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Hilbert series of PI relatively free G-graded algebras are rational functions

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Abstract

Given a finite group G and a field F of characteristic zero, we let $F\langle x_{1,g_1},\dots,x_{r,g_r}\rangle$ be the free G-graded F-algebra generated by homogeneous variables $\{x_{i,g_i}\}_{g_i\in G}$. Let $\mathcal{I}$ be a G-graded T-ideal of $F\langle x_{1,g_1},\dots,x_{r,g_r}\rangle$ which is PI (that is, the algebra $F\langle x_{1,g_1},\dots,x_{r,g_r}\rangle/\mathcal{I}$ is PI). We prove that the Hilbert series of $F\langle x_{1,g_1},\dots,x_{r,g_r}\rangle/\mathcal{I}$ is a rational function. More generally, we show that the Hilbert series which corresponds to any g-homogeneous component of $F\langle x_{1,g_1},\dots,x_{r,g_r}\rangle/\mathcal{I}$ is a rational function.

1. *Introduction*

The Hilbert series of an affine (that is, finitely generated) algebra and its computation is a topic which attracted a lot of attention in the last century, classically in commutative algebra (see, for example, [**29**, **30**]), but also (and in fact more importantly for the purpose of this paper) in non-commutative algebra (see, for example, [**5**]). In particular the question of whether or not the Hilbert series H_W of and algebra W is the Taylor expansion of a rational function is fundamental in the theory and has important applications to other growth invariants of W (see [**7**], [**13**–**15**], [**25**], [**26**], [**28**]).

In case the algebra W is a relatively free algebra, that is, isomorphic to the quotient of an affine free algebra $F\langle x_1,\dots,x_n\rangle$ by a T-ideal $\mathcal{I}$ it is known that $H_{F\langle x_1,\dots,x_n\rangle/\mathcal{I}}$ is a rational function (see [**8**, **20**]) and this fact has been successfully used in the estimation of the asymptotic behaviour of the co-character sequence of a PI algebra.

Specifically in [**11**] (based on explicit formulas for the co-character sequences which appear in [**10**]) the authors show that if A is a PI-algebra with 1 which satisfies a Capelli identity, then the asymptotic behaviour of the codimension sequence is of the form

$$c_n(A) = a n^g l^n,$$

where a is a scalar, $2g$ is an integer and l is a non-negative integer (we refer the reader to [**17**–**19**] for a comprehensive account on the codimension sequence of a PI algebra).

The rationality of the Hilbert series of an affine relatively free algebra has been established also in case W is a super algebra (that is, $\mathbb{Z}_2$-graded) and our goal in this paper is to extend these results to G-graded relatively free affine PI algebras where G is an arbitrary finite group.

Let $\mathcal{G} = (g_1,\dots,g_r)$ be an unordered r-tuple of elements of G (in particular we allow repetitions). Let $\mathcal{X}_{\mathcal{G}} = \{x_{(1,g_1)},\dots,x_{(r,g_r)}\}$ be a set of variables that correspond to the elements of $\mathcal{G}$ and let $F\langle\mathcal{X}_{\mathcal{G}}\rangle$ be the free algebra generated by $\mathcal{X}_{\mathcal{G}}$ over F, where F is a field of characteristic zero. In order to keep the notation as light as possible we may omit one index and

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write $\mathcal{X}_{\mathcal{G}} = \{x_{g_1}, \ldots, x_{g_r}\}$ with the convention that the variables x_{g_i} and x_{g_j} are different even if $g_i = g_j$ in G. We equip $F\langle \mathcal{X}_{\mathcal{G}} \rangle$ with a (natural) G-grading by setting the homogeneous degree of a monomial $x_{g_{s_1}} x_{g_{s_2}} \ldots x_{g_{s_m}}$ to be $g_{s_1}\, g_{s_2} \ldots g_{s_m} \in G$. We also consider its homogeneous degree $m \in \mathbb{N}$, namely the number of variables in the monomial. Let $\mathcal{I}$ be a G-graded T-ideal, that is, $\mathcal{I}$ is closed under G-graded endomorphisms of $F\langle \mathcal{X}_{\mathcal{G}} \rangle$.

REMARK 1.1. (1) It is convenient to view the ideal $\mathcal{I}$ as the evaluation on $F\langle \mathcal{X}_{\mathcal{G}} \rangle$ of a T-ideal I of the G-graded free algebra $F\langle X_G \rangle$, where X_G consists of *countably* many homogeneous variables of each degree $g \in G$. In fact I is nothing but the T-ideal of G-graded identities of $F\langle \mathcal{X}_{\mathcal{G}} \rangle / \mathcal{I}$.

(2) It is well known (since F is a field of zero characteristic) that I is generated by multilinear polynomials. Furthermore, by the G-grading on the algebra $F\langle X_G \rangle$, we have that I is generated as a G-graded T-ideal by multilinear strongly homogeneous polynomials, that is, polynomials whose monomials have all the same homogeneous degree $g \in G$ (see [**3**, paragraph preceding Theorem 1.1]).

(3) It follows that the ideal $\mathcal{I}$ of $F\langle \mathcal{X}_{\mathcal{G}} \rangle$ is generated by polynomials which are strongly homogeneous in the variables of $\mathcal{X}_{\mathcal{G}}$. This fact will play an important role in the sequel. Note that, in general, when passing from I to $\mathcal{I}$ (by evaluation) we lose the multilinearity condition.

We will assume in addition that the T-ideal $\mathcal{I}$ 'is PI'. By this we mean that the G-graded, relatively free algebra $F\langle \mathcal{X}_{\mathcal{G}} \rangle / \mathcal{I}$ is PI. Equivalently, the ideal I contains all G-graded polynomials obtained by assigning all possible degrees in G to the variables x_i of an ordinary non-zero polynomial $p(x_1, \ldots, x_n)$.

REMARK 1.2. Note, for instance, that the T-ideal of G-graded identities of any finite dimensional G-graded algebra is PI. On the other hand, if $G \neq \{e\}$, then the G-graded algebra W (and hence its T-ideal of G-graded identities) where W_e is a free non-commutative algebra and $W_g = 0$ for $g \neq e$ is G-graded PI but of course not PI.

Let $F\langle x_{g_1}, \ldots, x_{g_r} \rangle$ be the free G-graded algebra generated by homogeneous variables $\{x_{g_i}\}_{i=1}^{r}$. As above, we let $\mathcal{I}$ be a G-graded T-ideal which is PI and let $F\langle x_{g_1}, \ldots, x_{g_r} \rangle / \mathcal{I}$ be the corresponding relatively free G-graded algebra. Let Ω_n be the (finite) set of monomials of degree n on $\{x_{g_i}\}_{i=1}^{r}$ and let c_n be the dimension of the F-subspace of $F\langle x_{g_1}, \ldots, x_{g_r} \rangle / \mathcal{I}$ spanned by the monomials of Ω_n. We denote by

$$H_{F\langle x_{g_1}, \ldots, x_{g_r} \rangle / \mathcal{I}}(t) = \sum_n c_n t^n$$

the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r} \rangle / \mathcal{I}$ with respect to the generators represented by $\{x_{g_i}\}_{i=1}^{r}$.

THEOREM 1.3. *The series $H_{F\langle x_{g_1}, \ldots, x_{g_r} \rangle / \mathcal{I}}(t)$ is the Taylor series of a rational function.*

In our proof, we strongly use key ingredients that appear in the proof of representability of the G-graded relatively free algebras (see [**3**]). These ingredients include the existence of certain polynomials, called Kemer polynomials, which are 'extremal non-identities' (see [**3**]) and their existence relies on the fundamental fact that the Jacobson radical of an affine PI algebra is nilpotent (see [**4**, **12**, **22**, **27**]) (this parallels the Hilbert's Nullstellensatz in the commutative theory), and the solution of the Specht problem (see [**3**, **23**, **24**, **31**]) (which parallels the Hilbert basis theorem in the commutative theory). We emphasize however that the rationality of the Hilbert series is not a corollary of representability since there are examples of representable algebras which have a transcendental Hilbert series. Indeed, recall from [20] that the monomial algebra supported by monomials of the form $h = x_1 x_2^m x_3 x_4^n x_5$ with the extra relation that $h = 0$ if $m^2 - 2n^2 = 1$ is representable but has a transcendental Hilbert series. For more on this we refer the reader to [9, 21].

In Theorems 1.5 and 1.8 we present different generalizations of Theorem 1.3. Then, in Theorem 1.9, these are combined into one general statement. We start with multivariate Hilbert series.

Let $F\langle x_{g_1}, \ldots, x_{g_r}\rangle$ be the free G-graded algebra generated by homogeneous variables $\{x_{g_i}\}_{i=1}^r$. As above, we let $\mathcal{I}$ be a G-graded T-ideal which is PI and let $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ be the corresponding relatively free G-graded algebra. For any r-tuple of non-negative integers $(d_1, \ldots, d_r)$, we consider the (finite) set of monomials $\Omega_{(d_1,\ldots,d_r)}$ on $\{x_{g_i}\}_{i=1}^r$ where the variable x_{g_i} appears exactly d_i times, $i = 1, \ldots, r$. We denote by $c_{(d_1,\ldots,d_r)}$ the dimension of the F-subspace of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ spanned by the monomials in $\Omega_{(d_1,\ldots,d_r)}$.

Definition 1.4. Notation as above. The multivariate Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is given by

$$H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}}(t_1, \ldots, t_r) = \sum_{(d_1,\ldots,d_r)} c_{(d_1,\ldots,d_r)} t_1^{d_1} \ldots t_r^{d_r}.$$

The following result generalizes Theorem 1.3.

Theorem 1.5. *Notation as above. The Hilbert series $H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}}(t_1, \ldots, t_r)$ is a rational function.*

Remark 1.6. Clearly, Theorem 1.3 follows from Theorem 1.5 simply by replacing all variables t_i by a single variable t.

The next generalization of Theorem 1.3 is in a different direction. We consider the Hilbert series (in one variable t) of a unique homogeneous component. More precisely, we fix $g \in G$ and we consider the set of monomials $\Omega_{g,n}$ of degree n whose homogeneous degree is g. We let $c_{g,n}$ be the dimension of the subspace in $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ spanned by the monomials in $\Omega_{g,n}$ (or rather, by the elements in $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ they represent).

Definition 1.7. With the above notation, the Hilbert series of the g-component of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is given by

$$H_{g,F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}}(t) = \sum c_{g,n} t^n.$$

Theorem 1.8. *With the above notation, the Hilbert series $H_{g,F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}}(t)$ is a rational function.*

The case where $g = e$ is of particular interest. Moreover, we could consider the Hilbert series of any collection of g-homogeneous components and in particular the Hilbert series which corresponds to a subgroup H of G.

Finally, we combine the generalizations that appeared in Theorems 1.5 and 1.8. Fix an element $g \in G$. For any r-tuple $(d_1, \dots, d_r)$ of non-negative integers we consider the monomials with x_{g_1} appearing d_1 times, x_{g_2} appearing d_2 times, ..., whose homogeneous degree is $g \in G$. We denote this set of monomials by $\Omega_{g,(d_1,\dots,d_r)}$ and we let $c_{g,(d_1,\dots,d_r)}$ be the dimension of the space in $F\langle x_{g_1}, \dots, x_{g_r}\rangle/\mathcal{I}$ spanned by elements whose representatives are the monomials in $\Omega_{g,(d_1,\dots,d_r)}$.

THEOREM 1.9. *The Hilbert series*

$$H_{g,F\langle x_{g_1},\dots,x_{g_r}\rangle/\mathcal{I}}(t_1, \dots, t_r) = \sum c_{g,(d_1,\dots,d_r)} t_1^{d_1} \dots t_r^{d_r}$$

which corresponds to the g-component of the G-graded algebra $F\langle x_{g_1}, \dots, x_{g_r}\rangle/\mathcal{I}$ is a rational function.

REMARK 1.10. Clearly, Theorems 1.3, 1.5 and 1.8 are direct corollaries of Theorem 1.9.

Theorem 1.9 is proved in Section 2. As mentioned above the proof uses ingredients from the proof of the representability of relatively free affine G-graded PI algebras. For the convenience of the reader, we recall in the first part of the section the required results from [**3**] that are used in the proof.

In Section 3, we consider the T-ideal of G-graded identities I of the group algebra FG and show, in a rather direct way, the rationality of the Hilbert series of the corresponding relatively free algebra $F\langle x_{g_1}, \dots, x_{g_r}\rangle/\mathcal{I}$, where $\mathcal{I}$ (as above) is the evaluation of I on $F\langle x_{g_1}, \dots, x_{g_r}\rangle$. We also give in this case (see Corollary 3.4) the asymptotic behaviour of the corresponding codimension sequence. Furthermore, these results can be easily extended to twisted group algebras $F^\alpha G$, where $\alpha \in Z^2(G, F^*)$ (see Remark 3.5). The G-grading determined by twisted group algebras is called 'fine' and it plays an important role in the classification of G-graded simple algebras (see [**6**]).

We close the introduction by mentioning that one may consider the Hilbert series of an affine G-graded free algebra. It is easy to see that the corresponding Hilbert series is a rational function. Of course one may ask whether the different generalizations (Theorems 1.5 and 1.8) apply also in this case (that is, when $\mathcal{I} = 0$). It is easy to see that this is indeed the case when extending to multivariate series. As for the second generalization, we refer the reader to [**16**]. The authors prove the rationality of the Hilbert series which corresponds to the subalgebra of coinvariant elements, that is, in case $g = e$.

2. *Preliminaries and proofs*

We start this section by recalling some facts on G-graded algebras W over a field of characteristic zero F and their corresponding G-graded identities. We refer the reader to [**3**] for a detailed account on this topic.

Let W be an affine G-graded PI algebra over F. We denote by $I = \mathrm{id}_G(W)$ the ideal of G-graded identities of W. These are polynomials in the free G-graded algebra over F that are generated by X_G and that vanish upon any admissible evaluation on W. Here, $X_G = \bigcup X_g$ and X_g is a set of countably many variables of degree g. An evaluation is admissible if the variables from X_g are replaced only by elements of W_g. It is known that I is a G-graded T-ideal, that is, closed under G-graded endomorphisms of $F\langle X_G\rangle$.

We recall from [**3**] that the T-ideal $I = \mathrm{id}_G(W)$ is generated by multilinear polynomials and so it does not change when passing to $\bar{F}$, the algebraic closure of F, in the sense that the ideal of identities of $W_{\bar{F}}$ over $\bar{F}$ is the span (over $\bar{F}$) of the T-ideal of identities of W over F. It is easily checked that the Hilbert series remains the same when passing to the algebraic closure of F. Thus, from now on we assume $F = \bar{F}$.

Next, we recall some terminology and some facts from Kemer theory extended to the context of G-graded algebras as they appear in [3]. We start with the concept of alternating polynomial on a set of variables.

Let $f(x_{1,g}, \ldots, x_{r,g}; y_1, \ldots, y_n)$ be a multilinear polynomial with variables $x_{1,g}, \ldots, x_{r,g}$, homogeneous of degree g, and some other homogeneous variables $y_1, \ldots, y_n$ of unspecified degrees. We say that f is alternating on the set $x_{1,g}, \ldots, x_{r,g}$ if there is a multilinear polynomial $h(x_{1,g}, \ldots, x_{r,g}; y_1, \ldots, y_n)$ such that

$$f(x_{1,g}, \ldots, x_{r,g}; y_1, \ldots, y_n) = \sum_{\sigma \in \mathrm{Sym}(r)} (-1)^{\sigma} h(x_{\sigma(1),g}, \ldots, x_{\sigma(r),g}; y_1, \ldots, y_n).$$

We say that a polynomial f alternates on a collection of disjoint sets of homogeneous variables (each set constituting of variables of the same degree), if it is alternating on each set.

In what follows, we will need to consider multilinear polynomials f that alternate on d disjoint sets of g-elements, each of cardinality r. More generally, we will consider multilinear polynomials such that for any $g \in G$, f contains n_g disjoint sets of variables of homogeneous degree g and each set of cardinality d_g.

We recall from [3] that a G-graded polynomial with an alternating set of g-homogeneous variables that is 'large enough' is necessarily an identity. More precisely, for any affine G-graded PI algebra W and for any $g \in G$ there exists an integer d_g such that any G-graded polynomial which has an alternating set of g variables of cardinality exceeding d_g is necessarily a G-graded identity of W.

In particular, this holds for a finite-dimensional G-graded algebra A. Note that in this case, if a polynomial f has an alternating set of g-homogeneous elements whose cardinality exceeds the dimension of A_g, it is clearly an identity of A.

Next we recall that by the Wedderburn–Malcev decomposition theorem, a G-graded finite-dimensional algebra A over F, may be decomposed into the direct sum of $\bar{A} \oplus J$ (decomposition as vector spaces) where J is the Jacobson radical (G-graded) and $\bar{A}$ is a (semisimple) subalgebra of A isomorphic to A/J as G-graded algebras. As a consequence, we have decompositions of $\bar{A}$ and J to the corresponding g-homogeneous components.

Now, we know that in order to test whether a multilinear polynomial is an identity of an algebra, it is sufficient to evaluate its variables on a base and hence, applying the above decomposition, we may consider semisimple and radical G-graded evaluations. From these considerations we conclude that if a polynomial has sufficiently many alternating sets of g-homogeneous elements of cardinality that exceeds the dimension of the g-homogeneous component of $\bar{A}$, the polynomial is necessarily an identity of A. Indeed, in any evaluation, we either have a semisimple basis element that appears twice or at least one of the evaluations is radical. In the first case, we get zero as a result of the alternation of two elements that are equal, while in the second, we get zero if the number of alternating sets is at least the nilpotency index of J. We therefore see that if a G-graded polynomial f has a number of alternating sets (same cardinality) of g-variables that is at least the nilpotency index of J (and in particular if it has 'sufficiently many') then the cardinality of the sets must be bounded by the dimension of $\bar{A}_g$ if we know that f is a non (G-graded)-identity of A. Thus, if f is a non (G-graded)-identity of A with α_g disjoint alternating sets of g-homogeneous elements of cardinality $d_g + 1$, where $g \in G$ and $d_g = \dim(\bar{A}_g)$, then $\sum_g \alpha_g \leqslant n - 1$, where n is the nilpotency index of J.

In [3], the notion of a finite-dimensional G-graded *basic* algebra was introduced. For our exposition here, it is not necessary to recall its precise definition but only say (as a result of Kemer's Lemmas 1 and 2 for G-graded algebras (see Sections 5 and 6 in [3])) that any basic algebra A admits non-identities G-graded polynomials that have 'arbitrarily many' (say at least n, the nilpotency index of J) alternating sets of g-homogeneous variables of cardinality d_g and precisely (a total of) $n-1$ alternating sets of g-homogeneous variables of cardinality d_g+1 for some $g \in G$. These extremal non-identities are the so-called 'G-graded Kemer polynomials' of the basic algebra A.

Thus to each basic algebra A corresponds an $r+1$-tuple of non-negative integers $(d_{g_1}, \ldots, d_{g_r}; n-1)$ where d_g is the dimension of the g-component of the semisimple part of A and n is the nilpotency index of J, the radical of A. We refer to such a tuple as the Kemer point of the basic algebra A.

The representability theorem for affine G-graded PI algebras can be stated as follows.

Given an affine G-graded PI algebra W over F, where F is a field of zero characteristic, there exists a finite number of G-graded basic algebras $A_1, \ldots, A_m$ over a field extension K of F such that W satisfies the same G-graded identities as $A_1 \oplus \ldots \oplus A_m$. Note that since $\mathrm{id}_G(A_1 \oplus \ldots \oplus A_n) = \bigcap \mathrm{id}_G(A_i)$, we may assume $\mathrm{id}(A_i) \nsubseteq \mathrm{id}(A_j)$ for every $1 \leqslant i, j \leqslant m$ with $i \neq j$.

Remark 2.1. In fact, by passing to the algebraic closure of K, we may assume that the algebras A_i above are finite dimensional over the same field F.

Definition 2.2. With the above notation, we say that a finite-dimensional G-graded algebra A is subdirectly irreducible if it has no non-trivial, two-sided G-graded ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \cap \mathfrak{b} = (0)$.

Remark 2.3. Note that if the algebra $A_1 \oplus \ldots \oplus A_m$ is subdirectly irreducible, then $m = 1$.

A key ingredient in the proof of Theorem 1.9 is the existence of an essential Shirshov base for the relatively free algebra $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$.

For the convenience of the reader, we recall the necessary definitions and statements from [**3**], starting from the ordinary case (that is, ungraded).

Definition 2.4. Let W be an affine PI-algebra over F. Let $\{a_1, \ldots, a_s\}$ be a set of generators of W. Let m be a positive integer and let Y be the (finite!) set of all words in $\{a_1, \ldots, a_s\}$ of length *no greater than* m. We say that W has a *Shirshov* base of length m and of height h if elements of the form $y_{i_1}^{k_1} \ldots y_{i_l}^{k_l}$ where $y_{i_i} \in Y$ and $l \leqslant h$, span W as a vector space over F.

Theorem 2.5. *If W is an affine PI-algebra, then it has a Shirshov base for some m and h. More precisely, suppose W is generated by a set of elements of cardinality s and suppose it has PI-degree m (that is, there exists an identity of degree m and m is minimal), then W has a Shirshov base of length m and of height h where $h = h(m, s)$.*

In fact, we will need a weaker condition.

Definition 2.6 [**3**, Definition 7.8]. Let W be an affine PI-algebra. We say that a (finite) set Y as above is an *essential Shirshov* base of W (of length m and of height h) if there exists a finite set $D(W)$ such that the elements of the form $d_{i_1} y_{i_1}^{k_1} d_{i_2} \ldots d_{i_l} y_{i_l}^{k_l} d_{i_{l+1}}$ where $d_{i_j} \in D(W)$, $y_{i_j} \in Y$ and $l \leqslant h$ span W.

An essential Shirshov's base gives the following.

THEOREM 2.7 [3, Theorem 7.9]. *Let C be a commutative ring and let $W = C\langle\{a_1, \ldots, a_s\}\rangle$ be an affine algebra over C. If W has an essential Shirshov base (in particular, if W has a Shirshov base) whose elements are integral over C, then it is a finite module over C.*

Returning to G graded algebras we have the following.

PROPOSITION 2.8 [3, Theorem 7.10]. *Let W be an affine G-graded PI algebra. Then it has an essential G-graded Shirshov base of elements of W_e.*

For the proof of Theorem 1.9, we will assume below that there exists a G-graded T-ideal $\mathcal{I}$ which is non-rational (that is, the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is non-rational) and obtain a contradiction. The next two lemmas will be used to reduce the problem to the case where $\mathcal{I}$ is maximal with respect to being non-rational and also that the relatively free algebra $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is G-graded PI equivalent to one basic algebra (rather than to a direct sum of them).

Before stating the lemmas we simplify the terminology as follows.

REMARK 2.9. If V is a subspace of an algebra $\mathcal{U} = F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ spanned by strongly homogeneous polynomials, then we may consider naturally its corresponding multivariate Hilbert series. Then when we say 'Hilbert series of V' we mean 'multivariate G-graded Hilbert series which corresponds to the g-component of V' for any given $g \in G$.

LEMMA 2.10. *Let $\mathcal{J}$ be a G-graded T-ideal containing $\mathcal{I}$. Let $H_{\mathcal{U}}$, $H_{\mathcal{U}/\mathcal{J}}$ and $H_{\mathcal{J}/\mathcal{I}}$ be the Hilbert series of $\mathcal{U}$, $\mathcal{U}/\mathcal{J}$ and $\mathcal{J}/\mathcal{I}$, respectively. Then*

$$H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}} = H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{J}} + H_{\mathcal{J}/\mathcal{I}}.$$

Proof. This is clear since $\mathcal{J}$ is spanned by strongly homogeneous polynomials. $\square$

LEMMA 2.11. *Let $\mathcal{I}'$ and $\mathcal{I}''$ be two G-graded T-ideals that contain $\mathcal{I}$. Then the following holds*:

$$H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/(\mathcal{I}'\cap\mathcal{I}'')} = H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}'} + H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/\mathcal{I}''} - H_{F\langle x_{g_1},\ldots,x_{g_r}\rangle/(\mathcal{I}'+\mathcal{I}'')}.$$

Proof. The proof is similar to the proof of Lemma 9.40 in [20] and hence is omitted. $\square$

Let us assume now that there exists a G-graded T-ideal $\mathcal{K}$ of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle$ that is PI and such that the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{K}$ is non-rational.

PROPOSITION 2.12. *Under the above assumption there exists a G-graded T-ideal $\mathcal{I}$ of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle$ that is PI and*

1. *is maximal among T-ideals $\mathcal{K}$ that are PI and the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{K}$ is non-rational;*
2. *the relatively free algebra $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is G-graded PI equivalent to a G-graded basic algebra A.*

Proof. The first assertion follows from the G-graded Specht property (see [**3**, Section 12]). Indeed, if there is no such an ideal, then we get an infinite ascending sequence of ideals that does not stabilize and this contradicts the fact that the union of the T-ideals is finitely generated. We, thus, may assume that the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is non-rational and the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{J}$ for any G-graded T-ideal $\mathcal{J}$ that properly contains $\mathcal{I}$ is a rational function.

For the proof of the second assertion, let us show that the maximality of $\mathcal{I}$ already implies that $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is PI equivalent to a G-graded basic algebra. Assuming the converse, we have that $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is G-graded PI equivalent to a direct sum of basic algebras $A_1 \oplus A_2 \oplus \ldots \oplus A_m$ where $m \geqslant 2$ and $\mathrm{id}_G(A_i) \nsubseteq \mathrm{id}_G(A_j)$ for any $1 \leqslant i, j \leqslant m$ with $i \neq j$. It follows that the ideal $\mathrm{id}_G(F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}) = \mathrm{id}_G(A_1 \oplus A_2 \oplus \ldots \oplus A_m) = \bigcap \mathrm{id}_G(A_i)$ and $\mathrm{id}_G(F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}) \subsetneq \mathrm{id}_G(A_i)$ for any i. Now, consider the evaluations $\mathcal{I}_i$ of the T-ideals $\mathrm{id}_G(A_i)$ on $F\langle x_{g_1}, \ldots, x_{g_r}\rangle$. Clearly, $\mathcal{I}_i$ properly contains $\mathcal{I}$ and their intersection is $\mathcal{I}$. By Lemma 2.11, we conclude that the Hilbert series of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is rational. Contradiction. □

It is convenient to view the relatively free algebra $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ as an algebra of generic elements, G-graded embedded in a matrix algebra over a suitable rational function field over F. Indeed, by part (2) of Proposition 2.12 we have that $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ is G-graded PI equivalent to a G-graded basic algebra which we denote by A and which from now on will be viewed as a G-graded subalgebra of the $n \times n$ matrices over F (see [**3**]). If $\{v_{g,1}, \ldots, v_{g,s_g}\}$ is an F-basis of A_g, the g-homogeneous component of A, then we consider different sets of central indeterminates $\{t_{g,1}, \ldots, t_{g,s_g}\}$, $g \in G$ (one set for each generator x_g of the free algebra).

Let $K = F\langle\{t_{g,i}\}\rangle$ be the field of rational functions on the t's and A_K be the algebra over K obtained from A by extending scalars from F to K. The algebra of generic elements will be an F-subalgebra of A_K. For each variable $x_g = x_{g_j}$ in the generating set of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle$, we form an element $z_g = \sum_i t_{g,i} v_{g,i}$ in A_K. Following the embedding of A in $M_n(F)$ we view any generator z_g as an $n \times n$-matrix over the field K. Observe that these generating elements are matrices whose entries are homogeneous polynomials (on the t's) of degree one.

PROPOSITION 2.13. (*see* [**3**, *Lemma 7.4*]) *There is an F-isomorphism of G-graded algebras of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ with $\mathcal{A}$, the F-subalgebra of A_K generated by the elements $\{z_{g_1}, \ldots, z_{g_r}\}$.*

REMARK 2.14. In the rest of the proof, we use without further notice the identification of the relatively free algebra with the algebra of generic elements $\mathcal{A}$ embedded in $M_n(K)$.

Now we recall from Proposition 2.8 that the relatively free algebra $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ has an essential Shirshov base Θ that is contained in the e-component. Picking the natural set of generators of $F\langle x_{g_1}, \ldots, x_{g_r}\rangle/\mathcal{I}$ we note that Θ consists of elements represented by monomials on $x_{g_1}, \ldots, x_{g_r}$ and hence, viewed in $\mathcal{A}$, they consist of matrices over K whose entries are homogeneous polynomial in the t_i's. It then follows that the characteristic values of the elements in Θ (that is, the coefficients of the characteristic polynomials) are homogeneous polynomials on the t_i's.

Denote by C the algebra over F generated by these homogeneous polynomials. Note that C is an affine commutative algebra (the essential Shirshov base is finite). Moreover, if we extend the algebra of generic elements $\mathcal{A}$ to C we have (by Theorem 2.7) that $\mathcal{A}_{\mathcal{C}}$ is a finite module over C.

For $\mathcal{A}_{\mathcal{C}}$, we know the following result.

Lemma 2.15. *The Hilbert series of $\mathcal{A}_{\mathcal{C}}$ is rational. More generally, the rationality of the Hilbert series is independent of the integer degree given to the generators.*

Proof. Indeed, the algebra is a finitely generated module over an affine domain and hence its Hilbert series is rational (see [**20**, Proposition 9.33]). □

More generally, we may consider C-submodules M of $\mathcal{A}_{\mathcal{C}}$ that are generated by elements that are represented by strongly homogeneous polynomials on $x_{g_1}, \dots, x_{g_r}$.

Lemma 2.16. *The Hilbert series of M is rational.*

Proof. Since $\mathcal{A}_{\mathcal{C}}$ is a finite module over a Noetherian domain, the module M is finitely generated as well. The result now follows from the second part of Lemma 2.15. □

We can now complete the proof of Theorem 1.9. Let f be a G-graded Kemer polynomial of the basic algebra A and let $\mathcal{J}$ be the G-graded T-ideal it generates (evaluating on $F\langle x_{g_1}, \dots, x_{g_r}\rangle$) together with $\mathcal{I}$. Note that since f is a non-identity of A (and hence a non-identity of $F\langle x_{g_1}, \dots, x_{g_r}\rangle/\mathcal{I}$) the T-ideal $\mathcal{J}$ strictly contains $\mathcal{I}$, and hence, by the maximality of $\mathcal{I}$, we have that the Hilbert series of $F\langle x_{g_1}, \dots, x_{g_r}\rangle/\mathcal{J}$ is rational. The key property that we need here is that the ideal $\mathcal{J}/\mathcal{I}$ is closed under the multiplication of the coefficients of the characteristic polynomials of the elements in Θ (see [**3**, Proposition 8.2]). Hence, the ideal $\mathcal{J}/\mathcal{I}$ of the relatively free algebra is in fact a C-submodule of $\mathcal{A}_{\mathcal{C}}$. Hence, its Hilbert series is rational. Applying Lemma 2.10 the result follows. This completes the proof of Theorem 1.9.

3. *A special case*

In this section, we show by direct computations the rationality of the Hilbert series of the affine relative free G-graded algebra in case I is the T-ideal of G-graded identities of the group algebra FG. In addition, we obtain a precise estimation of the asymptotic behaviour of the G-graded codimension sequence for that case.

Let $\bar{\alpha} = (g_1, g_2, \dots, g_r)$ be an r-tuple in $G^{(r)}$. As in previous sections, we consider the free G-graded algebra $F\langle x_{1,g_1}, \dots, x_{r,g_r}\rangle$, where the x_{i,g_i} are non-commuting variables which are in one-to-one correspondence with the entries of $\bar{\alpha}$. As above, we may abuse notation by deleting the index i and simply write x_{g_i}.

Next, we consider the T-ideal of G-graded identities $\mathrm{id}_G(FG)$ of the group algebra FG. Recall from [**2**] that $\mathrm{id}_G(FG)$ is generated as a T-ideal by binomial identities of the form $x_{g_{i_1}} x_{g_{i_2}} \dots x_{g_{i_n}} - x_{g_{i_{\sigma(1)}}} x_{g_{i_{\sigma(2)}}} \dots x_{g_{i_{\sigma(n)}}}$ where σ is a permutation in S_n and the products $g_{i_1} g_{i_2} \cdots g_{i_n}$ and $g_{i_{\sigma(1)}} g_{i_{\sigma(2)}} \cdots g_{i_{\sigma(n)}}$ coincide in G. That is, two monomials are equivalent if and only if they have the same variables and they determine elements in the same g-homogeneous component. In particular, if the group G is abelian, then two monomials are equivalent if and only if they have the same variables.

Remark 3.1. Clearly, by passing to a subgroup of G if necessary, we may assume that the elements of $\bar{\alpha} = (g_1, g_2, \dots, g_r)$ generate the group G.

Fix a natural number n and consider monomials of degree n with n_1 variables x_{g_1}, n_2 variables x_{g_2}, $\dots$, n_r variables x_{g_r} where $n_1 + n_2 + \dots + n_r = n$. Now consider permutations of any monomial of that form. Clearly, any permutation determines the same element in the abelianization of G. In other words, the elements of G determined by two monomials that have the same variables lie in the same coset of the commutator G'. In the next lemma, we show that if the monomial is 'rich enough' we may obtain all elements of a G'-coset. In order to state the lemma we need the following notation.

We say that the word $\Sigma = g_{i_1} g_{i_2} \dots g_{i_n}$ is a presentation of $g \in G$ (in terms of the entries of $\bar{\alpha} = (g_1, g_2, \dots, g_r)$) if $g = g_{i_1} g_{i_2} \dots g_{i_n}$ in G. For any word Σ, we may consider the corresponding *monomial* (in the free algebra) $X_\Sigma = x_{g_{i_1}} x_{g_{i_2}} \dots x_{g_{i_n}}$. We say that the monomial X_Σ represents g in G.

LEMMA 3.2. *For every $z \in G$ there exists an integer n and a word $g_{i_1} g_{i_2} \dots g_{i_n}$ in G such that the set of words in*

$$\Omega_{g_{i_1} g_{i_2} \dots g_{i_n}} = \{g_\sigma = g_{i_{\sigma(1)}} g_{i_{\sigma(2)}} \dots g_{i_{\sigma(n)}}, \sigma \in \mathrm{Sym}(n)\}$$

represents all elements of the form zg where $g \in G'$ (that is, the full coset of G' in G represented by z).

Proof. Clearly, it is sufficient to find a word whose permutations yield all elements of G'. It is well known that the commutator subgroup G' of G is generated by commutators $[g, h] = ghg^{-1}h^{-1}$ where $g, h \in G$. For any commutator $[g, h]$, we write the elements g and h as words in the entries of $\bar{\alpha}$ and then we write g^{-1} and h^{-1} by inverting the corresponding words. We denote by $\Sigma_{[g,h]}$ the corresponding word in the entries of $\bar{\alpha}$ and their inverses (which is equal to $[g, h]$ in G). Clearly, the total degree (in $\Sigma_{[g,h]}$) of each entry g_i is zero and hence permuting the elements of $\Sigma_{[g,h]}$ we obtain the identity element e. It follows easily that taking products of such words we obtain a word $\Sigma = \Sigma_z$ whose different permutations yield all of G'. This would complete the proof of the lemma if we assume that whenever g is an entry in $\bar{\alpha}$, then also g^{-1} is. But clearly, if this is not the case, we can replace g^{-1} by $g^{ord(g)-1}$ and the result follows. $\square$

Consider the (r-dimensional) lattice $\Gamma_r = (\mathbb{Z}_+)^{(r)}$ of non-negative integer points, where r is the cardinality of $\bar{\alpha}$. We refer to Γ_r as the r-dimensional non-negative Euclidean lattice. Similarly, we will consider Γ_k, where $k \leqslant r$, the k-dimensional non-negative Euclidean lattices and their translations, $\overrightarrow{x} + \Gamma_k$ where $\overrightarrow{x} \in (\mathbb{Z}_+)^{(k)}$. We view the lattice Γ_r as a partial ordered set, where $A = (n_1, \dots, n_r) \prec B = (m_1, \dots, m_r)$, if and only if $n_i \leqslant m_i$ for $1 \leqslant i \leqslant r$. Clearly, this partial order inherits a partial order on Γ_k, $k \leqslant r$, and their translations. To any point $A = (n_1, \dots, n_r)$, $n_i \geqslant 0$ in Γ_r we attach all monomials X_Σ with the number of variables as prescribed by the point A, that is, the variable x_{1,g_1} appears n_1 times, x_{2,g_2} appears n_2 times and so on. Clearly, any word Σ determines a unique lattice point A in which case we write $A = A_\Sigma$ or $\Sigma \in A$. Clearly, the elements in G represented by all monomials that correspond to a point $A \in \Gamma_r$ lie in the same coset of G' and hence denoting by $N_A = \{g \in G : g$ is represented by monomials in $A\}$, we have that $1 \leqslant \mathrm{ord}(N_A) \leqslant \mathrm{ord}(G')$.

LEMMA 3.3. *The function $\mathrm{ord}(N(A)) : \Gamma_r \to \{1, \dots, \mathrm{ord}(G')\}$ is monotonic (increasing) with respect to partial ordering on Γ_r.*

In particular if $A_\Sigma \in \Gamma_r$, where Σ is a word in the entries of $\bar{\alpha}$ whose different permutations represent all elements of G' (as constructed in Lemma 3.2)), then for any word Π such that $A_\Pi \succeq A_\Sigma$ we have $\mathrm{ord}(N(A_\Pi)) = \mathrm{ord}(G')$.

Proof. This is clear. $\square$

We can now complete the proof that the Hilbert series of the corresponding relatively free algebra is rational.

Let $A = (n_1, \ldots, n_r) \in \Gamma_r$, where $n = n_1 + \ldots + n_r$, and P_A be the space spanned by all monomials X_Π where $\Pi \in A$. Since the T-ideal of identities is spanned by strongly homogeneous polynomials we have that the subspace of the relatively free algebra spanned by monomials of degree n is decomposed into the direct sum of spaces which are spanned by monomials in lattice points A of degree n.

We claim that for any integer λ, $1 \leqslant \lambda \leqslant \operatorname{ord}(G')$, the set of points $A \in \Gamma_r$ such that

$$\dim(P_A / P_A \cap (\mathrm{id}_G(FG))) = \lambda$$

is a *finite* union of *disjoint* sets which are translations of lattices Γ_k, $0 \leqslant k \leqslant r$. We present here a proof which was shown to us by Uri Bader. Consider the one point compactification $\widehat{\mathbb{Z}_+}$ of $\mathbb{Z}_+$. It is convenient to view the space $\widehat{\mathbb{Z}_+}$ as homeomorphic to the set of points $I_{\mathbb{N}} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ with the induced topology of the Lesbegue measure on the interval $[0, 1]$. The closed sets are either the finite sets or those that contain 0. Consequently, the sets that are closed and open are either the finite sets without 0 or sets that contain a set of the form $\{1/n : n \geqslant d\} \cup \{0\}$. Consider the r-fold cartesian product

$$I_{\mathbb{N}}^r = I_{\mathbb{N}} \times I_{\mathbb{N}} \times \ldots \times I_{\mathbb{N}}.$$

Clearly, $I_{\mathbb{N}}^r$ is compact. Furthermore, the function $\operatorname{ord}(N(A))$ with values in the finite set $T = \{1, \ldots, \operatorname{ord}(G')\}$ (viewed as a function on $I_{\mathbb{N}}^r$) is monotonic decreasing and hence continuous. It follows that the inverse image of any point in T is an open and closed subset of $I_{\mathbb{N}}^r$ and the result now follows easily.

Returning to $\Gamma_r = (\mathbb{Z}_+)^{(r)}$, we see that the rationality of the series will follow if we know the rationality of the Hilbert series which corresponds to the enumeration of lattice points of degree n in Γ_k, $k \leqslant r$. Thus, we need to check the rationality of the following power series:

$$\sum_n ((n + k - 1)!/(n!)(k - 1)!)t^n$$

and this is clear. This gives us a direct proof of Theorem 1.3 in case $I = \mathrm{id}_G(FG)$.

We close the article with an estimate of the sequence of codimensions that corresponds to the T-ideal $I = \mathrm{id}_G(FG)$. For the calculation, we consider the F-space P_n spanned by all G-graded multilinear monomials of degree n on the variables $\{x_{g,i}\}_{g \in G, i=1,\ldots,n}$ that are permutations of monomials of the form $x_{1,g_1} x_{2,g_2} \ldots x_{n,g_n}$, where $(g_1, \ldots, g_n) \in G^{(n)}$. Clearly, $\dim(P_n) = \operatorname{ord}(G)^n \times n!$. Let

$$c_G^n(FG) = \dim_F(P_n/(P_n \cap I)).$$

The integer $c_G^n(FG)$ is the n-codimension that corresponds to the G-graded T-ideal I.

COROLLARY 3.4. *Let $A = FG$ be the group algebra over a field F and let $\mathrm{id}_G(FG)$ be the G-graded T-ideal of identities. Let $c_G^n(FG)$ be the nth coefficient of the codimension sequence of $\mathrm{id}_G(FG)$. Then, for any integer n, we have*

(1)

$$ord(G)^n \leqslant c_G^n(FG) \leqslant \operatorname{ord}(G') \operatorname{ord}(G)^n,$$

(2)

$$\lim_{n\to\infty} (c_G^n(FG)/\operatorname{ord}(G') \operatorname{ord}(G)^n)) = 1.$$

In particular, $\exp_G(FG) = \lim_{n\to\infty} \sqrt[n]{c_G^n(FG)} = \operatorname{ord}(G)$ *(see* [1]*).*

Proof. Let $\mathbf{x} = x_{1,g_1}x_{2,g_2}\dots x_{n,g_n}$ be a G-graded monomial in P_n. As noted above, the set of all $n!$ permutations of $\mathbf{x}$ decomposes into at most G' equivalence classes where two monomials are equivalent if and only if the difference is a binomial identity of FG. It then follows easily that $\operatorname{ord}(G)^n \leqslant c_G^n(FG) \leqslant \operatorname{ord}(G')\operatorname{ord}(G)^n$.

Next, recall from the second part of Lemma 3.3 that if a monomial contains G-graded variables with degrees in G as in the word Σ (including multiplicities), then its permutations yield precisely G' non-equivalent classes. So the second part of the corollary will follow easily if we show that

$$\lim_{n\to\infty} \frac{d_n}{\operatorname{ord}(G)^n \times n!} = 0,$$

where d_n denotes the number of monomials in P_n that do not contain the set of elements of Σ (with repetitions). But this of course follows from an easy calculation which is omitted. $\square$

REMARK 3.5. One may replace the group algebra FG above by any twisted group algebra $F^{\alpha}G$, where α is a 2-cocycle of G with values in F^*. As above, also here, two monomials are equivalent if and only if they have the same variables and they determine elements in the same g-homogeneous component. The only difference (comparing to the case where $\alpha \equiv 1$) is that here, for two such monomials, the polynomial identity they determine has the form

$$x_{g_{i_1}}x_{g_{i_2}}\dots x_{g_{i_n}} - \gamma x_{g_{i_{\sigma(1)}}}x_{g_{i_{\sigma(2)}}}\dots x_{g_{i_{\sigma(n)}}},$$

where γ is a non-zero element of F which is determined by the 2-cocycle α and the words $g_{i_1}g_{i_2}\dots g_{i_n}$ and $g_{i_{\sigma(1)}}g_{i_{\sigma(1)}}\dots g_{i_{\sigma(1)}}$ (see [**2**]). One easily sees that the twisting by the 2-cocycle α has no effect neither on the Hilbert series nor on the sequence of codimensions. Details are left to the reader.

References

1. E. ALJADEFF, A. GIAMBRUNO and D. LA MATTINA, 'Graded polynomial identities and exponential growth', *J. reine angew. Math.* 650 (2011) 83–100.
2. E. ALJADEFF, D. HAILE and M. NATAPOV, 'Graded identities of matrix algebras and the universal graded algebra', *Trans. Amer. Math. Soc.* 362 (2010) 3125–3147.
3. E. ALJADEFF and A. KANEL-BELOV, 'Representability and Specht problem for G-graded algebras', *Adv. Math.* 225 (2010) 2391–2428.
4. S. A. AMITSUR, 'A generalization of Hilbert's Nullstellensatz'. *Proc. Amer. Math. Soc.* 8 (1957) 649–656.
5. D. J. ANICK, 'Noncommutative graded algebras and their Hilbert series', *J. Algebra* 78 (1982) 120–140.
6. YU. A. BAHTURIN, S. K. SEHGAL and M. V. ZAICEV, 'Finite-dimensional simple graded algebras', *Sb. Math.* 199 (2008) 965–983.
7. J. P. BELL, 'The Hilbert series of prime PI rings', *Israel J. Math.* 139 (2004) 1–10.
8. A. YA. BELOV, 'Rationality of Hilbert series with respect to free algebras', *Uspekhi Mat. Nauk* 52 (1997) 153–154 (Russian), translation in *Russian Math. Surveys* 52 (1997) 394–395.
9. A. BELOV, V. BORISENKO and V. LATISHEV, *Monomial algebras* (Plenum, NY, 1998) 5–190. *J. Math. Sci., NY* 87 (1997) 3463–3575.
10. A. BERELE, 'Applications of Belov's theorem to the cocharacter sequence of p.i. algebras', *J. Algebra* 298 (2006) 208–214.
11. A. BERELE and A. REGEV, 'Asymptotic behaviour of codimensions of p.i. algebras satisfying Capelli identities', *Trans. Amer. Math. Soc.* 360 (2008) 5155–5172.
12. A. BRAUN, 'The nilpotency of the radical in a finitely generated PI ring', *J. Algebra* 89 (1984) 375–396.
13. V. S. DRENSKY, 'Codimensions of T-ideals and Hilbert series of relatively free algebras', *C. R. Acad. Bulgare Sci.* 34 (1981) 1201–1204.
14. V. S. DRENSKY, 'On the Hilbert series of relatively free algebras', *Comm. Algebra* 12 (1984) 2335–2347.
15. V. S. DRENSKY, 'Codimensions of T-ideals and Hilbert series of relatively free algebras', *J. Algebra* 91 (1984) 1–17.
16. V. O. FERREIRA and L. S. I. MURAKAMI, 'On free associative algebras linearly graded by finite groups', *Groups, Algebras and Applications (Sao Pedro, Brazil 2009)*, Contemporary Mathematics 537 (American Mathematical Society, Providence, RI, 2011) 185–192.
17. A. GIAMBRUNO and M. ZAICEV, 'On codimension growth of finitely generated associative algebras', *Adv. Math.* 140 (1998) 145–155.

18. A. Giambruno and M. Zaicev, 'Exponential codimension growth of P.I. algebras: an exact estimate', *Adv. Math.* 142 (1999) 221–243.
19. A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, Mathematical Surveys and Monographs 122 (American Mathematical Society, Providence, RI, 2005).
20. A. Kanel-Belov and L. H. Rowen, *Computational aspects of polynomial identities* (A. K. Peters Ltd., Wellesley, MA, 2005).
21. A. Kanel-Belov and L. H. Rowen, *Perspectives on Shirshov's Height Theorem*, selected papers of A.I. Shirshov (Birkhüser, Basel, 2009) 3–20.
22. A. R. Kemer, 'Capelli identities and nilpotency of the radical of finitely generated PI-algebra', *Dokl. Akad. Nauk SSSR* 255 (1980) 793–797 (Russian).
23. A. R. Kemer, *Ideals of identities of associative algebras*, American Mathematical Society Translations of Monographs 87 (American Mathematical Society, Providence, RI, 1991).
24. A. R. Kemer, 'Identities of finitely generated algebras over an infinite field', *Izv. Akad. Nauk SSSR Ser. Mat.* 54 (1990) 726–753 (Russian), English translation in *Math. USSRIzv.* 37 (1991) 69–96.
25. Y. Kobayashi, 'The Hilbert series of some graded algebras and the Poincare series of some local rings', *Math. Scand.* 42 (1978) 19–33.
26. P. Koshlukov, 'Rational Hilbert series of relatively free Jordan algebras', *J. Algebra* 121 (1989) 301–309.
27. Ju. P. Razmyslov, 'The Jacobson Radical in PI-algebras', *Algebra i Logika* 13 (1974) 337–360, 365 (Russian).
28. J. B. Shearer, 'A graded algebra with a nonrational Hilbert series', *J. Algebra* 62 (1980) 228–231.
29. R. P. Stanley, 'Hilbert functions of graded algebras', *Adv. Math.* 28 (1978) 57–83.
30. R. P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics 41 (Birkhuser Boston, Inc., Boston, MA, 1983).
31. I. Sviridova, 'Identities of PI-algebras graded by a finite abelian group', *Comm. Algebra* 39 (2011) 3462–3490.

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