according to the configuration described above. Then the restriction to S of the natural map*

$$\widehat{W} \to \widehat{W}[\lambda_1, \dots, \lambda_n]/J_b$$

is an embedding.

Proof. We need to show that if $f \in S \cap J_b$ then $f = 0$. Being in J_b, f has the form

$$\sum p_i(X, \lambda)\bigl(b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b\bigr) q_i(X, \lambda)$$

for some p_i and q_i. Furthermore f can be written as sums of expressions of the form

$$p_1(X_1)\bigl(b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b\bigr) p_2(X_2) g(\lambda)$$

where

(1) $p_i(X_i)$ are polynomials in variables of $X_{(\alpha,s(\alpha),u)}$,
(2) from the definition of the ideal I_2 above we can assume that all variables of $X_{(\alpha,s(\alpha),u)}$ appear exactly once in either $p_1(X_1)$ or $p_2(X_2)$,
(3) the polynomials $p_i(X_i)$ are free of λ's,
(4) $g(\lambda)$ is $X_{(\alpha,s(\alpha),u)}$ free.

Let us alternate the variables of $X_{(\alpha,s(\alpha),u)}$ (according to its decomposition to small and big sets).

Note 10.7. Since the polynomial f is already alternating in the variables of $X_{(\alpha,s(\alpha),u)}$, the alternation above as the effect of multiplying the polynomial f by an integer π which is a product of factorials. Since the characteristic of the field F is zero we have $\pi \neq 0$. This is why the result of this section is not characteristic free. In positive characteristics (proceeding as above) one can conclude only that the *images* of the alternating operators form a representable space.

Applying this alternation on each summand

$$h = p_1(X_1)\big(b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b\big) p_2(X_2) g(\lambda) \in J_b$$

yields a polynomial

$$\widehat{h} = \sum \operatorname{sgn}(\sigma) p_1(X_{\sigma,1})\big(b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b\big) p_2(X_{\sigma,2}) g(\lambda)$$

that alternates on small sets and big sets of $X_{(\alpha,s(\alpha),u)}$. We will present an interpretation of the variables λ_i which annihilates $\widehat{h} = h(x_1, \ldots, x_n, y)$. But then, since f is free of λ's (that is the interpretation does not annihilate f) the result follows.

Recall the operators u_j^b from the Zubrilin–Razmyslov identity (Proposition 10.3). Factoring the algebra $\widehat{W}[\lambda_1, \ldots, \lambda_n]$ by the ideal generated by $\lambda_j - u_j^b$ applied to the polynomials of S yields the algebra

$$D = \widehat{W}[\lambda_1, \ldots, \lambda_n] / \big\langle \big(\lambda_j - u_j^b\big) S_0 \big\rangle.$$

Invoking the Interpretation Lemma (Lemma 10.4) for $\theta_j = u_j^b$, $j = 1, \ldots, n$ we have that $\widehat{W}$ and in particular S, are embedded in D and hence the interpretation does not annihilate f. Finally, let us see that the substitution $\theta_j = u_j^b$, $j = 1, \ldots, n$ annihilates

$$\widehat{h} = \sum \operatorname{sgn}(\sigma) p_1(X_{\sigma,1})\big(b^{n+1} + \lambda_1 b^n + \lambda_2 b^{n-1} + \cdots + \lambda_n b\big) p_2(X_{\sigma,2}) g(\lambda).$$

Indeed, this follows from Proposition 10.3 and the fact that $\widehat{h}$ is alternating on small and big sets which correspond to the Kemer point $(\alpha, s(\alpha))$. This completes the proof of Theorem 10.2. □

We close the section with the following general statement. The proof is similar to the proof of Theorem 10.2 and hence is omitted. Let W, $\overline{W}$, S as in the previous lemma.

Theorem 10.8. *The subspace S of $\widehat{W}$ is representable.*

11. Representability of affine G-graded algebras

In this section we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Choose a Kemer point $(\alpha, s(\alpha))$ of Γ and let $S_{(\alpha,s(\alpha))}$ be the T-ideal generated by all Kemer polynomial which correspond to the point $(\alpha, s(\alpha))$ with at least μ-folds of small sets. Consider the T-ideal $\Gamma' = \langle \Gamma + S_{(\alpha,s(\alpha))} \rangle$. Observe that the Kemer set of Γ' is strictly contained in the Kemer set of Γ (since $(\alpha, s(\alpha))$ is not a Kemer point of Γ'). Hence, applying induction (if $(\alpha, s(\alpha)) = 0$ is the only Kemer point of Γ then $\Gamma = \mathrm{id}_G(0)$), there exists a finite-dimensional algebra A' with $\Gamma' = \mathrm{id}_G(A')$. We show that Γ is *PI*-equivalent to the algebra $A' \oplus B_{(\alpha,s(\alpha))}$. Clearly, Γ is contained in the intersection of the T-ideals $\mathrm{id}_G(A')$

and $\mathrm{id}_G(B_{(\alpha,s(\alpha))})$. For the converse take an identity f of A' which is not in Γ. We can assume that f is generated by Kemer polynomials and hence by Corollary 9.6 it has a corollary f' which is Kemer. But then it has an evaluation in $\mathcal{W}'_\Gamma$ which yields a non-zero element of S_0. Applying Theorem 10.2 we have that $S_0 \cap \mathrm{id}_G(B_{(\alpha,s(\alpha))}) = 0$ and the result follows. This completes the proof of Theorem 1.1. □

Corollary 11.1 *(G-graded representability-affine). The relatively free G-graded algebra $\Omega_{F,G}/\mathrm{id}_G(W)$ is representable, that is, $\Omega_{F,G}/\mathrm{id}_G(W)$ can be embedded in a finite-dimensional algebra over a (sufficiently large) field K.*

Proof. By Theorem 1.1 we know that there exists a finite-dimensional G-graded algebra A with $\mathrm{id}_G(W) = \mathrm{id}_G(A)$. Consequently the corresponding relatively free algebras $\mathcal{W}$ and $\mathcal{A}$ are isomorphic. Since $\mathcal{A}$ is representable the result follows. □

We close the section with a theorem which is a corollary of Theorem 1.1, the reduction to direct products of basic algebras (Proposition 4.15) and Kemer's Lemma 2 (Lemma 6.7).

Theorem 11.2. *Every variety M_W of an affine algebra W can be generated by a finite-dimensional algebra which is a finite direct product of basic algebras $B_1, \ldots, B_n$.*

Note that we can view the basic algebras B_i as *adequate models* of the variety: this means that combinatorial parameters, namely, cardinalities of small sets and number of big sets of Kemer polynomials coincide with dimensions of graded components of the semisimple part of B_i and the nilpotency index of $J(B_i)$.

12. Specht problem for G-graded affine algebras

In this section we prove Theorem 1.2.

Let W be an affine *PI* G-graded algebra over F and let $\mathrm{id}_G(W)$ be its T-ideal of G-graded identities. Our goal is to find a finite generating set for $\mathrm{id}_G(W)$. Since we are assuming that W is (ungraded) *PI*, we have by Theorem 1.1 that $\mathrm{id}_G(W) = \mathrm{id}_G(A)$ where A is an algebra over K (a field extension of F), G-graded and finite-dimensional. If the dimension of A is m say, then clearly W satisfies c_{m+1}, the ungraded Capelli identity on $2(m+1)$ variables, or equivalently, the *finite* set of G-graded identities $c_{(G,m+1)}$ which follow from c_{m+1} by designating G-degrees to its variables.

Now, observe that any T-ideal of G-graded identities is generated by at most a countable number of graded identities (indeed, for each n the space of multilinear G-graded identities of degree n is finite-dimensional) hence we may take a sequence of graded identities $f_1, \ldots, f_n, \ldots$ which generate $\mathrm{id}_G(W)$. Clearly, since the set $c_{(G,m+1)}$ is finite, in order to prove the finite generation of $\mathrm{id}_G(W)$ it is sufficient to show that the ascending chain of graded T-ideals $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \cdots$, where Γ_n is the T-ideal generated by the polynomials $c_{(G,m+1)} \cup \{f_1, \ldots, f_n\}$, stabilizes.

Now by Section 7.1, for each n, the T-deal Γ_n corresponds to an affine algebra and hence invoking Theorem 1.1 we may replace each Γ_n by $\mathrm{id}_G(A_n)$ where A_n is a G-graded finite-dimensional algebra over a suitable field extension K_n of F. Clearly, extending the coefficients to a sufficiently large field K we may assume all algebras A_n are finite-dimensional over an algebraically closed field K.

So we need to show that the sequence $\mathrm{id}_G(A_1) \subseteq \mathrm{id}_G(A_2) \subseteq \cdots$ stabilizes in $\Omega_{F,G}$ or equivalently, that the sequence stabilizes in $\Omega_{K,G}$. Consider the Kemer sets of the algebras $\{A_n\}$, $n \geqslant 1$. Since the sequence of ideals is increasing, the corresponding Kemer sets are monotonically decreasing (recall that this means that for any Kemer point (α, s) of A_{i+1} there is a Kemer point (α', s') of A_i with $(\alpha, s) \preceq (\alpha', s')$). Furthermore, since these sets are finite, there is a subsequence $\{A_{i_j}\}$ whose Kemer points (denoted by E) coincide. Clearly it is sufficient to show that the subsequence $\{\mathrm{id}_G(A_{i_j})\}$ stabilizes and so, in order to simplify notation, we replace our original sequence $\{\mathrm{id}_G(A_i)\}$ by the subsequence.

Choose a Kemer point (α, s) in E. Clearly we may replace the algebra A_i by a direct product of basic algebras $A'_{i,1} \times A'_{i,2} \times \cdots \times A'_{i,u_i} \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}$ where the $A'_{i,j}$'s correspond to the Kemer point (α, s) and the $\widehat{A}_{i,l}$ have Kemer index $\neq (\alpha, s)$ (note that their index may or may not be in E).

Our goal is to replace (for a subsequence of indices i_k) the direct product $A'_{i,1} \times A'_{i,2} \times \cdots \times A'_{i,u_i}$ (the basic algebras that correspond to the Kemer point (α, s)) by a certain G-graded algebra B such that

$$\mathrm{id}_G(B \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}) = \mathrm{id}_G(A_i)$$

for all i.

Let us show how to complete the proof assuming such B exists. Replace the sequence of indices $\{i\}$ by the subsequence $\{i_k\}$. (Clearly, it is sufficient to show that the subsequence of T-ideals $\{\mathrm{id}_G(A_{i_k})\}$ stabilizes.)

Let I be the T-ideal generated by Kemer polynomials of B which correspond to the Kemer point (α, s). Note that the polynomials in I are identities of the basic algebras $\widehat{A}_{i,l}$'s. It follows that the Kemer sets of the T-ideals $\{(\mathrm{id}_G(A_i) + I)\}$ do not contain the point (α, s) and hence are strictly smaller. By induction we obtain that the sequence of T-ideals

$$\bigl(\mathrm{id}_G(A_1) + I\bigr) \subseteq \bigl(\mathrm{id}_G(A_2) + I\bigr) \subseteq \cdots$$

stabilizes.

On the other hand we claim that $I \cap \mathrm{id}_G(A_i) = I \cap \mathrm{id}_G(A_j)$ for any i, j. This follows at once since $A_i = B \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}$ and $I \subseteq \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}$.

Combining the last statements the result follows.

Let us show now the existence of the algebra B.

Let A be a G-graded basic algebra which corresponds to the Kemer point (α, s). Let $A = \overline{A} \oplus J(A)$ be the decomposition of A into the semisimple and radical components. As shown in Section 5, $\alpha_g = \dim(\overline{A}_g)$ for every $g \in G$ and so, in particular, the dimension of $\overline{A}$ is determined by α. The following claim is key (see [7]).

Proposition 12.1. *The number of isomorphism classes of G-graded semisimple algebras of a given dimension is finite.*

Clearly it is sufficient to show that the number of isomorphisms classes of G-graded semisimple algebras C of a given dimension, which are G-simple, is finite. To see this recall that the G-graded structure is given by a subgroup H of G, a 2nd cohomology class in $H^2(H, K^*)$ and a k-tuple $(g_1, \ldots, g_k)$ in G^k where $k^2 \leqslant \dim(\overline{C})$. Clearly the number of subgroups H of G and the number of k-tuples are both finite. For the cardinality of $H^2(H, K^*)$, note that since K is algebraically closed the cohomology group $H^2(H, K^*)$ coincides with the Schur multiplier of H which is known to be finite. This proves the proposition.

We obtain:

Corollary 12.2. *The number of G-graded structures on the semisimple components of all basic algebras which correspond to the Kemer point (α, s) is finite.*

It follows that by passing to a subsequence $\{i_s\}$ we may assume that all basic algebras that appear in the decompositions above and correspond to the Kemer point (α, s) have G-graded isomorphic semisimple components (which we denote by C) and have the same nilpotency index $(= s)$.

Consider the G-graded algebras

$$\widehat{C}_i = C * K\langle X_G\rangle/(I_i + J)$$

where X_G is a set of G-graded variables of cardinality $(s-1)\cdot \mathrm{ord}(G)$ (that is $s-1$ variables for each $g \in G$), I_i is the ideal generated by all evaluations of $\mathrm{id}_G(A_i)$ on $C * K\langle X_G\rangle$ and J is the ideal generated by all words in $C * K\langle X_G\rangle$ with s variables from X_G.

Proposition 12.3.

(1) *The ideal generated by variables from X_G is nilpotent.*
(2) *For any i, the algebra $\widehat{C}_i$ is finite-dimensional.*
(3) *For any i,* $\mathrm{id}_G(\widehat{C}_i \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}) = \mathrm{id}_G(A_i)$.

Proof. The first part is clear. In order to prove (2) consider a typical non-zero monomial which represents an element of the algebra $\widehat{C}_i$. It has the form

$$a_{t_1} x_{t_1} a_{t_2} x_{t_2} \cdots a_{t_r} x_{t_r} a_{t_{(r+1)}}.$$

Since the set X_G is finite and also the number of variables appearing in a non-zero monomial is bounded by $s-1$, we have that the number of different configurations of these monomials (namely, the number of different tuples $x_{t_1}, \ldots, x_{t_r}$) is finite. In between these variables we have the elements a_{t_j}, $j = 1, \ldots, r+1$, which are taken from the finite-dimensional algebra C. This proves the second part of the proposition. We now show the 3rd part of the proposition.

Clearly, $\mathrm{id}_G(\widehat{A}_{i,j}) \supseteq \mathrm{id}_G(A_i)$ for $j = 1, \ldots, r_i$. Also, from the definition of $\widehat{C}_i$ we have that $\mathrm{id}_G(\widehat{C}_i) \supseteq \mathrm{id}_G(A_i)$ and so $\mathrm{id}_G(\widehat{C}_i \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i}) \supseteq \mathrm{id}_G(A_i)$. For the converse we show that $\mathrm{id}_G(\widehat{C}_i) \subseteq \mathrm{id}_G(A'_{i,j})$ for every $j = 1, \ldots, u_i$. To see this let us take a multilinear, G-graded polynomial $p = p(x_{i_1,g_{i_1}}, \ldots, x_{i_t,g_{i_t}})$ which is a graded non-identity of $A'_{i,j}$ and show that p is in fact a graded non-identity of $\widehat{C}_i$. Fix a non-vanishing evaluation of p on $A'_{i,j}$ where $x_{j_1,g_{j_1}} = z_1, \ldots, x_{j_k,g_{j_k}} = z_k$ $(k \leqslant s-1)$ are the variables with the corresponding radical evaluations and $x_{q_1,g_{q_1}} = c_1, \ldots, x_{q_k,g_{q_k}} = c_k$ are the other variables with their semisimple evaluations. Consider the G-graded map

$$\eta : C * K\langle X_G\rangle \to A'_{i,j}$$

where

(1) C is mapped isomorphically.
(2) A subset of k variables $\{y_1, \dots, y_k\}$ of X_G (with appropriate G-grading) are mapped onto the set $\{z_1, \dots, z_k\}$. The other variables from X_G are mapped to zero.

Note that η vanishes on $(I_i + J)$ and hence we obtain a G-graded map $\bar{\eta} : \widehat{C}_i \to A'_{i,j}$. Clearly, the evaluation of the polynomial $p(x_{i_1,g_{i_1}}, \dots, x_{i_t,g_{i_t}})$ on $\widehat{C}_i$ where $x_{q_1,g_{q_1}} = c_1, \dots, x_{q_k,g_{q_k}} = c_k$ and $x_{j_1,g_{j_1}} = y_1, \dots, x_{j_k,g_{j_k}} = y_k$ is non-zero and the result follows. □

At this point we have a sequence of T-ideals

$$\begin{aligned}&\mathrm{id}_G(\widehat{C}_1 \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{1,r_1}) \subseteq \cdots \subseteq \mathrm{id}_G(\widehat{C}_i \times \widehat{A}_{i,1} \times \cdots \times \widehat{A}_{i,r_i})\\ &\quad \subseteq \mathrm{id}_G(\widehat{C}_{i+1} \times \widehat{A}_{i+1,1} \times \cdots \times \widehat{A}_{i+1,r_{i+1}}) \subseteq \cdots.\end{aligned}$$

In order to complete the construction of the algebra B (and hence the proof of the Specht problem) we will show that in fact, by passing to a subsequence, all $\widehat{C}_i$ are G-graded isomorphic. Indeed, since $\mathrm{id}_G(A_i) \subseteq \mathrm{id}_G(A_{i+1})$ we have a (surjective) map

$$\widehat{C}_i = C * K\langle X_G\rangle/(I_i + J) \to \widehat{C}_{i+1} = C * K\langle X_G\rangle/(I_{i+1} + J).$$

Since the algebras $\widehat{C}_i$'s are finite-dimensional the result follows.

13. Non-affine algebras

In this section we prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We proceed as in [11] where the Hopf algebra H is replaced by $(FG)^*$, the dual Hopf algebra of the group algebra FG. Let W be a *PI* G-graded (possibly) non-affine algebra. We consider the algebra $W^* = W \otimes E$ where E is the Grassmann algebra. Note that the algebra W^* is $\mathbb{Z}/2\mathbb{Z} \times G$-graded where the G-grading comes from the G-grading on W and the $\mathbb{Z}/2\mathbb{Z}$-grading comes from the $\mathbb{Z}/2\mathbb{Z}$-grading on E.

By [11, Lemma 1], there exists an affine $\mathbb{Z}/2\mathbb{Z} \times G$-graded algebra W_{affine} such that $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}\times G}(W^*) = \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}\times G}(W_{\mathit{affine}})$ and hence by Theorem 1.1, it coincides with $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}\times G}(A)$ where A is a finite-dimensional $\mathbb{Z}/2\mathbb{Z} \times G$-graded algebra. Applying the $*$ operator to W^* (and using the fact that $\mathrm{id}_G(W) = \mathrm{id}_G(W^{**})$) we obtain that $\mathrm{id}_G(W) = \mathrm{id}_G(A^*)$ as desired. □

Proof of Theorem 1.4. Let Γ be the T-ideal of G-graded identities of W. Let $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ be an ascending sequence of T-ideals whose union is Γ. Since W is assumed to be *PI* (as in the affine case) we can add to all Γ_i's a finite set of G-graded identities so that the ideals obtained correspond to T-ideals of *PI* G-graded algebras. By Theorem 1.3 these T-ideals correspond to Grassmann envelopes of finite-dimensional $\mathbb{Z}/2\mathbb{Z} \times G$-graded algebras A_i, that is we obtain an ascending chain of the form $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}\times G}((A_1)^*) \subseteq \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}\times G}((A_2)^*) \subseteq \cdots$. Applying the $*$ operator, we get an ascending chain of T-ideals of identities of finite-dimensional algebras so it must stabilize. The result now follows from the fact that $*$ is an involution. □

We conclude the section with the theorem corresponding to Theorem 11.2.

Theorem 13.1. *Every variety M of G-graded algebras can be generated by a finite direct product $B_1^* \times \cdots \times B_r^* \times B_1' \times \cdots \times B_s'$ of Grassmann envelopes B_i^* of basic algebras B_i and basic algebras B_j'.*

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Appendix A. Polynomials and finite-dimensional algebras

In this section, F will denote an algebraically closed field of characteristic zero.

Our goal here is to explain some of the basic ideas the relate the structures of polynomials and finite-dimensional algebras. Recall that the Capelli polynomial c_n is defined by

$$c_n = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots x_{\sigma(n)} y_n.$$

We say that the Capelli polynomial is alternating in the x's. More generally, let $f(X;Y) = f(x_1, \dots, x_m; Y)$ be a polynomial which is multilinear in the set of variables X. We say that $f(X,Y)$ is alternating in the set X (or that the variables of X alternate in f) if there exists a polynomial $h(X;Y) = h(x_1, x_2, \dots, x_m; Y)$ such that

$$f(x_1, x_2, \dots, x_m; Y) = \sum_{\sigma \in \mathrm{Sym}(m)} \mathrm{sgn}(\sigma) h(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)}; Y).$$

Following the notation in [23], if $X_1, X_2, \dots, X_p$ are p disjoint sets of variables, we say that a polynomial $f(X_1, \dots, X_p; Y)$, is alternating in the sets $\{X_1, \dots, X_p\}$, if it is alternating in each set X_i.

Clearly, in order to test whether a multilinear polynomial (and in particular c_n) is an identity of a finite-dimensional algebra A, it is sufficient to evaluate the variables on basis elements. It follows that if A is an algebra over F of dimension n then $c_{n+1} \in \mathrm{id}(A)$. Clearly, we cannot expect that Capelli polynomials detect precisely the dimension of a finite-dimensional algebra since on one hand we can just take a commutative algebra A of arbitrary dimension over F, and on the other hand $c_2 \in \mathrm{id}(A)$. This simple fact will lead us to consider (below) minimal or adequate models.

Given an algebra A, finite-dimensional over a field F, it is well known that A decomposes as a vector space into $A \cong \overline{A} \oplus J(A)$ where $\overline{A}$ is semisimple and $J(A)$ is the radical of A. Moreover, $\overline{A}$ is closed under multiplication. As mentioned above, in order to test whether a multilinear polynomial f is an identity of A it is sufficient to evaluate the variables on any chosen basis of A over F and hence we may take a basis consisting of elements which belong either to $\overline{A}$ or $J(A)$. We refer to these evaluations as semisimple or radical evaluations respectively. Our aim is to present a set of polynomials which detect the dimension of $\overline{A}$ over F and also the nilpotency index of $J(A)$.

Denote by $n = \dim_F(\overline{A})$ and by s the nilpotency index of $J(A)$.

For every integer r consider the set of multilinear polynomials with r-folds of alternating sets of variables of cardinality m. Let us denote these sets of variables by $X_1, X_2, \dots, X_r$. Clearly, if

$n < m$ each alternating set will assume at least one radical evaluation and hence if $r \geqslant s$ we will have at least s radical evaluations. This shows that the polynomial vanishes upon any evaluation. It follows that if $n < m$ and $r \geqslant s$ then f is an identity of A. In other words if we know that for every positive integer r there is a non-identity, multilinear polynomial with alternating sets of cardinality m, then $n \geqslant m$. The question which arises naturally is whether a finite-dimensional algebra A, where $n = \dim_F(\overline{A})$, will always admit, for arbitrary large integer r, non-identities (multilinear) with r alternating sets of cardinality n. The answer is clearly negative since again, on one hand we can take a semisimple commutative algebra of arbitrary dimension over F, and on the other hand, the cardinality of alternating sets cannot exceed 1. Again, this leads us to consider adequate models. The following terminology is not standard and will be used only in this appendix.

Definition A.1. A finite-dimensional algebra A is *weakly adequate* if for every integer r there is a multilinear polynomial, non-identity of A, which has r alternating sets of cardinality $n = \dim_F(\overline{A})$.

An important result due to Kemer implies:

Lemma A.2 *("Kemer's Lemma 1"). Any finite-dimensional algebra A is PI-equivalent (i.e. the same T-ideal of identities) to a direct product $A_1 \times \cdots \times A_k$ where A_i is weakly adequate.*

The lemma allows us to control the dimension of the semisimple component of A (after passing to direct products of weakly adequate algebras) in terms of noncommutative polynomials. But we need more. We would like to control also the nilpotency index in terms of noncommutative polynomials. For this we need to strengthen the definition of weakly adequacy.

Definition A.3. A finite-dimensional algebra A is *adequate* if for every integer r there is a multilinear polynomial, non-identity of A, which has r alternating sets of cardinality $n = \dim_F(\overline{A})$ and precisely $s - 1$ alternating sets of variables of cardinality $n + 1$.

As noted above a non-identity of A cannot have more than $s - 1$ alternating sets of cardinality $n + 1$.

A key result of Kemer ("Kemer's Lemma 2") implies:

Theorem A.4 *(Adequate model theorem). Any finite-dimensional algebra A is PI-equivalent to a direct product $A_1 \times \cdots \times A_k$ where A_i is adequate.*

Remark A.5. In fact one shows by a sequence of reductions, that any finite-dimensional algebra A is *PI*-equivalent to a direct product of algebras which are called *basic*. Kemer's Lemma 2 says that any basic algebra is adequate.

Remark A.6. It should be emphasized that our main application of Kemer's lemma is in the "reverse direction": We start with Γ, the T-ideal of identities of an affine algebra W. First one shows that there exists a finite-dimensional algebra A such that $\Gamma \supseteq \mathrm{id}(A)$. Then one shows easily that there exist a pair (n, s) of non-negative integers, such that for any integer r there exist polynomials f *outside* Γ (called Kemer polynomials for Γ) which have r sets of alternating variables of cardinality n and $s - 1$ sets of alternating variables of cardinality $n + 1$. Moreover the pair (n, s) is maximal with respect to the usual lexicographic order. The point of Kemer's Lemma's is that one can find a basic algebra which "realizes" these parameters, i.e. a finite-dimensional algebra A where $n = \dim_F(\overline{A})$, s is the nilpotency index of $J(A)$ and such that the Kemer polynomials for Γ are *outside* $\mathrm{id}(A)$. As pointed out in Section 1, this is the connection which allows us to prove the Phoenix property for Kemer polynomials.

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