



# Kemer's theorem for affine PI algebras over a field of characteristic zero <sup>☆</sup>



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## ABSTRACT

We present a proof of Kemer's representability theorem for affine PI algebras over a field of characteristic zero.

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## 1. Introduction

In this article we provide a proof of the representability theorem for affine PI algebras over a field of characteristic zero which is due to Kemer.

**Theorem 1.1.** *Let  $W$  be an affine PI algebra over a field  $F$  of characteristic zero. Then there exists a field extension  $L$  of  $F$  and a finite dimensional algebra  $A$  over  $L$  which is PI equivalent to  $W$ , that is  $\text{Id}_L(W_L) = \text{Id}_L(A)$ , where  $W_L = W \otimes_F L$ .*

Kemer's proof may be found in his original article [7] and in his monograph [8].

What is the objective of the present paper? Kemer's theorem is considered as one of the fundamental theorems in the theory of polynomial identities (PI). It is the key step in the positive solution of the Specht problem (the  $T$ -ideal of identities of a PI-algebra over a field of characteristic zero is finitely based). Its proof, apart of being rather involved, contains a remarkable number of beautiful new ideas. Kemer's theory opened new avenues of research in the study of varieties of algebras (associative, Lie, Jordan,  $G$ -graded, general nonassociative) as well as in the study of varieties of group and semigroups. We have no objection on

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the correctness of Kemer’s proof. However we believe that Kemer’s expositions (mentioned above) as well as others which can be found in the literature are not detailed enough (see Procesi’s request for clarity in [9]). It is therefore our first objective here to present a proof with *many more details*. In addition we supplement the proof with comments and remarks explaining in each step “what we did”, “what we plan to do” and “how this fits in the general program”. It is our hope that the detailed proof as well as the explanatory remarks will facilitate the reading of the article and make the proof of Kemer’s theorem (considerable more) accessible.

Our exposition differs from Kemer’s in its “second part”. In the first part of the proof we exhibit a finite dimensional algebra  $A$  with (i)  $\text{Id}(A) \subseteq \text{Id}(W)$  (that is the  $T$ -ideal of identities of  $A$  is contained in the corresponding  $T$ -ideal of  $W$ ). (ii) The  $T$ -ideals of identities of  $A$  and  $W$  coincide in the value of certain invariants, namely the Kemer index and Kemer polynomials. In the second part of the proof we exhibit a finite algebra  $B$  which, as the algebra  $A$  above, has the same Kemer polynomials as  $W$  (and therefore the same Kemer index) but now  $\text{Id}(B)$  contains  $\text{Id}(W)$ . In this part (unlike Kemer’s) we make use of a construction made by Razmyslov and Zubrilin (see [13,14]). It seems to us that using “Razmyslov–Zubrilin traces” make the proof more reachable. Then, with these two parts we complete the proof of [Theorem 1.1](#).

Let us also comment on the paper [1] (see also [12]). The paper [1] provides a generalization of Kemer’s Theorem for PI  $G$ -graded algebras  $W$  over a field of zero characteristic where  $G$  is an arbitrary finite group. The proof of the “representability theorem” for  $G$ -graded algebras is considerably more technical and requires some additional ideas (ideas which are unnecessary in case the group  $G$  is trivial). Of course, in our exposition here, these additions are not needed. This allows us to fully “concentrate” on the essential part of the proof, namely “representability” for affine PI algebras. To the best of our knowledge the extension of Kemer’s result to “nonaffine” PI-algebras (using the infinite dimensional Grassmann algebra) as well as the application of “Representability” to the solution of the Specht problem are well understood and so they are not presented here.

## 2. Sketch of the proof

In this short section we outline the main steps of the proof of Kemer’s theorem.

- (1) Step 1. We show there exists a finite dimensional algebra  $A$  with  $\text{Id}(A) \subseteq \Gamma = \text{Id}(W)$ . This is a highly nontrivial result which uses Razmyslov, Kemer, Braun theorem on the nilpotency of the Jacobson radical of an affine PI algebra, Kaplansky’ theorem on primitive PI algebras and Lewin’s theorem on products of  $T$ -ideals.
- (2) Step 2. Definition of  $\text{Ind}(\Gamma)$ , the Kemer index of any  $T$ -ideal  $\Gamma$  which contains the  $T$ -ideal of a finite dimensional algebra  $A$ . In case  $\Gamma = \text{Id}(A)$ , where  $A$  is a finite dimensional algebra, we refer to the index of  $\Gamma$  as the index of  $A$  and denote it by  $\text{Ind}(A)$ . The Kemer index of  $\Gamma$  consists of two parameters (nonnegative integers)  $(\alpha, s)$ , which provide a “measure” of the extreme alternating properties of polynomials which are not in  $\Gamma$ . It is clear that any multilinear polynomial which is alternating on a set of cardinality  $> \dim_F(A)$  must be in  $\text{Id}(A) = \Gamma$ . It follows that the cardinality of alternating sets of any multilinear polynomial  $f$  which is not in  $\Gamma$  is uniformly bounded (e.g. by  $\dim_F(A)$ ). Strictly speaking none of the parameters  $\alpha$  or  $s$  (where  $\text{Ind}(\Gamma) = (\alpha, s)$ ) measures the maximal cardinality of an alternating set in a polynomial  $f \notin \Gamma$ . They measure more subtle invariants related to alternation in polynomials  $f \notin \Gamma$ .  
As mentioned above, the Kemer index  $\text{Ind}(\Gamma) = (\alpha, s)$  is an element in  $\Omega = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . We consider the lexicographic ordering on  $\Omega$  and denote it by  $\leq$ . From the definition of  $\text{Ind}(\Gamma)$  it will be easy to conclude that if  $\Gamma_1 \subseteq \Gamma_2$  then  $\text{Ind}(\Gamma_2) \leq \text{Ind}(\Gamma_1)$  (reverse ordering). In particular  $\text{Ind}(\Gamma) \leq \text{Ind}(A)$ .
- (3) Step 3. Definition of Kemer polynomials of a  $T$ -ideal  $\Gamma$ . These are extremal polynomials which are not in  $\Gamma$  whose alternation realize the Kemer index  $\text{Ind}(\Gamma)$ .

- (4) Step 4. Construction of basic algebras. These are finite dimensional algebras which on one hand are “abundant” enough so that any finite dimensional algebra is PI equivalent to the direct product of finitely many basic algebras and on the other hand they are “special” enough so that the parameters  $\alpha$  and  $s$  (of  $\text{Ind}(A)$ ) coincide respectively with the integers  $\dim_F(\bar{A})$  and  $n_A - 1$ , where  $\bar{A}$  is the semisimple subalgebra of  $A$  which supplements the radical  $J(A)$  and  $n_A$  is the nilpotency index of  $J(A)$ .
- (5) Step 5. From the connection between the parameters of the Kemer index of any basic algebra  $A$  and its geometrical properties (namely  $\dim_F(\bar{A})$  and  $n_A - 1$ ) we obtain the Phoenix property of Kemer polynomials of  $A$ . By definition, Kemer polynomials satisfy the Phoenix property if for any Kemer polynomial  $f$  of  $\Gamma$  (in particular  $f \notin \Gamma$ ) and any polynomial  $f' \in \langle f \rangle$  which is not in  $\Gamma$ , there is  $f'' \in \langle f' \rangle$  which is Kemer of  $\Gamma$  (i.e. Kemer polynomials regenerate themselves). By the fact that any finite dimensional algebra  $A$  is PI equivalent to the product of some basic algebras we obtain the Phoenix property for Kemer polynomials of finite dimensional algebras.
- (6) Step 6. We find a finite dimensional algebra  $B$  with  $\text{Id}(A) \subseteq \text{Id}(B) \subseteq \Gamma$  so that  $\text{Id}(B)$  and  $\Gamma$  have the same Kemer index and have the same Kemer polynomials. From that we obtain the Phoenix property for any Kemer polynomial of  $\Gamma$ .
- (7) Step 7. Construction of a representable algebra  $B_{(\alpha,s)}$  over  $F$  with  $\text{Id}(B_{(\alpha,s)}) \supseteq \Gamma$  and such that all Kemer polynomials of  $\Gamma$  are nonidentities of  $B_{(\alpha,s)}$  (an algebra over  $F$  is representable if it embeds in the algebra of  $n \times n$ -matrices over some field extension  $L$  of  $F$  and some integer  $n$ ). It follows that  $\Gamma$  and  $B_{(\alpha,s)}$  have the same Kemer polynomials and in particular the same index. It is interesting to note that also from that part one can deduce the Phoenix property of Kemer polynomials of  $\Gamma$ .
- (8) Step 8. We consider  $\Gamma' = \Gamma + S$  where  $S$  is the  $T$ -ideal generated by all Kemer polynomials of  $\Gamma$ . This will imply that  $\text{Ind}(\Gamma') < \text{Ind}(\Gamma)$  and hence by induction on the Kemer index there exists a finite dimensional algebra  $A'$  (over a field extension  $L$  of  $F$ ) with  $\Gamma' = \text{Id}(A')$ .
- (9) Step 9. We show that all polynomials of  $S$  (which are not in  $\Gamma$ ) are nonidentities of  $B_{(\alpha,s)}$  (that is, not merely the elements of  $S$  which are also Kemer polynomials, as shown in Step 7). This is achieved by the Phoenix property of Kemer polynomials. Since any nonidentity  $f'$  of  $\Gamma$  which is in  $S$ , produces (by the  $T$ -operation) a Kemer polynomial which by Step 7 is not in  $\text{Id}(B_{(\alpha,s)})$  we have also that  $f' \notin \text{Id}(B_{(\alpha,s)})$ . From that one concludes that  $\Gamma = \text{Id}(A' + B_{(\alpha,s)})$ . Being the algebra  $A' + B_{(\alpha,s)}$  sum of representable algebras, it is representable and so we are done.

### 3. Start

The starting point of the Representability Theorem for affine PI algebras over a field  $F$  of characteristic zero is the existence of a finite dimensional  $F$ -algebra  $A$  whose ideal of identities  $\text{Id}_F(A)$  is contained in  $\Gamma$  (the  $T$ -ideal of identities of the given affine PI  $F$ -algebra  $W$ ). The existence of such algebra  $A$  relies on three fundamental theorems of PI theory. In what follows we prove this assertion and state (without proof) along the way the theorems when they are needed. For future reference let us formulate the main theorem of this section.

**Theorem 3.1.** *Let  $\Gamma$  be the  $T$ -ideal of identities of an affine PI algebra  $W$  over a field  $F$  of characteristic zero. Then there exists a finite dimensional algebra  $A$  over  $F$  such that  $\text{Id}(A) \subseteq \Gamma$ .*

Consider the relatively free algebra  $\mathcal{W} = F\langle X \rangle / \Gamma$ . Denote by  $\mathcal{J}$  its Jacobson radical and by  $J$  the pre-image of  $\mathcal{J}$  inside  $F\langle X \rangle$ . It is clear that  $J$  contains  $\Gamma$ .

First, we claim that  $J$  is a  $T$ -ideal (an ideal of  $F\langle X \rangle$  is a  $T$ -ideal if it is invariant under any endomorphism of the  $F$ -algebra  $F\langle X \rangle$ ). For this recall that an element  $r$  is in  $J(C)$  (the Jacobson radical of an  $F$ -algebra  $C$ ) if and only if any element of  $rC$  is right-quasi-invertible (i.e. for every  $x \in rC$  there is  $y_x \in C$  (quasi-inverse of  $x$ ) such that  $xy_x - x - y_x = 0$ ). Therefore,  $J(C)$  is the unique maximal right-quasi-regular right-ideal of  $C$

(a right ideal  $U$  of  $C$  is right-quasi-regular if every element  $z \in U$  is quasi-invertible). It follows that if  $I \triangleleft \mathcal{W}$  is a right-quasi-regular right-ideal, then so is the image of  $I$  (inside  $\mathcal{W}$ ) by any surjective endomorphism of  $\mathcal{W}$ . Hence, by the maximality of  $\mathcal{J}$ , we deduce that the image of  $\mathcal{J}$  under any epimorphism  $\mathcal{W} \rightarrow \mathcal{W}$  is contained in  $\mathcal{J}$ . We conclude that the image of  $J$  under any surjective endomorphism of  $F \langle X \rangle$  is contained in  $J$ . We have proved the invariance of the ideal  $J$  under endomorphisms of  $F \langle X \rangle$  which are epic. In order to show the invariance of  $J$  under arbitrary endomorphisms of  $F \langle X \rangle$ , let  $\phi \in \text{End}_F F \langle X \rangle$ , and define  $\phi_n \in \text{End}_F F \langle X \rangle$  by

$$\begin{aligned} \phi_n(x_i) &= \phi(x_i), \quad i = 1, 2, \dots, n \\ \phi_n(x_i) &= x_{i-n}, \quad i = n + 1, n + 2, \dots \end{aligned}$$

Note that  $\phi_n$  is onto  $F \langle X \rangle$  and  $\phi_n|_{F \langle x_1, \dots, x_n \rangle} = \phi|_{F \langle x_1, \dots, x_n \rangle}$ . Thus,  $\phi(J \cap F \langle x_1, \dots, x_n \rangle) \subseteq J$ . This holds for every  $n$ , so the claim follows.

Next, consider the algebra  $\mathcal{W}' = \mathcal{W}/\mathcal{J} = F \langle X \rangle / J$ . This is the relatively free algebra associated with  $J$ . Since clearly the Jacobson radical of  $\mathcal{W}'$  is zero, this algebra is a sub-direct product of  $A_{\text{Ind}} = \prod_{i \in \text{Ind}} A_i$ , where  $\text{Ind}$  is some index set and each  $A_i$  is a primitive  $F$ -algebra. Note that all the  $A_i$  satisfy the identities of  $\mathcal{W}$  and  $\text{Id}_F(A_{\text{Ind}}) = \text{Id}_F(\mathcal{W}')$ .

**Theorem 3.2.** (See Kaplansky [5].) *Suppose  $A$  is a primitive  $F$ -algebra satisfying a PI of degree  $d$ . Then  $A$  is a central simple algebra and  $\dim_{\text{Cent}(A)} A \leq \lfloor d/2 \rfloor$ .*

Hence,  $\mathcal{W}'$  has the same identities as  $\prod_{i \in \text{Ind}} M_{n_i}(F_i)$ , where  $F_i$  are fields containing  $F$  and all the  $n_i$  are uniformly bounded. The ideal of identities of this algebra is equal to  $\cap_{i \in \text{Ind}} \text{Id}_F(M_{n_i}(F))$  ( $\text{Id}_F(M_{n_i}(F_i)) = \text{Id}_F(M_{n_i}(F))$ ), thus there is some  $n_0$  such that this ideal is equal to  $\text{Id}_F(M_{n_0}(F))$ . We conclude that  $\text{Id}_F(M_{n_0}(F)) = \text{Id}_F(\mathcal{W}') = J$ .

In order to continue we need to know that  $\mathcal{J}$  is nilpotent. In characteristic zero this is a result of Razmyslov and Kemer [6]. Braun extended their result to any characteristic [3].

**Theorem 3.3.** *If  $A$  is an affine PI  $F$ -algebra, then its Jacobson radical is nilpotent.*

Note that we cannot apply [Theorem 3.3](#) directly to  $\mathcal{W}$  since (in general) it is nonaffine. However, since an element  $c$  of an algebra  $C$  is in  $J(C)$  if and only if  $cC$  consists of right-quasi-regular elements, we deduce that the evaluations of  $J$  on  $W$  are inside  $J(W)$ . Since  $W$  is affine, there is an integer  $r$  for which  $J(W)^r = 0$ , thus  $J^r \subseteq \text{Id}_F(W) = \Gamma$ .

We complete the proof by showing that  $J^r$  is the  $T$ -ideal of a certain finite dimensional algebra over  $F$ . Recall that  $J$  is the  $T$ -ideal of identities of  $M_{n_0}(F)$ .

**Theorem 3.4.** (See [4], Theorem 2.) *Let  $B = UT_F(d_1, \dots, d_m)$  be the upper block triangular matrix algebra over  $F$ . Then,  $\text{Id}_F(B) = \text{Id}_F(M_{d_1}(F)) \cdots \text{Id}_F(M_{d_m}(F))$ .*

Therefore,  $J^r$  is the ideal of identities of the finite dimensional  $F$ -algebra  $UT_F(\underbrace{n_0, \dots, n_0}_{r \text{ times}})$ . This completes the proof of [Theorem 3.1](#).

It follows that there exists an  $n$  such that any polynomial which alternates on a set of cardinality  $n$  is an identity of  $W$ . In that case we say that  $c_n$ , the  $n$ th Capelli polynomial, is in  $\Gamma$  or that  $W$  satisfies the  $n$ th Capelli identity.

It is well known (and easy to prove) that any  $T$ -ideal over a field of characteristic zero is generated by multilinear polynomials and hence  $T$ -ideals are stable under field extensions. This means that if  $\Gamma = \text{Id}(W)$

where  $W$  is an algebra over  $F$ , then  $\Gamma \otimes_F L = \text{Id}_L(W \otimes_F L)$ . We therefore assume, by extension of scalars, that  $F$  is algebraically closed.

#### 4. The index of $T$ -ideals

The following terminology will be used frequently.

**Definition 4.1.** Let  $f(X, Y)$  be a polynomial in noncommuting variables where  $X = \{x_1, \dots, x_n\}$  and  $Y$  is an arbitrary finite set. Suppose further that  $f$  is multilinear on the set  $X$ . We say that  $f$  is alternating on  $X$  (or the set  $X$  is alternating for  $f$ ) if there exists a polynomial  $h = h(X, Y)$  (multilinear on the set  $X$ ) such that

$$f(x_1, \dots, x_n, Y) = \sum_{\sigma \in \text{Sym}(n)} (-1)^\sigma h(x_{\sigma(1)}, \dots, x_{\sigma(n)}, Y).$$

**Terminology 4.2** (*Multialternating polynomial*). We consider polynomials with  $\nu$  disjoint alternating sets of variables each of cardinality  $d$  and  $\mu$  disjoint alternating sets (also disjoint to the previous sets) each of cardinality  $l$ . In particular we consider the case  $l = d + 1$ .

Given a  $T$ -ideal of identities  $\Gamma$  of an affine PI algebra  $W$  we have by the previous section that there exists an integer  $k$  such that any polynomial  $f$  with an alternating set of  $k$  variables is necessarily in  $\Gamma$  (indeed, any  $k > \dim_F(A)$  will do). So if  $d$  is an integer for which there exist polynomials with alternating sets of cardinality  $d$  which are not in  $\Gamma$ , then clearly  $d < k$ . Fix a nonnegative integer  $\nu$  and consider the largest integer  $d_\nu$  for which there exist multilinear polynomials **not** in  $\Gamma$  with  $\nu$  disjoint alternating sets of cardinality  $d_\nu$ . Clearly, the function  $d_\nu$  is nonnegative, integer valued and nonincreasing. Denote by  $d = \lim d_\nu$ . Furthermore, we let  $\mu$  be the integer  $\nu$  for which the function  $d_\nu$  stabilizes (i.e. reaches the limit). Note that  $d$  may be zero as it is in the case of nilpotent algebras.

By the definition of the integer  $d$  it follows that there exist multilinear polynomials not in  $\Gamma$  with arbitrary many disjoint alternating sets of variables of cardinality  $d$ . On the other hand (again from the definition of  $d$ ) there is a bound on the number of alternating disjoint sets of cardinality  $d + 1$  that one can find in multilinear polynomials outside  $\Gamma$ . Let us denote by  $s_\nu$  the maximal number of alternating sets of cardinality  $d + 1$  which appear in polynomials not in  $\Gamma$  which have  $\nu$  alternating sets of cardinality  $d$ . Again, the function  $s_\nu$  is nonincreasing, nonnegative integer valued and hence has a limit which we denote by  $s$ . So we have constructed three integers, namely  $d$ ,  $\mu$  and  $s$  (we take  $\mu$ , to be large enough so that the two functions  $d_\nu$  and  $s_\nu$  reach the limit). By construction we can find multilinear polynomials outside  $\Gamma$  with arbitrary many alternating disjoint sets of cardinality  $d$  and precisely  $s$  sets of alternating sets of cardinality  $d + 1$ . Furthermore, as long as the number of alternating sets of cardinality  $d$  exceeds (or equal)  $\mu$  we will not find multilinear polynomials outside  $\Gamma$  with more than  $s$  alternating sets of cardinality  $d + 1$ .

**Proposition 4.3.** *Let  $W$  be a PI algebra and denote by  $\Gamma = \text{Id}(W)$  its  $T$ -ideal of identities. Then  $W$  is nilpotent with nilpotency index  $= n$  ( $W^{n-1} \neq 0$  and  $W^n = 0$ ) if and only if  $d(\Gamma) = 0$  and  $s(\Gamma) = n - 1$ .*

**Proof.** Note that  $W$  is a nilpotent algebra with nilpotency index  $n \geq 1$  if and only if  $x_1 \cdots x_n \in \Gamma$ , and  $n$  is minimal. It follows there are no multilinear polynomials outside  $\Gamma$  with arbitrary many alternating sets of cardinality 1 and hence  $d = 0$ . On the other hand we have at most  $n - 1$  (alternating) sets of cardinality 1 in a nonidentity and hence  $s \leq n - 1$ . Since  $x_1 \cdots x_{n-1} \notin \Gamma$ ,  $s(\Gamma) = n - 1$ . Conversely, if  $d = 0$ , there is a bound on the number of alternating sets of cardinality 1 and hence there is a bound on the length of multilinear monomials. If  $m$  is the largest length, it is clear that the nilpotency index is  $m + 1$  as required.  $\square$

Note that if  $W$  is an algebra over  $F$ , affine and nilpotent, then it is clearly finite dimensional over  $F$  and hence representable (see definition in section 2, Step 7). This allows us to focus (in certain proofs) on affine PI algebras with index  $(\alpha, \gamma)$  where  $\alpha \geq 1$ .

**Definition 4.4.** We say that a multilinear polynomial  $f$  is a Kemer polynomial of  $\Gamma$  if (a)  $f$  is not in  $\Gamma$  (b)  $f$  has at least  $\mu$  alternating disjoint sets of cardinality  $d$  (*small sets*) (c)  $f$  has exactly  $s$  alternating disjoint sets of cardinality  $d + 1$  (*big sets*).

For a  $T$ -ideal of identities  $\Gamma$  of an affine PI algebra we have defined two parameters, namely  $d$  and  $s$ . We let  $\text{Ind}(\Gamma) = (d, s)$  and call it the Kemer index of  $\Gamma$ . The pair  $\text{Ind}(\Gamma)$  is viewed as a point in the set  $\Omega = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  with the (left) lexicographic ordering (denoted by  $\leq$ ). So any  $T$ -ideal of identities of an affine PI algebra determines a point in  $\Omega$ .

If  $W$  is non-PI, we have  $\Gamma = 0$  and  $d$  is not bounded. We are assuming this is not the case. If  $W = 0$ , the  $T$ -ideal is the free algebra and hence  $\text{Ind}(\Gamma) = (0, 0)$ . If  $W$  is an affine PI algebra and nonzero then  $\text{Ind}(\Gamma) > (0, 0)$ . If  $W$  is PI but nonaffine, then it may satisfy a Capelli identity and it may not. If it satisfies a Capelli identity then it has an index in  $\Omega$  and if it doesn't satisfy any Capelli identity it does not have an index in  $\Omega$ . Note that by [Theorem 3.1](#) any affine PI algebra satisfies a Capelli identity  $c_n$  for some  $n$ .

**Lemma 4.5.** If  $\Gamma_1 \subseteq \Gamma_2$  then  $\text{Ind}(\Gamma_1) \geq \text{Ind}(\Gamma_2)$  i.e. the order is reversed.

Before proving the lemma let us explain once again (in more formal terminology) the functions  $d_\nu$  and  $s_\nu$  considered above.

**Explanation 4.6** (*The functions  $d_\nu, s_\nu$* ). We are assuming that there exists a finite dimensional algebra  $A$  (say of dimension  $n$ ) such that  $\text{Id}(A) \subseteq \Gamma$ . The algebra  $A$  satisfies  $c_{n+1}$  meaning that any multilinear polynomial  $p(X)$  which is alternating on a set of cardinality  $n + 1$  is in  $\Gamma$ . In other words, outside  $\Gamma$  there is no multilinear polynomial which is alternating on a set of cardinality  $n + 1$ .

Fix an integer  $\nu = 1, 2, \dots$ . Let  $d_\nu$  be the maximal nonnegative integer such that there exists *outside*  $\Gamma$  a multilinear polynomial which is alternating on  $\nu$  disjoint sets of cardinality  $d_\nu$ . Note that  $d_\nu < n + 1$  since any polynomial with at least one alternating set of cardinality  $n + 1$  is already in  $\Gamma$ . This says that we can find outside  $\Gamma$  a multilinear polynomial with  $\nu$  alternating (disjoint) sets of cardinality  $d_\nu$  but any polynomial with  $\nu$  alternating sets of cardinality  $d_\nu + 1$  is already in  $\Gamma$ . In that sense  $d_\nu$  is maximal.

Now take  $\rho > \nu$ . Then, by definition, there is a multilinear polynomial  $q(X) \notin \Gamma$  with  $\rho$  alternating sets of cardinality  $d_\rho$ . Let us show  $d_\rho \leq d_\nu$ . Suppose  $d_\rho > d_\nu$ . This would contradict the maximality of  $d_\nu$  since the polynomial  $q(X)$  has  $\rho$  sets (and hence  $\nu$  sets) of cardinality  $d_\rho$  which are alternating. Consequently, the function  $d : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$  is nonincreasing and hence has a limit which we denote by  $d$ . We denote by  $\mu$  the minimal integer with  $d_\mu = d_{\mu+1} = d_{\mu+2} = \dots$ .

Here is the interpretation of  $d$  and  $\mu$ . There exist polynomials outside  $\Gamma$  with arbitrary many alternating sets (disjoint) of cardinality  $d$ . On the other hand we will not find polynomials outside  $\Gamma$  with arbitrary many alternating sets of cardinality  $d + 1$ . By the definition of  $\mu$  above, the number of alternating sets of cardinality  $d + 1$  we can find in polynomials outside  $\Gamma$  is bounded by  $\mu - 1$ .

Let us return to the above considerations with somewhat more formal notation and prove [Lemma 4.5](#). For  $\nu \in \mathbb{N}$  consider the set of nonnegative integers

$$\Delta_\nu = \{r \in \mathbb{N} \cup \{0\} : \exists p(X) \notin \Gamma, \text{ alternating on } \nu \text{ disjoint sets of cardinality } r\}.$$

The set  $\Delta_\nu$  is bounded by  $n$  (the dimension of  $A$ ). We denote by  $d_\nu$  its maximum. Since any polynomial with  $\rho$  alternating sets (of a given cardinality) contains  $\nu$  alternating sets (of the same cardinality) for any  $\nu < \rho$  we have that  $\Delta_\rho \subseteq \Delta_\nu$  and hence  $d_\rho \leq d_\nu$ . We denote by  $d$  the limit of  $d_\nu$ .

In order to construct the second parameter of the Kemer index of  $\Gamma = \text{Id}(W)$  ( $W$  affine and PI), we know by the definition of the parameter  $d$ , that for any  $\nu \geq 1$ , there is a multilinear polynomial  $p_\nu(X) \notin \Gamma$  which alternates on  $\nu$  sets of cardinality  $d$ . Clearly, the parameter  $d$  depends on the  $T$ -ideal  $\Gamma$  so in what follows we may write  $d_\Gamma$ . From the definition of the parameter  $d_\Gamma$  we know that the set

$S_\nu^{(\Gamma)} = \{j \in \mathbb{N} \cup \{0\} : \exists p(X) \notin \Gamma, \text{ alternating on } \nu \text{ disjoint sets of cardinality } d_\Gamma \text{ and } j \text{ disjoint sets (and disjoint to the previous sets) of cardinality } d_\Gamma + 1\}$  is nonempty. Furthermore, by the maximality of  $d_\Gamma$ , we know the set  $S_\nu^{(\Gamma)}$  is bounded for every  $\nu$  and we denote by  $s_\nu = \max(S_\nu^{(\Gamma)})$ . As it is for the sequence  $\{d_\nu\}$ , also the sequence  $\{s_\nu\}$  is nonincreasing and we let  $s = s_\Gamma = \lim_{\nu \rightarrow \infty} s_\nu$ . We let  $\mu$  be an integer where the sequences  $\{d_\nu\}$  and  $\{s_\nu\}$  reach the limit.

In what follows it *will not be important* to keep the precise value  $\mu$  (i.e. the minimal value) where the sequences  $\{d_\nu\}$  and  $\{s_\nu\}$  stabilize. We can take larger integers.

**Proof of Lemma 4.5.** Consider the sets  $\Delta_\nu^{(\Gamma_1)}$  and  $\Delta_\nu^{(\Gamma_2)}$  which correspond to the integer  $\nu$  and the  $T$ -ideals  $\Gamma_1$  and  $\Gamma_2$  respectively. Since  $\Gamma_1 \subseteq \Gamma_2$  we have  $\Delta_\nu^{(\Gamma_1)} \supseteq \Delta_\nu^{(\Gamma_2)}$ . Consequently,  $(d_{\Gamma_1})_\nu = \max(\Delta_\nu^{(\Gamma_1)}) \geq (d_{\Gamma_2})_\nu = \max(\Delta_\nu^{(\Gamma_2)})$  for every  $\nu$  and hence  $d_{\Gamma_1} \geq d_{\Gamma_2}$ . In order to complete the proof of the lemma, we need to show that if  $d_{\Gamma_1} = d_{\Gamma_2}$  then  $s_{\Gamma_1} \geq s_{\Gamma_2}$ . To see this, note that in that case  $S_\nu^{(\Gamma_1)} \supseteq S_\nu^{(\Gamma_2)}$  for every  $\nu$  and hence  $(s_{\Gamma_1})_\nu = \max(S_\nu^{(\Gamma_1)}) \geq (s_{\Gamma_2})_\nu = \max(S_\nu^{(\Gamma_2)})$  for every  $\nu$ . Taking the limit we have  $s_{\Gamma_1} \geq s_{\Gamma_2}$  and we are done.  $\square$

### 5. The index of finite dimensional algebras

We start this section with the definition of the Phoenix property.

**Definition 5.1** (*Phoenix property*). Let  $\Gamma$  be a  $T$ -ideal of an affine PI algebra. Let  $P$  be any property which may be satisfied by polynomials (e.g. being Kemer). We say that  $P$  is “ $\Gamma$ -Phoenix” (or in short “Phoenix”) if given a multilinear polynomial  $f$  satisfying  $P$  which is not in  $\Gamma$  and *any*  $f'$  in  $\langle f \rangle$  (the  $T$ -ideal generated by  $f$ ) which is not in  $\Gamma$  as well, there exists a multilinear polynomial  $f''$  in  $\langle f' \rangle$  which is not in  $\Gamma$  and satisfies  $P$ . We say that  $P$  is “*strictly*  $\Gamma$ -Phoenix” if any multilinear polynomial  $f' \in \langle f \rangle$  which is not in  $\Gamma$ , satisfies  $P$ .

If  $S_P$  is the set of polynomials defined by property  $P$ , we say by abuse of language that the polynomials in  $S_P$  satisfy the Phoenix property (resp. strictly satisfy the Phoenix property).

**Example 5.2.** “Being multilinear” is a Phoenix property. Indeed, if  $f$  is any multilinear polynomial not in  $\Gamma$  and  $f' \in \langle f \rangle$  which is not in  $\Gamma$ , applying multilinearization to  $f'$  there exists a multilinear polynomial  $f'' \in \langle f' \rangle$  which is not in  $\Gamma$ .

Let us pause for a moment and summarize what we have at this point. We are given a  $T$ -ideal  $\Gamma$  (the  $T$ -ideal of identities of an affine algebra  $W$ ). We assume that  $W$  is PI and hence as shown in Section 3 there exists a finite dimensional algebra  $A$  with  $\Gamma \supseteq \text{Id}(A)$ . To the  $T$ -ideal  $\Gamma$  we attach the corresponding Kemer index in  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . Similarly, we may consider the Kemer index of  $\text{Id}(A)$  which by abuse of notation we denote it by  $\text{Ind}(A)$ . By Lemma 4.5, we have  $\text{Ind}(\Gamma) \leq \text{Ind}(A)$ .

One of our main goals (in the first part of the proof of Theorem 1.1) is to replace the algebra  $A$  by a representable algebra  $A'$  with a larger  $T$ -ideal such that

- (1)  $\Gamma \supseteq \text{Id}(A')$ .
- (2)  $\Gamma$  and  $\text{Id}(A')$  have the same Kemer index.
- (3)  $\Gamma$  and  $\text{Id}(A')$  have the “same” Kemer polynomials.

The terminology “the same Kemer polynomials” needs a clarification. Suppose  $\Gamma_1 \supseteq \Gamma_2$  are  $T$ -ideals with  $\text{Ind}(\Gamma_1) = \text{Ind}(\Gamma_2)$ . We say that  $\Gamma_1$  and  $\Gamma_2$  have the same Kemer polynomials if there exists an integer  $\mu$  such that all Kemer polynomials of  $\Gamma_2$  with at least  $\mu$  alternating small sets are not in  $\Gamma_1$ .

**Remark 5.3.** Statements (1)–(3) above will establish the important connection between the combinatorics of the Kemer polynomials of  $\Gamma$  and the structure of finite dimensional algebras. The “Phoenix” property for the Kemer polynomials of  $\Gamma$  will follow from that connection.

Let  $A$  be a finite dimensional algebra over  $F$  and let  $J(A)$  be its Jacobson radical. We know that  $\bar{A} = A/J(A)$  is semisimple. Moreover by the Wedderburn–Malcev Principal Theorem there exists a semisimple subalgebra  $\bar{A}$  of  $A$  such that  $A = \bar{A} \oplus J(A)$  as vector spaces. In addition, the subalgebra  $\bar{A}$  may be decomposed as an algebra into the direct product of (semisimple) simple algebras  $\bar{A} \cong A_1 \times A_2 \times \cdots \times A_q$ .

This decomposition enables us to consider “semisimple” and “radical” substitutions. More precisely, since in order to check whether a given multilinear polynomial is an identity of  $A$  it is sufficient to evaluate the variables on any (given) spanning set, we may take a basis consisting of elements of  $\bar{A} \cup J(A)$ . We refer to such evaluations as semisimple or radical evaluations respectively. Moreover, the semisimple substitutions may be taken from the simple components. Evaluations from the set  $\cup_{i=1}^q A_i \cup J(A)$  are called *elementary*.

*In what follows, whenever we evaluate a polynomial on a finite dimensional algebra, we consider only elementary evaluations.*

For any finite dimensional algebra  $A$  over  $F$  we let  $d(A)$  be the dimension of the semisimple subalgebra  $\bar{A}$  and  $n_A$  the nilpotency index of  $J(A)$ . We denote by  $\text{Par}(A) = (d(A), n_A - 1)$  the parameter of the algebra  $A$ .

**Lemma 5.4.** *Let  $A$  be a finite dimensional algebra over  $F$  and  $\text{Par}(A) = (d(A), n_A - 1)$  its parameter. If  $f$  is a multilinear polynomial with at least  $n_A$  alternating sets  $\{X_l\}_l$ , each of cardinality  $d(A) + 1$ , then  $f$  is an identity of  $A$ .*

**Proof.** If one of the sets  $X_l$  is evaluated only with semisimple elements, by the pigeonhole principle we must have repetitions, and hence by the alternation the polynomial vanishes. Otherwise all sets  $X_l$  get at least one radical evaluation and again the polynomial vanishes since their number is at least the nilpotency index of  $J(A)$ .  $\square$

**Proposition 5.5.** *Let  $(\alpha, s)$  be the index of  $A$ . Then  $(\alpha, s) \leq (d(A), n_A - 1)$ .*

**Proof.** By the definition of the parameter  $\alpha$ , there exist nonidentity polynomials with arbitrary many alternating sets of cardinality  $\alpha$ . This says that  $\alpha \leq d(A)$  for if  $\alpha > d(A)$ , by the previous lemma we cannot have in a nonidentity more than  $n(A) - 1$  alternating sets of cardinality  $\alpha$ . In order to complete the proof of the proposition we need to show (by the lexicographic ordering) that if  $\alpha = d(A)$  then  $s \leq n_A - 1$ . If not, by the definition of  $\text{Ind}(A)$  there exists a nonidentity of  $A$  with  $s > n_A - 1$  alternating sets of cardinality  $\alpha + 1 (= d(A) + 1)$  which is again impossible by the previous lemma. This proves the proposition.  $\square$

In the next example we show that the index of  $A$  may be quite far from  $\text{Par}(A) = (d(A), n_A - 1)$ .



**Example 5.6.** Let  $B$  be a finite dimensional simple algebra,  $r > 1$ ,  $B(r) = B \times \cdots \times B$  ( $r$  times). Clearly  $\text{Id}(B) = \text{Id}(B(r))$  and hence  $B$  and  $B(r)$  have the same Kemer index. On the other hand  $\text{Par}(B(r))$  increases with  $r$ :  $\text{Par}(B) = (d(B), 1) < (r \cdot d(B), 1) = \text{Par}(B(r))$ .

In order to establish a precise relation between the index of a finite dimensional algebra  $A$  and its structure we need to find appropriate finite dimensional algebras which will serve as *minimal models* for a given Kemer index.

**Definition 5.7.** Let  $A$  be a finite dimensional algebra over  $F$ . We say  $A$  is *basic* if  $A$  is not PI equivalent to an algebra  $B$  where  $B = B_1 \times \cdots \times B_r$ ,  $B_i$  is a finite dimensional algebra over  $F$  and  $\text{Par}(B_i) < \text{Par}(A)$  for  $i = 1, \dots, r$ .

**Remark 5.8.** The above definition of a basic algebra, as well as some definitions below, are different from those in [1].

In Proposition 5.5 we showed that  $\text{Ind}(A) \leq \text{Par}(A)$  for any finite dimensional algebra. In the next lemma we show that if  $A$  is not basic then the inequality is strict.

**Lemma 5.9.** *Let  $A$  be a finite dimensional nonbasic algebra. Then  $\text{Ind}(A) < \text{Par}(A)$ .*

**Proof.** If  $A$  is nonbasic, there exists an algebra  $B = B_1 \times \cdots \times B_r$ , PI equivalent to  $A$ , where  $B_i$  is a finite dimensional algebra over  $F$  and  $\text{Par}(B_i) < \text{Par}(A)$  for  $i = 1, \dots, r$ . We know by Prop. 5.5 that  $\text{Ind}(B_i) \leq \text{Par}(B_i)$ . Suppose  $\text{Ind}(A) = \text{Par}(A) = (d(A), n_A - 1)$ . Then for any  $\mu$  (by definition of the Kemer index), there exists a multilinear polynomial  $f$ , nonidentity of  $A$ , with  $\mu$ -folds of alternating sets of cardinality  $d(A)$  and precisely  $n_A - 1$  alternating sets of cardinality of  $d(A) + 1$  (a Kemer polynomial). We claim however, that if  $\mu$  is sufficiently large the polynomial  $f$  is an identity of  $B_i$ ,  $i = 1, \dots, r$ , and hence an identity of  $B = B_1 \times \cdots \times B_r$  showing that the algebras  $A$  and  $B$  are not PI equivalent. Indeed, since  $\text{Par}(B_i) < \text{Par}(A)$  we either have  $d(B_i) < d(A)$  or else  $d(B_i) = d(A)$  and  $n_{B_i} < n_A$ . In the first case we have  $d(B_i) + 1 \leq d(A)$  and hence by Lemma 5.4  $f$  is an identity as long as  $\mu$ , the number of alternating sets of cardinality  $d(A)$ , is  $\geq n_{B_i}$ . If  $d(B_i) = d(A)$  and  $n_{B_i} < n_A$ , then again by Lemma 5.4,  $f$  is an identity of  $B_i$  since it has  $n_A - 1 \geq n_{B_i}$  alternating sets of cardinality  $d(B_i) + 1 = d(A) + 1$ . We see that any Kemer polynomial of  $A$  with  $\mu$  alternating sets of cardinality  $d(A)$ ,  $\mu \geq \max\{n_{B_1}, \dots, n_{B_r}\}$ , is an identity of  $B$ . This completes the proof of the lemma.  $\square$

Our main task in this section and the next two is to show that if  $A$  is basic then  $\text{Ind}(A) = \text{Par}(A)$  (in fact we will show the two conditions are equivalent for a finite dimensional algebra  $A$ ) (see 7.13). For the proof of that statement we introduce two properties of finite dimensional algebras, named *full* and *property K*.

**Definition 5.10.** We say that a finite dimensional algebra  $A$  is *full* with respect to a multilinear polynomial  $f$  if the following hold:

- (1)  $f$  is a nonidentity of  $A$ .
- (2) Every simple component of  $A$  is represented (among the semisimple substitutions) on *every* nonvanishing elementary evaluation of  $f$  on  $A$ . We also say the nonzero evaluation of  $f$  *visits* any simple component of  $A$ .

A finite dimensional algebra  $A$  is said to be full if it is full with respect to some multilinear polynomial  $f$ .

We wish to show that any finite dimensional algebra may be decomposed (up to PI-equivalence) into the direct product of full algebras.

**Lemma 5.11.** *Let  $A$  be a finite dimensional algebra over  $F$  with  $q$  simple components. If the algebra  $A$  is not full, then  $A$  is PI-equivalent to a finite dimensional algebra  $B = B_1 \times \cdots \times B_q$ , where (1)  $B_i$  has fewer simple components than  $A$  for  $i = 1, \dots, q$ . (2)  $d(B_i) < d(A)$  for  $i = 1, \dots, q$  and hence  $\text{Par}(B_i) < \text{Par}(A)$ .*

**Proof.** Note that if  $q = 0$ , i.e.  $A$  is radical (nonzero) then  $A$  is full with respect to any multilinear polynomial, nonidentity of  $A$ . We therefore may assume that  $q > 0$  and suppose  $A$  is not full. This means that any multilinear polynomial, nonidentity of  $A$ , has a nonvanishing evaluation which misses the simple component  $A_i$  of  $A$  for some  $i$ .

For any  $i = 1, \dots, q$ , consider the subalgebra  $B_i = \langle A_j, J : j \neq i \rangle$  (i.e. the subalgebra generated by all elements of  $J$  and  $A_j$ ,  $j \neq i$ ). Note that  $J(B_i) = J(A) = J$  and  $B_i/J \cong \overline{B}_i = \prod_{j \neq i} A_j$ . Consider the product  $B = B_1 \times \cdots \times B_q$ . We know that  $\text{Id}(A) \subseteq \text{Id}(B_i)$ , any  $i$ , and hence  $\text{Id}(A) \subseteq \bigcap_i \text{Id}(B_i) = \text{Id}(B)$ . For the converse, let  $f$  be a multilinear polynomial, nonidentity of  $A$ . By our assumption above  $f$  has a nonzero evaluation on  $A$  which misses  $A_i$  for some  $i$  and hence  $f$  is a nonidentity of  $B_i$ . We obtain that  $f$  is a nonidentity of  $B$  as desired.  $\square$

**Corollary 5.12.** *Every finite dimensional algebra  $A$  is PI equivalent to a finite dimensional algebra  $B = B_1 \times \cdots \times B_m$ , where  $B_i$  is full for  $i = 1, \dots, m$ .*

**Proof.** This follows from the corollary above (either from part (1) or (2)).  $\square$

**Corollary 5.13.** *Let  $A$  be a finite dimensional algebra. If  $A$  is basic then  $A$  is full.*

**Proof.** If  $A$  is not full, by the previous corollary,  $A$  is PI equivalent to  $B = B_1 \times \cdots \times B_m$ , where  $d(B_i) < d(A)$  and hence, by definition,  $A$  is not basic.  $\square$

Roughly speaking, a finite dimensional algebra  $A$  is *full* if its simple components are “connected” via all nonzero evaluations of a suitable nonidentity. “Property K” (see Section 7), concerns with the number of radical evaluations of nonidentity polynomials.

## 6. Kemer’s Lemma 1

The task in this section is to show that if  $A$  a finite dimensional algebra which is full then the first parameter of  $\text{Ind}(A)$  and the first parameter of  $\text{Par}(A)$  coincide.

For the proof we need to show that for an arbitrary large integer  $\nu$  there exists a multilinear nonidentity  $f$  that contains  $\nu$  folds of alternating sets of cardinality  $\dim_F(\overline{A})$ . Since the algebra  $A$  is full, there is a multilinear polynomial  $f(x_1, \dots, x_q, \overline{y})$ , which does not vanish upon an evaluation of the form  $x_j = \overline{x}_j \in \overline{A}_j$ ,  $j = 1, \dots, q$  and the variables of  $\overline{y}$  get values in  $A$  (either in simple components or in the radical). The idea is to produce polynomials  $\widehat{f}$ ’s in the  $T$ -ideal generated by  $f$  which remain nonidentities of  $A$  and that reach eventually the desired form. The way one checks that the polynomials  $\widehat{f}$ ’s are nonidentities is by presenting suitable evaluations on which they do not vanish.

**Lemma 6.1** (*Kemer’s Lemma 1 for finite dimensional algebras*). *Notation as above. Let  $A$  be a finite dimensional algebra which is full with respect to the polynomial  $f = f(x_1, \dots, x_q, \overline{y})$ . Then for any integer  $\nu$  there exists a multilinear polynomial  $f'$  in the  $T$ -ideal generated by  $f$  with the following properties:*

- (1)  $f' \notin \text{Id}(A)$ ,
- (2)  $f'$  has  $\nu$ -folds of alternating sets of cardinality  $\dim_F(\bar{A})$ .

**Proof.** We note that if the algebra is radical, then the lemma is clear.

Let  $f_0$  be the polynomial obtained from  $f$  by multiplying (on the left say) each one of the variables  $x_1, \dots, x_q$  by variables  $z_1, \dots, z_q$  respectively. Note that the polynomial obtained, denoted by  $f_1$ , is a non-identity since the variables  $z_i$ 's may be evaluated by the elements  $1_{\bar{A}_i}$ 's where

$$1_{\bar{A}_i} = E_{1,1}^i + \dots + E_{k_i,k_i}^i.$$

(Here we use the notation,  $\bar{A}_i = M_{k_i}(F)$ .)

By linearity there exists a nonzero evaluation where the variables  $z_1, \dots, z_q$  take values of the form  $E_{j_1,j_1}^1, \dots, E_{j_q,j_q}^q$  where  $1 \leq j_i \leq k_i$  and  $i = 1, \dots, q$ .

Our aim is to replace each one of the variables  $z_1, \dots, z_q$  by polynomials  $Z_1, \dots, Z_q$  such that:

- (1) For every  $i = 1, \dots, q$ , the polynomial  $Z_i$  is alternating in  $\nu$ -folds of sets of cardinality  $\dim_F(\bar{A}_i)$ .
- (2) For every  $i = 1, \dots, q$ , the polynomial  $Z_i$  assumes the value  $E_{j_i,j_i}^i$ .

Once this is accomplished, we complete the construction by alternating the corresponding  $p$ th sets,  $p = 1, \dots, \nu$ , which come from different  $Z_i$ 's. Clearly, the polynomial  $f'$  obtained

- (1) is a nonidentity since any nontrivial alternation of the evaluated variables (as described above) vanishes,
- (2) has  $\nu$ -folds of alternating sets of cardinality  $\dim_F(\bar{A})$ .

We now show how to construct the polynomials  $Z_i$ .

In order to simplify the notation we put  $\hat{A} = \bar{A}_i (\cong M_k(F))$  where  $\bar{A}_i$  is the  $i$ -th simple component.

Fix  $1 \leq t \leq k$  and consider a product of the  $k^2$  different matrix units  $E_{i,j}$  of  $M_k(F)$  with value  $E_{t,t}$  (it is not difficult to show the such a product exists). We refer to these matrix units  $E_{i,j}$  as *designated matrices*. Next, we border each matrix unit  $E_{i,j}$  with idempotents  $E_{i,i}$  and  $E_{j,j}$ . We refer to these idempotent matrices as *frames*. Clearly the product of all matrices, namely designated matrices and frames is  $E_{t,t}$ . Now we construct a Capelli polynomial which corresponds to that word of matrix units: We construct a monomial whose variables are in 1–1 correspondence with the matrix units just mentioned, namely the designated and frame matrices. We make a distinction between variables that correspond to designated matrices  $(u_1, \dots, u_n, \dots)$  and variables that correspond to frame matrices  $(y_1, \dots, y_m, \dots)$ . We denote that monomial by  $\Sigma_1 t$ . The subscript 1 stands for the fact that the monomial has exactly one set of designated variables whereas the subscript  $t$  stands for the fact that there is an evaluation with value  $E_{t,t}$ . Next we consider the product of  $\nu$  monomials  $\Sigma_1 t$  (with different variables). We denote the long monomial obtained by  $\Sigma_\nu t$ . In order to complete the construction of the polynomials  $Z_i$ , we consider the polynomial  $\hat{Z}_\nu t$  obtained by alternating each set of designated variables *separately*. We let  $Z_i = \hat{Z}_\nu j_i$ , for each  $i = 1, \dots, q$ . Once again for each  $p = 1, \dots, \nu$ , we consider the  $p$ th set of designated variables in each polynomial  $Z_i$  and as indicated above we alternate these variables among the different  $i = 1, \dots, q$ . It is clear that if we evaluate the variables accordingly, any nontrivial permutation yields a zero value and hence get a nonidentity of  $A$  of the desired form. This completes the proof of the lemma and also of [Proposition 6.2](#).  $\square$

From Kemer’s Lemma 1 and the discussion that precedes it we obtain

**Proposition 6.2.** *Let  $A$  be a finite dimensional algebra which is full. Let  $\text{Ind}(A) = (\alpha, s)$  and  $\text{Par}(A) = (d(A), n_A - 1)$ . Then  $\alpha = d(A)$ .*

**Remark 6.3.** Let us return once again to the definition of the *full* property of a finite dimensional algebra  $A$ . By definition, a finite dimensional algebra  $A$  is full if there exists a multilinear polynomial, nonidentity of  $A$  such that all simple components are represented in any nonzero evaluation on  $A$ . Nevertheless, for the proof of Kemer's Lemma 1 we used a seemingly weaker condition, namely the existence of a multilinear nonidentity of  $A$  which has a nonzero evaluation which visits all simple components of  $A$ . These two conditions are indeed nonequivalent for a given polynomial. However, as we see below, it follows from Kemer's Lemma 1, the  $T$ -ideal generated by a nonidentity  $f$  which satisfies the weaker condition contains a polynomial  $f'$  which satisfies the stronger condition.

**Corollary 6.4.** *Let  $A$  be a finite dimensional algebra. Then  $A$  is full if and only if there exists a multilinear polynomial  $f$ , nonidentity of  $A$  which admits a nonzero evaluation which visits every simple component of  $A$ .*

**Proof.** Clearly, if  $A$  is full the condition is satisfied. In order to prove the opposite direction we need to show that if  $f$  is a multilinear polynomial, nonidentity of  $A$  with a nonzero evaluation on  $A$  which visits any simple component then there exists a multilinear polynomial  $f'$ , nonidentity of  $A$ , which visits every simple component of  $A$  on every nonzero evaluation. Indeed, by Kemer Lemma 1 we know there exists a multilinear polynomial  $f'$  in the  $T$ -ideal  $\langle f \rangle$ , nonidentity of  $A$ , which alternates on  $\nu$  disjoint sets of variables of cardinality  $d(A)$ , where  $\nu$  is arbitrary. Now, if we take  $\nu \geq n_A$ , then in any nonzero evaluation of  $f'$ , at least one alternating set of cardinality  $d(A)$  must be evaluated by only semisimple elements for otherwise in each alternating set there is a variable which gets a radical value. Of course any such evaluation vanishes since at least  $n_A$  variables of  $f'$  get radical values. But if in any nonzero evaluation of  $f'$ , there is one alternating set of cardinality  $d(A)$  which gets only semisimple values, by the alternation, these values must be linearly independent over  $F$  and hence consist of a basis of the semisimple subalgebra  $\bar{A}$ . This shows that any nonzero evaluation of  $f'$  visits all simple components as desired.  $\square$

## 7. Kemer's Lemma 2

In this section we prove Kemer's Lemma 2 (Lemma 7.8 below). Before stating the precise statement we need an additional reduction which enables us to control the number of radical evaluations in certain nonidentities.

Let  $f$  be a multilinear polynomial which is not in  $\text{Id}(A)$ . Clearly, any nonzero evaluation cannot have more than  $n_A - 1$  radical evaluations.

**Lemma 7.1.** *Let  $A$  be an algebra which is full. Let  $\text{Ind}(A) = (\alpha, s)$  be its Kemer index. Then  $s \leq n_A - 1$ .*

**Proof.** By Kemer's Lemma 1 we can find nonidentity polynomials with arbitrary many alternating sets of cardinality  $d(A)$  and since this is the maximum possible, we have that  $\alpha = d(A)$ . It follows that in alternating sets of cardinality  $d(A) + 1$  we must have at least one radical evaluation and hence we cannot have more than  $n_A - 1$  in a nonidentity.  $\square$

**Remark 7.2.** Although not needed later in the paper, it is worth noting that the result of the lemma above holds also for arbitrary finite dimensional algebras. Indeed we know (Corollary 5.12) that the algebra  $A$  is PI equivalent to the direct product of algebras  $B_1 \times \cdots \times B_m$ , where  $B_i$  is full for  $i = 1, \dots, m$ . For each  $B_i$  we consider the dimension of the semisimple part  $d(B_i)$ . Applying Kemer Lemma 1 we have that  $\alpha \geq \max_i(d(B_i))$ . On the other hand if  $\alpha > d(B_i)$ , any multilinear polynomial with more than  $n_{B_i} - 1$  alternating sets of cardinality  $\alpha$  is in  $\text{Id}(B_i)$  (any alternating set must have at least one radical evaluation) and hence if  $\alpha > \max_i(d(B_i))$ , any polynomial as above is an identity of  $B_1 \times \cdots \times B_m$  and hence of  $A$ .

This contradicts the definition of the parameter  $\alpha$  and hence  $\alpha = \max_i(d(B_i))$ . Now take an alternating set of cardinality  $\alpha + 1$ . In every such set we must have a radical evaluation or elements from different full algebras. If they come from different full algebras we get zero. If we get a radical element then we cannot pass  $n_A - 1$ .

The next definition is key in the proof of Kemer’s Lemma 2.

**Definition 7.3.** Notation as above. Let  $f$  be a multilinear polynomial which is not in  $\text{Id}(A)$ . We say that  $A$  has property  $K$  with respect to  $f$  if  $f$  vanishes on any evaluation on  $A$  with less than  $n_A - 1$  radical substitutions.

We say that a finite dimensional algebra  $A$  has property  $K$  if it satisfies the property with respect to some nonidentity multilinear polynomial.

**Proposition 7.4.** *Let  $A$  be a finite dimensional basic  $F$ -algebra. Then it has property  $K$ .*

Before proving the proposition we introduce a construction which will enable us to put some “control” on the nilpotency index of (the radical of) finite dimensional algebras which are PI equivalent.

Let  $B$  be any finite dimensional algebra and let  $B' = \overline{B} * F\{x_1, \dots, x_n\}$  be the algebra of polynomials in the variables  $\{x_1, \dots, x_n\}$  with coefficients in  $\overline{B}$ , the semisimple component of  $B$  (in case  $\overline{B} = 0$ ,  $B' = F\langle x_1, \dots, x_n \rangle$  is the nonunital free  $F$ -algebra generated by the variables  $\{x_1, \dots, x_n\}$ ). The number of variables we take is at least the dimension of  $J(B)$ . Let  $I_1$  be the ideal of  $B'$  generated by all evaluations of polynomials of  $\text{Id}(B)$  on  $B'$  and let  $I_2$  be the ideal generated by all variables  $\{x_1, \dots, x_n\}$ . Consider the algebra  $\widehat{B}_u = B'/(I_1 + I_2^u)$ .

**Proposition 7.5.** *The following hold.*

- (1)  $\text{Id}(\widehat{B}_u) = \text{Id}(B)$  whenever  $u \geq n_B$  ( $n_B$  denotes the nilpotency index of  $J(B)$ ). In particular  $\widehat{B}_u$  and  $B$  have the same index.
- (2)  $\widehat{B}_u$  is finite dimensional.
- (3) The nilpotency index of  $J(\widehat{B}_u)$  is  $\leq u$ .

**Proof.** Note that by the definition of  $\widehat{B}_u$  (modding  $B'$  by an ideal that contains  $I_1$ ),  $\text{Id}(\widehat{B}_u) \supseteq \text{Id}(B)$ . On the other hand there is a surjection  $\widehat{B}_u \rightarrow B$  which maps the variables  $\{x_1, \dots, x_n\}$  onto a spanning set of  $J(B)$  and  $\overline{B}$  is mapped isomorphically. Indeed, we have such a map from  $B'$ , namely there is a surjective map  $\phi : B' \rightarrow B$  which maps  $\overline{B}$  isomorphically and the variables  $\{x_1, \dots, x_n\}$  onto a spanning set of  $J(B)$ . The ideal  $I_1$  consists of all evaluation of  $\text{Id}(B)$  on  $B'$  and hence is contained in  $\ker(\phi)$ . Also the ideal  $I_2^u$  is contained in  $\ker(\phi)$  since  $u \geq n_B$  and  $\phi(x) \in J$ . This shows (1).

To see (2) observe that any element in  $\widehat{B}_u$  is represented by a sum of elements of the form  $b_1 z_1 b_2 z_2 \cdots b_j z_j b_{j+1}$  where  $j < u$ ,  $b_i \in \overline{B}$  and  $z_i \in \{x_i\}$  (we allow also consecutive  $z_i$ ’s). Clearly, the subspace spanned by monomials for a given configuration of the  $z_i$ ’s (and arbitrary  $b_i$ ’s) has finite dimension. On the other hand the number of different configurations is finite and so the result follows. In order to prove the 3rd statement, note that  $I_2$  generates a radical ideal in  $\widehat{B}_u$  and since  $B'/I_2 \cong \overline{B}$  we have that  $\widehat{B}_u/I_2 \cong B'/(I_1 + I_2^u + I_2) = B'/(I_1 + I_2) \cong (B'/(I_2))/I_1 = \overline{B}/I_1 = \overline{B}$  (the last equality follows from the fact that  $\overline{B} \subseteq B$ ). We therefore see that  $I_2$  generates the radical in  $\widehat{B}_u$ , and hence its nilpotency index is bounded by  $u$  as claimed.  $\square$

**Proof of Proposition 7.4.** Suppose  $A$  is a basic algebra for which property  $K$  fails. This means that any multilinear polynomial that vanishes on any evaluation with less than  $n_A - 1$  radical evaluations must be in  $\text{Id}(A)$ . Consider the algebra  $\widehat{A}_u = A'/(I_1 + I_2^u)$  (from the proposition above). We claim that  $\text{Id}(\widehat{A}_{n_A-1}) =$

$\text{Id}(A)$ . This will complete the proof since the algebras  $\widehat{A}_{n_A-1}$  and  $A$  have isomorphic semisimple parts and the nilpotency index of  $J(\widehat{A}_{n_A-1})$  is bounded by  $n_A - 1$ . This shows that  $\text{Par}(\widehat{A}_{n_A-1}) < \text{Par}(A)$  and hence  $A$  is not basic.

In order to prove the claim note that by construction  $\text{Id}(A) \subseteq \text{Id}(\widehat{A}_{n_A-1})$ . For the converse inclusion take a multilinear polynomial  $f$  which is not in  $\text{Id}(A)$ . Then by assumption, there is a nonzero evaluation  $\tilde{f}$  of  $f$  on  $A$  with less than  $n_A - 1$  radical substitutions (say  $k$ ). Following this evaluation we refer to the variables of  $f$  that get semisimple (radical) values as semisimple (radical) variables respectively. Consider the evaluation  $\hat{f}$  of  $f$  on  $A' = \bar{A} * \{x_1, \dots, x_n\}$  where semisimple variables are evaluated as in  $\tilde{f}$  whereas the radical variables are evaluated on  $\{x_1, \dots, x_n\}$  respecting the surjection  $\phi : A' \rightarrow A$  (by abuse of language “we replace the radical values in  $\tilde{f}$  by indeterminates”).

We claim  $\hat{f} \notin I_1 + I_2^{n_A-1}$ . This will show  $f \notin \text{Id}(\widehat{A}_{n_A-1})$  which is what we want. Clearly, the map  $\phi$  induces  $\bar{\phi} : A'/I_1 \rightarrow A$  and hence  $\hat{f} \notin I_1$ . Next, note that an element in  $A' = \bar{A} * \{x_1, \dots, x_n\}$  is in  $I_1$  if and only if each one of its multihomogeneous components in the variables of  $\{x_1, \dots, x_n\}$  is in  $I_1$ . But by construction  $\hat{f}$  is multihomogeneous of degree  $k < n_A - 1$  in the variables  $\{x_1, \dots, x_n\}$  whereas any element of  $I_2^{n_A-1} \subseteq A'$  is the sum of multihomogeneous elements degree  $\geq n_A - 1$ . We therefore have that  $\hat{f} \in I_1 + I_2^{n_A-1}$  if and only if  $\hat{f} \in I_1$  and we are done.  $\square$

Let  $A$  be a basic algebra. Let  $\text{Par}_A = (d(A), n_A - 1)$  where  $d(A)$  is the dimension of the semisimple part of  $A$  and  $n_A$  the nilpotency index of  $J(A)$ . By Proposition 7.4 the algebra satisfies property  $K$  with respect to a nonidentity polynomial  $f$ , that is  $f$  vanishes on any evaluation whenever it has less than  $n_A - 1$  radical substitutions. Furthermore, there is possibly a different nonidentity polynomial  $h$  with respect to which  $A$  is full, that is  $h$  has a nonzero evaluation which “visits” each one of the simple components of  $\bar{A}$ . In order to proceed we need both properties to be satisfied by the same polynomial.

**Lemma 7.6.** *Let  $A$  be a basic algebra. Then there exists a multilinear polynomial  $f$ , nonidentity of  $A$ , which visits every simple component in any nonzero evaluation and has property  $K$*

**Proof.** We prove the lemma by showing that there exists a multilinear polynomial, nonidentity of  $A$ , which visits every simple component in any nonzero evaluation and in addition realizes the property  $K$  of  $A$ , that is, vanishes on any evaluation with less than  $n_A - 1$  radical evaluations.

Suppose first  $A$  is a radical algebra. Then, any nonidentity of  $A$  which satisfies property  $K$ , visits (in an empty way) all simple components of  $A$  in every nonzero evaluation and so we are done in that case. Suppose now  $q > 0$  (number of simple components of  $A$ ) and suppose by way of contradiction that the lemma is false, that is any multilinear polynomial  $f$ , nonidentity of  $A$ , has a nonzero evaluation with less than  $n_A - 1$  radical evaluations or has a nonzero evaluation which does not visit all simple components of  $A$ .

Consider the subalgebras  $B_i, i = 1, \dots, q$  constructed in the proof of Lemma 5.11, the algebra  $B = B_1 \times \dots \times B_q$  and the algebra  $\widehat{A}_{n_A-1}$  (see proof of Proposition 7.4). Let  $C = B_1 \times \dots \times B_q \times \widehat{A}_{n_A-1}$ . We claim the algebras  $A$  and  $C$  are PI equivalent. This will contradict the fact that  $A$  is basic since  $\text{Par}(\widehat{A}_{n_A-1}) < \text{Par}(A)$  and  $\text{Par}(B_i) < \text{Par}(A)$  for  $i = 1, \dots, q$ . In order to prove the claim note that by the definition of the  $B_i$ 's and  $\widehat{A}_{n_A-1}$  we have  $\text{Id}(A) \subseteq (\cap \text{Id}(B_i)) \cap \text{Id}(\widehat{A}_{n_A-1}) = \text{Id}(C)$ . For the opposite direction let  $f \notin \text{Id}(A)$ , multilinear. By assumption,  $f$  has either a nonvanishing evaluation on  $A$  which does not visit all simple components of  $A$  (say  $A_j$ ) or has a nonvanishing evaluation with less than  $n_A - 1$  radical evaluations. In the first case  $f$  is a nonidentity of  $B_j$  whereas in the other case  $f$  is a nonidentity of  $\widehat{A}_{n_A-1}$ . In both cases  $f \notin \text{Id}(C)$  and the lemma is proved.  $\square$

In the next example we exhibit two nonidentities  $p$  and  $q$  of an algebra  $A$  such that  $p$  admits nonzero evaluations which visit any simple component and also nonzero evaluations which do not visit every simple component whereas any nonzero evaluation of  $q$  must visit any simple component.

**Example 7.7.** Let  $A$  be the algebra over  $F$  of upper triangular  $2 \times 2$ -matrices. Consider the polynomial  $p(x, y, z) = xyz$ . It is clear that  $p$  has a nonzero evaluation which visits the two simple components of  $A$  ( $x = e_{11}, y = e_{12}, z = e_{22}$ ). On the other one can easily find a nonzero evaluation of  $p$  which visits only one simple component ( $x = e_{11}, y = e_{11}, z = e_{11}$ ). Next we construct a polynomial which visits both simple components on every nonzero evaluation. Let  $q(x_1, x_2, x_3) = \sum (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$ . The polynomial  $q$  is alternating on a set of cardinality 3, the variables  $x_1, x_2, x_3$  must get linearly independent elements of  $A$  in any nonzero evaluation. Since  $A$  is of dimension 3, the elements  $e_{11}, e_{12}, e_{22}$  must appear as values and hence any nonzero evaluation visits all simple components. It remains to show that there is at least one nonzero evaluation. Indeed, it is easily verified that the evaluation  $x_1 = e_{11}, x_2 = e_{12}, x_3 = e_{22}$  is nonzero.

We can now state and prove Kemer’s Lemma 2.

**Lemma 7.8 (Kemer’s Lemma 2).** *Let  $A$  be a finite dimensional. Suppose  $A$  is basic with index ( $d = d(A), n_A - 1$ ). Then for any integer  $\nu$  there exists a multilinear, nonidentity polynomial  $f$  with  $\nu$ -alternating sets of cardinality  $d$  and precisely  $n_A - 1$  alternating sets of variables of cardinality  $d + 1$ .*

The alternating sets of cardinality  $d$  are called *small sets* whereas the sets of cardinality  $d + 1$  are called *big sets*.

Any nonzero evaluation of such  $f$  must consists only of semisimple evaluations in the  $\nu$ -folds and each one of the big sets must have exactly one radical evaluation.

**Proof of Lemma 7.8.** The theorem is clear either in case  $A$  is radical or semisimple (i.e. simple). Hence for the proof we assume that  $q \geq 1$  (the number of simple components of  $A$ ) and  $n_A > 1$ .

By Lemma 7.6, there exists a multilinear polynomial  $f$  with respect to which  $A$  is full and has property  $K$ . Let us fix a nonzero evaluation  $x \mapsto \hat{x}$  realizing the “full” property. Note that by the construction of  $f$ , being the evaluation nonzero, precisely  $n_A - 1$  variables must obtain radical values, and hence the rest of the variables obtain semisimple values. Let us denote by  $w_1, \dots, w_{n_A-1}$  the variables that obtain radical values (in the evaluation above) and by  $\hat{w}_1, \dots, \hat{w}_{n_A-1}$  their corresponding values. By abuse of language we refer to the variables  $w_1, \dots, w_{n_A-1}$  as *radical variables*.

Note that by Kemer Lemma 1 we could assume at this point that  $f$  is alternating on  $\nu$ -folds of alternating sets of cardinality  $\dim_F(\bar{A})$ , but since it will be important to know where these alternating sets are located (with respect to the radical evaluations), our starting polynomial in the proof below is merely assumed to realize property  $K$  and the nonzero evaluation (fixed above) to realize the full property of  $A$ .

We will consider four cases. These correspond to whether  $A$  has or does not have an identity element and whether  $q$  (the number of simple components)  $> 1$  or  $q = 1$ .

**Case (1, 1)** ( $A$  has an identity element and  $q > 1$ ).

By linearity we may assume the evaluation of any radical variable  $w_i$  is of the form  $1_{A_{j(i)}} \hat{w}_i 1_{A_{\bar{j}(i)}}$ ,  $i = 1, \dots, n_A - 1$ , where  $1_{A_k}$  is the identity element of the simple component  $A_k$  (we refer to  $1_{A_{j(i)}}, 1_{A_{\bar{j}(i)}}$ ,  $i = 1, \dots, n_A - 1$  as *bordering elements*). Note that the evaluation remains full (i.e. visits every simple component of  $A$ ).

Choose a monomial  $X$  of  $f$  which does not vanish upon the above evaluation.

Notice that the variables of  $X$  which get semisimple evaluations from *different* simple components must be separated by radical variables.

**Claim 7.9.** *The elements  $1_{A_{j(i)}}, 1_{A_{\bar{j}(i)}}$ ,  $i = 1, \dots, n_A - 1$ , which appear in the borderings above, represent all simple components of  $\bar{A}$ .*

Indeed, suppose that the component  $A_1$  (say) is not represented among the  $1_{A_k}$ ’s. Since our original evaluation is full, there is a variable which is evaluated by an element  $u$  of  $A_1$ . “Moving” along the monomial

$X$  to the left or right of  $u$  we will hit a bordering value  $1_{A_k}$  before we hit any radical evaluation. But this is possible only if both  $u$  and  $1_{A_k}$  belong to the same simple component. This proves the claim.

But we need more: Consider the radical evaluations which are bordered by pairs of elements  $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$  where  $j(i) \neq \tilde{j}(i)$  (i.e. belong to different simple components).

**Claim 7.10.** *Every simple component is represented by one of the elements in these pairs.*

Again, assume that  $A_1$  is not represented among these pairs. By the preceding claim  $A_1$  is represented in some pair and so it must be represented by both partners in each pair it appears. Take such a pair  $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$ , where  $j(i) = \tilde{j}(i) = 1$ . Moving along the monomial  $X$  to the left of  $1_{A_{j(i)}}$  or to the right  $1_{A_{\tilde{j}(i)}}$  we will hit a value in a different simple component. But before that we must hit a radical evaluation which is bordered by a pair where one of the partners is from  $A_1$  and the other from a different simple component. This contradicts our assumption and hence the claim is proved.

For  $t = 1, \dots, q$  we fix a variable  $w_{r_t}$  whose radical value is  $1_{A_{j(r_t)}} \widehat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$  where

- (1)  $j(r_t) \neq \tilde{j}(r_t)$  (i.e. different simple components).
- (2) One of the element  $1_{A_{j(r_t)}}, 1_{A_{\tilde{j}(r_t)}}$  is the identity element of the  $t$ -th simple component. We refer to that element as the *idempotent attached* to the simple component  $A_t$ .

**Remark 7.11.** Note that we may have  $w_{r_t} = w_{r_{t'}}$  even if  $t \neq t'$ .

Next by the  $T$  operation we replace the variables  $w_{r_t}$ ,  $t = 1, \dots, q$ , by  $y_{r_t} w_{r_t}$  or  $w_{r_t} \tilde{y}_{r_t}$  according to the location of the primitive idempotent attached to the  $t$ -th simple component. Clearly, by evaluating the variable  $y_{r_t}$  by  $1_{A_{j(r_t)}}$  (or the variable  $\tilde{y}_{r_t}$  by  $1_{A_{\tilde{j}(r_t)}}$ ) the value of the polynomial remains the same and in particular nonzero.

Applying Lemma 6.1 we can replace the variable  $y_{r_t}$ ,  $t = 1, \dots, q$ , by a  $\nu$ -fold alternating polynomial (on the sets  $U_l^t$ )  $Z_{r_t} = Z_{r_t}(U_1^t, \dots, U_\nu^t; Y_t)$ , (or replace  $\tilde{y}_{r_t}$  by a  $\nu$ -fold alternating polynomial  $\tilde{Z}_{r_t}$ ). Here, the sets  $U_l^t$ ,  $l = 1, \dots, \nu$  are each of cardinality  $\dim_F(A_t)$ . Now, if we further alternate the sets  $U_l^1, \dots, U_l^q$  for  $l = 1, \dots, \nu$ , we obtain a nonidentity polynomial with  $\nu$ -folds of (small) sets of alternating variables where each set is of cardinality  $\dim_F(\bar{A})$ . In the sequel we fix an evaluation of the polynomials  $Z_{r_t}$  (or  $\tilde{Z}_{r_t}$ ) so the entire polynomial obtains a nonzero value. As in Kemer’s Lemma 1, also here the alternating variables  $U_l^t$  are bordered by variables (called *frames*) whose values are primitive idempotents in  $A_t$ .

Our next task is to construct such polynomial with an extra  $n_A - 1$  alternating sets of cardinality  $d + 1$  (big sets). Consider the radical variables  $w_{r_t}$ ,  $t = 1, \dots, q$  with radical evaluations  $1_{A_{j(r_t)}} \widehat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$ ,  $j(r_t) \neq \tilde{j}(r_t)$  (i.e. different simple components).

We attach each variable  $w_{r_t}$  to one alternating set  $U_l^1, \dots, U_l^q$  (some  $l$ ). We see that any nontrivial permutation of  $w_{r_t}$  with one of the variables of  $U_l^1, \dots, U_l^q$ , keeping the evaluation above, will yield a *zero value* since the primitive idempotents values in frames variables of each variable of  $U_l^1, \dots, U_l^q$  belong to the same simple components whereas the pair of idempotents in  $1_{A_{j(r_t)}} \widehat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$  belong to different simple components. Thus we may alternate the variable  $w_{r_t}$  with  $U_l^1, \dots, U_l^q$ ,  $t = 1, \dots, q$  and obtain a multilinear nonidentity of  $A$ . Next we proceed in a similar way with any variable  $w_i$  whose evaluation is  $1_{A_{j(i)}} \widehat{w}_i 1_{A_{\tilde{j}(i)}}$  and  $j(i) \neq \tilde{j}(i)$ .

Finally we need to attach the radical variables  $w_i$  whose evaluation is  $1_{A_{j(i)}} \widehat{w}_i 1_{A_{\tilde{j}(i)}}$  where  $j(i) = \tilde{j}(i)$  (i.e. the same simple component) to some small set. We claim also here that if we attach the variable  $w_i$  to the sets  $U_l^1, \dots, U_l^q$  (some  $l$ ), any nontrivial permutation yields a zero value, and hence the value of the entire polynomial remains unchanged. If we permute  $w_i$  with an element  $u_0 \in U_l^k$  which is bordered by idempotents different to  $1_{A_{j(i)}}$  we obtain zero. On the other hand we claim that any permutation of  $w_i$  with an element  $u_0 \in U_l^k$  which is bordered by the idempotent  $1_{A_{j(i)}}$  corresponds to an evaluation of the original



polynomial with fewer radical values and then we will be done by the property  $K$ . In order to simplify our notation let  $\{U_l^1, \dots, U_l^q\} = \{U^1, \dots, U^q\}$  (omit the index  $l$ ) and suppose without loss of generality, that  $u_0 \in U^1$ . Permuting the variables  $w_i$  and  $u_0$  (with their corresponding evaluations) we see that the polynomial  $Z_{r_1} = Z_{r_1}(U^1 = U_1^1, \dots, U_\nu^t; Y_t)$  (or  $\tilde{Z}_{r_1}$ ) with  $w_i$  replacing  $u_0$ , obtains a radical value which we denote by  $\widehat{w}$ . Returning to our original polynomial, we obtain the same value if we evaluate the variable  $w_i$  by a suitable semisimple element, the variable  $w_{r_1}$  by  $\widehat{w}\widehat{w}_{r_1}$  (or  $\widehat{w}_{r_1}\widehat{w}$ ) and the evaluation of any semisimple variable remains semisimple. It follows that if we make such a permutation for a unique radical variable  $w_i$ , the value amounts to an evaluation of the original polynomial with  $n_A - 2$  radical evaluations and hence vanishes. Clearly, composing  $p > 0$  permutations of that kind yields a value which may be obtained by the original polynomial  $f$  with  $n_A - 1 - p$  radical evaluations and hence vanishes by property  $K$ . This completes the proof of the lemma where  $A$  has identity and  $q$ , the number of simple components, is  $> 1$ .

**Case (1, 2)** ( $A$  has an identity element and  $q = 1$ ). We start with a nonidentity  $f$  which satisfies property  $K$ . Clearly we may multiply  $f$  by a variable  $x$  and get a nonidentity (since  $x$  may be evaluated by 1). Again by Lemma 6.1 we may replace  $x$  by a polynomial  $h$  with  $\nu$ -folds of alternating sets of cardinality  $d$ . Consider the polynomial  $hf$ . We attach the radical variables of  $f$  to some of the small sets in  $h$ . Any nontrivial permutation vanishes because  $f$  satisfies property  $K$ . This completes the proof of the Lemma 7.8 in case  $A$  has an identity element.

**Case (2, 1)**. Suppose now  $A$  has no identity element and  $q > 1$ . The proof in this case is basically the same as in the case where  $A$  has an identity element. Let  $e_0 = 1 - 1_{A_1} - 1_{A_2} - \dots - 1_{A_q}$  ( $1 \in F$ ) and attach  $e_0$  to the set of elements which border the radical values  $\widehat{w}_j$ . Of course  $e_0$  is not an element of  $A$  but the product  $e_0a$ ,  $a \in A$  is well defined. A similar argument shows that also here every simple component  $(A_1, \dots, A_q)$  is represented in one of the bordering pairs where the partners are *different* (the point is that one of the partners (among these pairs) may be  $e_0$ ). Now we complete the proof exactly as in case (1, 1).

**Case (2, 2)**. In order to complete the proof of the lemma we consider the case where  $A$  has no identity element and  $q = 1$ . The argument in this case is different. For simplicity we denote by  $e_1 = 1_{A_1}$  and  $e_0 = 1 - e_1$ . Let  $f(x_1, \dots, x_n)$  be a nonidentity of  $A$  which satisfies property  $K$  and let  $f(\widehat{x}_1, \dots, \widehat{x}_n)$  be a nonzero evaluation for which  $A$  is full. If  $e_1f(\widehat{x}_1, \dots, \widehat{x}_n) \neq 0$  (or  $f(\widehat{x}_1, \dots, \widehat{x}_n)e_1$ ) we proceed as in case (1, 2). To treat the remaining case we may assume further that

$$e_0f(\widehat{x}_1, \dots, \widehat{x}_n)e_0 \neq 0.$$

First note, by linearity, that each one of the radical values  $\widehat{w}$  may be bordered by one of the pairs  $\{(e_0, e_0), (e_0, e_1), (e_1, e_0), (e_1, e_1)\}$  so that if we replace the evaluation  $\widehat{w}$  (of  $w$ ) by the corresponding element  $e_i\widehat{w}e_j$ ,  $i, j = 0, 1$ , we get nonzero.

Now, if one of the radical values (say  $\widehat{w}_0$ ) in  $f(\widehat{x}_1, \dots, \widehat{x}_n)$  allows a bordering by the pair  $(e_0, e_1)$  (and remains nonzero), then replacing  $w_0$  by  $w_0y$  yields a nonidentity (since we may evaluate  $y$  by  $e_1$ ). Invoking Lemma 6.1 we may replace the variable  $y$  by a polynomial  $h$  with  $\nu$ -folds of alternating (small) sets of cardinality  $\dim_F(\overline{A}) = \dim_F(A_1)$ . Then we attach the radical variable  $w_0$  to a suitable small set. Clearly, the value of any nontrivial permutation of  $w_0$  with any element of the small set is zero since the borderings are different. Similarly, attaching radical variables  $w$  whose radical value is  $e_i\widehat{w}e_j$  where  $i \neq j$ , to small sets yields zero for any nontrivial permutation and hence the value of the polynomial remains nonzero. The remaining possible values of radical variables are either  $e_0\widehat{w}e_0$  or  $e_1\widehat{w}e_1$ . Note that since semisimple values can be bordered only by the pair  $(e_1, e_1)$ , any alternation of the radical variables whose radical value is  $e_0\widehat{w}e_0$  with elements of a small set vanishes and again the value of the polynomial remains unchanged. Finally (in order to complete this case, namely where one of the radical variables, say  $w_0$ , is bordered by the pair  $(e_0, e_1)$ ) we attach the remaining radical variables (whose values are bordered by  $(e_1, e_1)$ ) to suitable small sets in  $h$ . Here, the value of any nontrivial permutation of  $w_0$  with elements of the small set is zero because of property  $K$  (as in case (1, 1)). This settles the case where the bordering pair of  $\widehat{w}_0$  is  $(e_0, e_1)$ .

Obviously, the same holds if the bordering pair of  $\widehat{w}_0$  above is  $(e_1, e_0)$ . The outcome is that we may assume that all radical values may be bordered by either  $(e_0, e_0)$  or  $(e_1, e_1)$ .

**Claim 7.12.** *Under the above assumption, all pairs that border radical values are equal, that is are all  $(e_0, e_0)$  or all  $(e_1, e_1)$ .*

Indeed, if we have of both kinds, we must have a radical value which is bordered by a mixed pair since the semisimple variables can be bordered only by the pair  $(e_1, e_1)$  (and in particular they cannot be bordered by mixed pairs). This of course contradicts our assumption.

A similar argument shows that we cannot have radical variables  $w$  with values  $e_0\widehat{w}e_0$  since again, semisimple values can be bordered only by  $(e_1, e_1)$  and this will force the existence of a radical value bordered by mixed idempotents (we remind the reader that here  $q = 1$  and  $A$  is full with respect the evaluation  $f(\widehat{x}_1, \dots, \widehat{x}_n)$ ).

The remaining case is the case where all values (radical and semisimple) are bordered by the pair  $(e_1, e_1)$  and this contradicts the assumption  $e_0f(\widehat{x}_1, \dots, \widehat{x}_n)e_0 \neq 0$ . This completes the proof of the Lemma.  $\square$

We are now ready to prove the following important proposition.

**Proposition 7.13.** *The following conditions are equivalent for a finite dimensional algebra  $A$ .*

- (1)  $A$  is basic.
- (2)  $A$  is full and satisfies property  $K$ .
- (3)  $\text{Ind}(A) = \text{Par}(A)$ .

**Proof.** If  $A$  is basic we know that  $A$  is full and satisfies property  $K$  ([Corollary 5.13](#), [Proposition 7.4](#)).

Kemer’s Lemma 2 shows that if  $A$  is full and satisfies property  $K$  there exist nonidentities of  $A$  with arbitrary many alternating sets of cardinality  $d(A)$  and precisely  $n_A - 1$  alternating sets of cardinality  $d(A) + 1$ . This shows that  $\text{Ind}(A) = \text{Par}(A)$ . In order to complete the proof of the proposition we show that if a finite dimensional algebra  $A$  admits a multilinear polynomial  $f$  with  $\mu$  alternating sets of cardinality  $d(A)$  and precisely  $n_A - 1$  alternating sets of cardinality  $d(A) + 1$  then  $A$  is basic. Suppose not. Then  $A$  is PI equivalent to an algebra  $B = B_1 \times \dots \times B_m$  where  $\text{Par}(B_i) < \text{Par}(A)$ ,  $i = 1, \dots, m$ . But this implies that  $f \in \text{Id}(B_i)$ ,  $i = 1, \dots, m$  and hence  $f \in \text{Id}(B)$ . Contradiction.  $\square$

We close this section by establishing the Phoenix property of Kemer polynomials (see [Definition 5.1](#)). This seemingly “unimportant” property is in fact key for the entire proof of the representability theorem.

**Lemma 7.14.** *Let  $A$  be a finite dimensional algebra over  $F$  and suppose  $A$  is basic. The following hold.*

- (1) *Let  $f \notin \text{Id}(A)$  be a multilinear polynomial and suppose  $A$  is full with respect to any nonzero evaluation of  $f$  on  $A$ , that is, in any nonzero evaluation of  $f$  on  $A$  we must have semisimple values from all simple components. Then if  $f' \in \langle f \rangle$  is multilinear ( $\langle f \rangle = T$ -ideal generated by  $f$ ) is a nonidentity of  $A$  then it is full with respect to any nonzero evaluation on  $A$ .*
- (2) *Let  $f \notin \text{Id}(A)$  be multilinear and suppose  $f$  vanishes on any evaluation on  $A$  with less than  $n_A - 1$  radical evaluations. Then if  $f' \in \langle f \rangle$  is multilinear (and nonidentity of  $A$ ) then it vanishes on any evaluation on  $A$  with less than  $n_A - 1$  radical evaluations.*

**Proof.** Suppose  $f(x_1, \dots, x_n)$  is a multilinear polynomial which satisfies the condition in (1). It is sufficient to show the condition remains valid if  $f'$  is multilinear and has the form (a)  $f' = \sum_i g_i \cdot f \cdot h_i$  (b)  $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$  where  $Z = z_1 \dots z_t$  is a multilinear monomial consisting of

variables disjoint to the variables of  $f(x_1, \dots, x_n)$ . If  $f' = \sum_i g_i \cdot f \cdot h_i$  then any nonzero evaluation of  $f'$  arises from a nonzero evaluation of  $f$  and so the claim is clear in this case. Let  $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$  and suppose  $x_i = \hat{x}_i$  and  $z_i = \hat{z}_i$  is a nonvanishing evaluation of  $f'$ . If a simple component  $A_1$  say, is not represented, then the same simple component is not represented in the evaluation  $x_1 = \hat{z}_1 \cdots \hat{z}_t, x_2 = \hat{x}_2, \dots, x_n = \hat{x}_n$  and hence  $f$  vanishes. We see that  $f'$  vanishes on any evaluation which misses a simple component.

We now turn to the proof of the 2nd part of the lemma. If  $f' = \sum_i g_i \cdot f \cdot h_i$  then it is clear that if an evaluation of  $f'$  has less than  $n_A - 1$  radical evaluations then with that evaluation  $f$  has less than  $n_A - 1$  radical evaluations and hence vanishes. This implies the vanishing of  $f'$ . If an evaluation of  $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$  has less than  $n_A - 1$  radicals, then this corresponds to an evaluation of  $f(x_1, \dots, x_n)$  with less than  $n_A - 1$  radicals and hence vanishes.  $\square$

**Corollary 7.15.** *Let  $A$  be a basic algebra, then Kemer polynomials satisfy the Phoenix property.*

**Proof.** Let  $f$  be a Kemer polynomial of  $A$  and let  $f' \in \langle f \rangle$  be a multilinear nonidentity of  $A$ . We need to show there exists a Kemer polynomial in  $\langle f' \rangle$ .

We know that  $A$  is full and satisfies property  $K$  with respect to the Kemer polynomial  $f$ . Furthermore, applying the previous lemma, we have that  $A$  is full and has property  $K$  with respect to  $f'$ . Finally, invoking Kemer’s Lemma 2 there exists  $f'' \in \langle f' \rangle$  which is Kemer.  $\square$

**8. Finite generation of the relatively free algebra**

Recall that an algebra  $W$  satisfies the  $m$ th Capelli identity if any multilinear polynomial having an alternating set of cardinality (at least)  $m$  is an identity of  $W$ . The purpose of this section is to prove that for any such algebra one can assume that the corresponding relatively free algebra  $\mathcal{W}$  is generated by (only)  $m - 1$  variables. More precisely, we will show that if

$$\mathcal{W} = F \langle x_1, \dots, x_{m-1} \rangle / \text{Id}(\mathcal{W}) \cap F \langle x_1, \dots, x_{m-1} \rangle$$

then  $\text{Id}(\mathcal{W}) = \text{Id}(W)$ . To this end we recall some basic results (and fix notation) from the representation theory of  $S_n$  (the symmetric group on  $n$  elements) and their application to PI theory.

Since  $FS_n$  is a semisimple algebra, we can write  $FS_n$  as a direct sum of its minimal two sided ideals. It is a basic fact that the number of such is equal to the number of conjugacy classes of  $S_n$ . Each conjugacy class can be described by a partition  $\mu$  of  $n$ , i.e. a finite sequence of nondecreasing natural numbers which sum up to  $n$ . As we shall see, each partition encodes the structure of some minimal ideal.

Suppose  $\mu = (\mu_1, \dots, \mu_m)$  is a partition of  $n$ . A Young diagram of  $\mu$  is a finite subset of  $\mathbb{Z} \times \mathbb{Z}$  defined as  $D_\mu = \{(i, j) \mid i = 1 \dots m, j = 1, \dots, \mu_i\}$ . This is not more than  $m$  rows of boxes, such that the length of the  $i$ th row is  $\mu_i$ . A Young tableau  $T_\mu$  associated to  $\mu$  is a filling of the boxes of  $D_\mu$  with the integers  $1, \dots, n$  (repetitions are not allowed). We say that a tableau  $T_\mu$  is *standard* if the numbers in each row and column of  $T_\mu$  are increasing from left to right and from up to bottom. To each tableau we associate two subgroups of  $S_n$  as follows: Let  $R_{T_\mu(1)}, \dots, R_{T_\mu(m)}$  denote the rows of  $T_\mu$  (i.e. the numbers appearing in each row) and  $C_{T_\mu(1)}, \dots, C_{T_\mu(t)}$  denote the columns of  $T_\mu$ . We denote by  $\mathcal{R}_{T_\mu(i)}$  and by  $\mathcal{C}_{T_\mu(j)}$  the symmetric groups  $S_{R_{T_\mu(i)}}$  and  $S_{C_{T_\mu(j)}}$  respectively (i.e. the symmetric groups acting on the numbers in the  $i$ th row and  $j$ th column of  $T_\mu$  respectively). Finally, we denote by  $\mathcal{R}_{T_\mu}$  and by  $\mathcal{C}_{T_\mu}$  the subgroups of  $S_n$  which are the row and column stabilizers of  $T_\mu$ . Clearly, with notation above, we have  $\mathcal{R}_{T_\mu} = \mathcal{R}_{T_\mu(1)} \times \cdots \times \mathcal{R}_{T_\mu(m)}$  and  $\mathcal{C}_{T_\mu} = \mathcal{C}_{T_\mu(1)} \times \cdots \times \mathcal{C}_{T_\mu(t)}$ .

For each tableau  $T_\mu$ , consider the left ideal  $V_\mu = FS_n e_{T_\mu}$ , where

$$e_{T_\mu} = \sum_{\sigma \in \mathcal{R}_{T_\mu}, \tau \in \mathcal{C}_{T_\mu}} (-1)^\tau \sigma \tau.$$

Now let  $I$  be a minimal 2-sided ideal of  $FS_n$ . It is well known that  $I$  is equal to the sum of all minimal left ideals isomorphic to some (fixed) minimal left ideal  $V$ . The following results are classical (see [10]).

**Theorem 8.1.** *The following hold:*

- (1) *Let  $V$  be a minimal left ideal of  $FS_n$ . Then there exists a partition  $\mu$  such that  $V \cong V_\mu$ .*
- (2) *If we denote the corresponding minimal two sided ideal by  $I_\mu$ , then it is the direct sum of minimal ideals  $V_\mu$  which correspond to standard tableaux  $T_\mu$  of  $\mu$ .*
- (3) *The map  $T_\mu \mapsto V_\mu$  establishes a one to one correspondence between Young tableaux associated to Young diagram  $D_\mu$  and minimal left ideal isomorphic to  $V_\mu$ .*

Now let us apply the theorem above to PI theory. Let  $P_n(W) = P_n / (P_n \cap \text{Id}(W))$ , where  $P_n$  is the space (of  $\dim_F(P_n) = n!$ ) of all multilinear polynomials with variables  $x_1, \dots, x_n$ . The group  $S_n$  acts on  $P_n(W)$  via  $\sigma \cdot \overline{x_{i_1} \cdots x_{i_n}} = \overline{x_{\sigma(i_1)} \cdots x_{\sigma(i_n)}}$  and hence we may consider its decomposition into irreducible submodules. By the theorem above, any such submodule can be written as  $FS_n e_{T_\mu} \cdot f$ , where  $f$  is some polynomial in  $P_n(W)$ . Clearly, if  $f \in P_n(W)$  is nonzero, then there is some partition  $\mu$  and a (standard) tableau  $T_\mu$  such that  $e_{T_\mu} \cdot f$  is nonzero.

We are ready to prove the main result of this section.

**Theorem 8.2.** *Let  $W$  be an algebra which satisfies the  $m$ th Capelli identity. Then  $\text{Id}(\mathcal{W}) = \text{Id}(W)$  where  $\mathcal{W}$  is the relatively free algebra of  $W$  generated by  $m - 1$  variables.*

**Proof.** It is clear that  $\text{Id}(W) \subset \text{Id}(\mathcal{W})$ . For the other direction suppose  $f$  is a multilinear nonidentity of  $W$  of degree  $n$ . Then, by the theorem above, there is a partition  $\mu$  of  $n$  and a tableau  $T_\mu$  such that  $g = e_{T_\mu} \cdot f$  is a nonidentity of  $W$ .

Let  $g_0 = \sum_{\tau \in \mathcal{C}_{T_\mu}} (-1)^\tau \tau \cdot f = \sum_{\tau \in \mathcal{C}_{T_\mu(1)}} (-1)^\tau \tau \cdot \left( \sum_{k=1}^l (-1)^{\tau_k} \tau_k \cdot f \right)$ , where  $\tau_1, \dots, \tau_l$  is a full set of representatives of  $\mathcal{C}_{T_\mu(1)}$ -cosets in  $\mathcal{C}_{T_\mu}$ .

Let  $h_\mu$  (the height of  $\mu$ ) denote the number of rows in the Young diagram  $D_\mu$ . If  $h_\mu \geq m$ , the polynomial  $g_0$  is alternating on the variables of the first column and hence by assumption is an identity of  $W$ . But in that case also the polynomial  $g = \sum_{\sigma \in \mathcal{R}_{T_\mu}} \sigma \cdot g_0$  is in  $\text{Id}(W)$  contradicting our assumption and so  $h_\mu$  must be smaller than  $m$ .

Let us now focus on the rows of  $D_\mu$ . Since  $g = \sum_{\sigma \in \mathcal{R}_{T_\mu}} \sigma \cdot g_0$ , it is symmetric in the variables corresponding to any row of  $T_\mu$  and so if for any  $i = 1, \dots, h_\mu$  we replace by  $y_i$  all variables in  $g$  corresponding to the  $i$ th row we obtain a polynomial  $\hat{g}$  which yields  $g$  by multilinearization. In particular  $g \in \text{Id}(W)$  if and only if  $\hat{g} \in \text{Id}(W)$ . In order to conclude, note that  $\hat{g}$  can be regarded as an element of  $\mathcal{W}$  (at most  $m - 1$  variables) and nonzero. This shows that  $g$  is nonidentity of  $\mathcal{W}$  and hence also  $f$ . This proves the theorem.  $\square$

In the sequel, if  $W$  satisfies the  $m$ th Capelli identity, we will consider affine relatively free algebras  $\mathcal{W}$  with at least  $m - 1$  generating variables.

**Definition 8.3.** Suppose  $W$  is an affine algebra. Any algebra of the form

$$F \langle x_1, \dots, x_m \rangle / \text{Id}(W) \cap F \langle x_1, \dots, x_m \rangle$$

having the same  $T$ -ideal as  $W$  is called *affine relatively free algebra of  $W$* .

We close this section with the following useful lemma.

**Lemma 8.4.** *Let  $W$  be a PI algebra which satisfies the  $m$ th Capelli polynomial and let  $\mathcal{W} = F\langle x_1, \dots, x_n \rangle / \widehat{\text{Id}(W)}$  be an affine relatively free algebra where  $n \geq m - 1$ . Let  $I$  be any  $T$ -ideal and denote by  $\widehat{I}$  the ideal of  $\mathcal{W}$  generated by all evaluations on  $\mathcal{W}$  of elements of  $I$ . Then  $\text{Id}(\mathcal{W}/\widehat{I}) = \text{Id}(W) + I$ .*

Note that in fact  $\widehat{I}$  consists of all evaluations of elements of  $I$  on  $\mathcal{W}$ .

**Proof.** Clearly,  $\text{Id}(\mathcal{W}/\widehat{I}) \supseteq \text{Id}(W)$  and  $\text{Id}(\mathcal{W}/\widehat{I}) \supseteq I$  so  $\text{Id}(\mathcal{W}/\widehat{I})$  contains  $\text{Id}(W) + I$ . For the converse note that

$$\mathcal{W}/\widehat{I} = (F\langle x_1, \dots, x_n \rangle) / \widehat{\text{Id}(W)} / \widehat{I} = F\langle x_1, \dots, x_n \rangle / (\widehat{\text{Id}(W)} + \widehat{I}) = F\langle x_1, \dots, x_n \rangle / \widehat{\text{Id}(W) + I}.$$

Then, since  $F\langle x_1, \dots, x_n \rangle / \widehat{\text{Id}(W)} + I$  satisfies the  $n$ th Capelli identity we obtain

$$\text{Id}(F\langle x_1, \dots, x_n \rangle / \widehat{\text{Id}(W)} + I) = \text{Id}(W) + I$$

as desired.  $\square$

### 9. Shirshov base

**Definition 9.1.** Let  $W$  be an affine algebra over  $F$ . Let  $a_1, \dots, a_s$  be a generating set of  $W$ . Let  $m$  be a positive integer and let  $Y$  be the set of words in  $a_1, \dots, a_s$  of length  $\leq m$ . We say that  $W$  has a Shirshov base of length  $m$  and of height  $h$  if  $W$  is spanned (over  $F$ ) by elements of the form  $y_1^{n_1} \cdots y_l^{n_l}$ , where  $y_i \in Y$  and  $l \leq h$ .

The following fundamental theorem was proved by Shirshov.

**Theorem 9.2.** (See [11].) *If an affine algebra  $W$  has a multilinear PI of degree  $m$ , then it has a Shirshov base of length  $m$  and some height  $h$  where  $h$  depends only on  $m$  and the number of generators of  $W$ .*

In fact, there is an important special case where we can get even “closer” to representability.

**Lemma 9.3.** *Let  $C$  be a commutative algebra over  $F$  and let  $W = C\langle a_1, \dots, a_s \rangle$ . Suppose  $W$  has a Shirshov base. If for every  $i = 1, \dots, s$ , the element  $a_i$  is integral over  $C$ , then  $W$  is a finitely generated  $C$ -module.*

If in addition, our commutative algebra  $C$  is Noetherian and unital we reach our goal, as the next theorem shows.

**Theorem 9.4.** (See Beidar [2].) *Let  $W$  be an  $F$  algebra and let  $C$  be a unital commutative Noetherian  $F$  algebra. If  $W$  is a finite module over  $C$ , then  $W$  is representable (See def. in Section 2, Step 7).*

**Proof.** The simplest case is when  $C$  is a local Artinian ring and  $W$  is of finite length (over  $C$ ). Let  $P$  be the unique maximal ideal of  $C$ . Since  $C$  is Artinian we know that  $P$  is nilpotent, thus  $C$  is complete in the  $P$ -adic metric. Thus  $C$  contains a field  $(F \subset)K$  such that  $K + P = C$ , and so  $K$  is isomorphic to  $C/P$ . Let  $W = W_{(0)} \supseteq W_{(1)} \supseteq \cdots \supseteq W_{(m)} = 0$  be a composition series of  $W$  (as a  $C$  module). We have that  $W_{(i)}/W_{(i+1)}$  is isomorphic to  $C/P$ , so it is a one dimensional  $K$ -space. It follows that  $W$  is a finite dimensional  $K$ -space.

Now assume that  $W$  has finite length as a  $C$  module (without any other assumptions on the ring  $C$ ). This yields that  $C' = C/\text{ann}_C(W)$  is Artinian. Therefore, we may decompose  $C'$  into a (finite) direct product of local Artinian rings  $C' = \times_{i=1}^m C_i$ . This decomposition induces a decomposition of  $W$ , namely  $W = \oplus_{i=1}^m C_i W$ , and since each  $C_i W$  is a  $C_i$  module of finite length, we are back to the first case (note that here we make use of the fact that a finite number of fields with the same characteristic can be embedded in a larger field).

We proceed now to the general case. Recall that the set of associated primes  $\text{Ass}_C(W)$  is a finite set. Let  $P_1, \dots, P_k$  be its maximal elements and let  $S = C - \cup_{i=1}^k P_i$  (a multiplicative set). Since  $S$  is the set of nonzero divisors of  $W$ , the localization by  $S$ ,  $C_0 = S^{-1}C$ , induces an embedding of  $W$  in  $W_0 = S^{-1}W$ . Moreover, the maximal ideals of  $C_0$  are  $S^{-1}P_1, \dots, S^{-1}P_k$  and  $\text{Ass}_{C_0}(W_0) = \{S^{-1}P \mid P \in \text{Ass}_C(W)\}$  (the last equality is due to the maximality of the  $P_i$ 's in  $\text{Ass}_C(W)$ ). Thus we may assume from the beginning that  $C$  is a semilocal ring whose maximal ideals are contained in  $\text{Ass}_C(W)$ .

We continue by induction on  $d_C(W)$ , where  $d_C(W)$  represents the greatest length of a descending chain of prime ideals from  $\text{Ass}_C(W)$ . If  $d_C(W) = 1$ , then all the ideals of  $\text{Ass}_C(W)$  are the maximal ideals of  $C$ . Since  $\text{Ass}_C(W)$  contains all minimal primes containing  $\text{ann}_C(W)$ , we conclude that all primes which contain  $\text{ann}_C(W)$  are maximal. Hence  $W$  is of finite length and we are done.

Suppose now that  $d_C(W) > 1$ . Let  $Q_1, \dots, Q_l$  be all maximal elements in the set  $H = \text{Ass}_C(W) - \{P_1, \dots, P_k\}$  and denote  $T = C - \cup Q_i$ . Write  $C_1 = T^{-1}C$ ,  $W_1 = T^{-1}W$  and let  $U$  be the kernel of the canonical map  $W \rightarrow W_1$ . It is easy to calculate the associated primes:

$$\text{Ass}_{C_1}(W_1) = \{T^{-1}P \mid P \in H\}, \text{Ass}_C(U) = \{P_1, \dots, P_k\}.$$

Since  $d_{C_1}(W_1) < d_C(W)$ , we know by the induction hypothesis that  $W_1$  is representable.

There is  $r > 0$  for which  $J^r U = 0$ , where  $J = \cap P_i$  is the Jacobson radical of  $C$ . Using the Artin–Rees lemma we obtain some  $r'$  for which  $J^{r'} W \cap U = 0$ . Thus,  $W$  is a sub direct product of  $W/U$  and  $W/J^{r'} W$ . Finally, these two algebras are representable:  $W/U$  – since it is contained in  $W_1$ , and  $W/J^{r'} W$  – since it is of finite length (recall that there is a product of primes which annihilate it).  $\square$

### 10. The trace ring

Suppose  $A$  is an  $M$ -dimensional  $F$ -algebra and let  $\mathcal{A}$  be an affine relatively free algebra, say generated by the set  $\{x_1 + \text{Id}(A), \dots, x_v + \text{Id}(A)\}$ , where  $x_1, \dots, x_v$  are noncommuting variables.

It is well known that  $\mathcal{A}$  may be interpreted as an algebra of “generic elements”. Let us recall briefly the construction.

Let  $K = F(\{t_{i,j} \mid i = 1, \dots, v ; j = 1, \dots, M\})$  and suppose  $a_1, \dots, a_M$  is an  $F$ -basis of  $A$ . Consider the  $F$ -subalgebra  $\mathcal{A}'$  of  $A_K = A \otimes_F K$  generated by

$$y_i = \sum_{j=1}^M t_{i,j} a_j, i = 1, \dots, v.$$

It is well known that the map  $\phi : \mathcal{A}' \rightarrow \mathcal{A}$  determined by  $\phi(y_i) = x_i + \text{Id}(A)$ , is an  $F$ -algebra isomorphism. Henceforth we will not distinguish between  $\mathcal{A}$  and  $\mathcal{A}'$  and we denote both by  $\mathcal{A}$ . It follows from the construction of the algebra of generic elements that  $\mathcal{A}$  is representable and so we fix for the rest of the paper an embedding  $\mathcal{A} \subseteq A_K$ .

Suppose  $A = A_1 \times \dots \times A_k$ , where  $A_r$  is finite dimensional over  $F$ , for  $r = 1, \dots, k$ . Fix a  $K$ -vector space decomposition  $(A_r)_K = (A_r)_K^{ss} \oplus (J_r)_K$  where  $(A_r)_K^{ss}$  is a maximal semisimple subalgebra of  $(A_r)_K$  and  $(J_r)_K$  is its Jacobson radical. Consider the embedding

$$(A_r)_K^{ss} \hookrightarrow \text{End}_K((A_r)_K^{ss})$$

given by  $a(x) = ax(a, x \in (A_r)_{K}^{ss})$ . Let  $\hat{\mathcal{A}}$  be the  $F$ -subalgebra of  $A_K$  generated by the projections of  $\mathcal{A}$  into  $A_K^{ss} = (A_1)_{K}^{ss} \times \cdots \times (A_k)_{K}^{ss}$  and  $J_K = (J_1)_K \times \cdots \times (J_k)_K$ . It is clear that  $\hat{\mathcal{A}}$  is an affine  $F$ -algebra containing (the image of)  $\mathcal{A}$ . Moreover, we may choose generators of  $\hat{\mathcal{A}}$  such that the corresponding Shirshov base is  $Y = Y_{ss} \sqcup Y_J$ , where  $Y_{ss} \subset A_K^{ss}$  and  $Y_J \subset J_K$  (indeed, choosing generators  $b_1, \dots, b_s$  of  $\hat{\mathcal{A}}$  either from  $A_K^{ss}$  or  $J_K$ , then  $b_{i_1} b_{i_2} \cdots b_{i_t} \in Y_{ss}$ ,  $t \leq m$ , if and only if  $b_{i_j} \in A_K^{ss}$ , for all  $j = 1, \dots, t$ ). Since  $J_K$  is nilpotent, it is clear that the elements of  $Y_J$  are integral over  $F$ , however this may not be the case for elements in  $Y_{ss}$ . Our goal is to extend  $F$  in a suitable way so that the elements of  $Y_{ss}$  “become” integral. Consider the elements of  $Y_{ss}$  as elements of  $\text{End}_K((A_1)_{K}^{ss}) \times \cdots \times \text{End}_K((A_k)_{K}^{ss})$  (via the above embedding) and define  $R$  to be the unital  $F$ -subalgebra of  $K \times \cdots \times K$  ( $k$  times) generated by  $\text{Tr}(Y_{ss}) = \{(\text{Tr}(p_1(u)), \dots, \text{Tr}(p_k(u)) : u \in Y_{ss})\}$ , where  $p_r$  is the projection  $\text{End}_K((A_1)_{K}^{ss}) \times \cdots \times \text{End}_K((A_k)_{K}^{ss}) \rightarrow \text{End}_K((A_r)_{K}^{ss})$ . Since  $Y_{ss}$  is a finite set, it is clear that  $R$  is a commutative unital Noetherian  $F$ -algebra.

Let us define an action of  $K \times \cdots \times K$  ( $k$ -times) (and hence also of  $R$ ) on  $A_K = (A_1)_K \times \cdots \times (A_k)_K$  by

$$(c_1, \dots, c_k) \cdot (b_1, \dots, b_k) = (c_1 b_1, \dots, c_k b_k),$$

where  $(c_1, \dots, c_k) \in K^{(k)}$  and  $b_1 \in (A_1)_K, \dots, b_k \in (A_k)_K$ .

Let  $d_r = \dim_F A_r^{ss}$  and denote by  $d = \max\{d_r : r = 1, \dots, k\}$ .

**Theorem 10.1.**  $\mathcal{A}_R = R \cdot \mathcal{A}$  is a finite module over  $R$ .

**Proof.** The Cayley–Hamilton theorem implies that for  $a \in Y_{ss}$

$$p_r(a^{d+1}) + \sum_{i=0}^{d-1} q_i (\text{Tr}(p_r(a^1)), \dots, \text{Tr}(p_r(a^d))) p_r(a^{i+1}) = 0,$$

where  $q_0, \dots, q_{d-1}$  are polynomials on commutative variables.

It follows that

$$a^{d+1} + \sum_{i=0}^{d-1} q_i (\text{Tr}(a^1), \dots, \text{Tr}(a^d)) a^{i+1} = 0$$

proving that  $Y_{ss}$ , and hence also  $Y$ , is integral over  $R$ .

**Remark 10.2.** The exponent  $d + 1$  is needed since the generic algebra  $\mathcal{A}$  has no identity.

Applying Lemma 9.3 we obtain that  $R \cdot \hat{\mathcal{A}}$  is a finite module over  $R$ . Since  $R$  is Noetherian,  $R \cdot \hat{\mathcal{A}}$  is a Noetherian  $R$ -module, and hence  $\mathcal{A}_R \subseteq R \cdot \hat{\mathcal{A}}$  is a finite module over  $R$ . This proves the theorem.  $\square$

**Corollary 10.3.** The algebra  $\mathcal{A}_R$  is representable. Furthermore, if  $I$  is an ideal of  $\mathcal{A}$  which is closed under multiplications by elements of  $R$ , that is  $I_R = RI = I$ , then  $\mathcal{A}/I$  is representable.

**Proof.** Since  $\mathcal{A}_R$  is a finite module over  $R$ , where  $R$  is unital, commutative and Noetherian, it is representable (by Theorem 9.4). For the same reason, if  $I$  is any ideal of  $\mathcal{A}$ , the quotient module  $\mathcal{A}_R/I_R$  (being finite) is representable. Now suppose  $I_R = I$ . Then we get  $\mathcal{A}/I \subseteq \mathcal{A}_R/I = \mathcal{A}_R/I_R$ . Since the later is representable, the result follows.  $\square$

We would like to apply the above results to the case where  $A = A_1 \times \cdots \times A_k$  is a product of basic algebras but before that we need the following basic lemma.

**Lemma 10.4.** *Let  $F\langle X \rangle$  be the free algebra over  $F$  where  $X$  is a countable set of variables. Let  $W$  be any algebra and  $S$  a set of polynomials in  $F\langle X \rangle$ . Let  $I = \langle S \rangle$ , the  $T$ -ideal generated by  $S$ . Denote by  $\mathcal{I}$  and  $\mathcal{S}$  the sets of all evaluations on  $W$  of polynomials of  $I$  and  $S$  respectively. Then  $\mathcal{I} = \langle \mathcal{S} \rangle$  (the ideal generated by  $\mathcal{S}$ ).*

**Proof.** We show first  $\mathcal{I}$  is an ideal of  $W$ . Let  $z, w \in \mathcal{I}$  and let  $p_z$  and  $p_w$  be polynomials in  $I$  with evaluations  $z$  and  $w$  respectively. By the  $T$ -property of  $I$  we may change variables and so we may assume  $p_z$  and  $p_w$  have disjoint sets of variables. Then there is an evaluation of  $p_z + p_w$  which is  $z + w$ . Next, let  $z \in \mathcal{I}$  and  $u \in W$ . If  $p_z \in I$  with value  $z$ , we may take a variable  $x$  which is not in  $p_z$  and get  $uz$  as an evaluation of  $xp_z$ .

Now, obviously  $\mathcal{I} \supseteq \langle \mathcal{S} \rangle$ . For the converse, consider the algebra  $\overline{W} = W/\langle \mathcal{S} \rangle$ . Clearly, elements of  $S$  are identities of  $\overline{W}$  and hence  $I = \langle S \rangle \subseteq \text{Id}(\overline{W})$ . It follows that all evaluations of  $I$  on  $W$  are contained in  $\langle \mathcal{S} \rangle$  as desired.  $\square$

**Proposition 10.5.** *Let  $A$  be a product of basic algebras as above and let  $I$  be the  $T$ -ideal generated by some Kemer polynomials of  $A$ . Denote by  $\hat{I}$  the ideal of  $\mathcal{A}$  obtained by all evaluation of the polynomials of  $I$  on  $\mathcal{A}$ . Then  $\hat{I}$  is closed under multiplication of  $R$ .*

**Proof.** We need to show that for any element  $f \in I$ , any evaluation  $\bar{f} \in \hat{I}$  (of  $f$  on  $\mathcal{A}$ ) and any  $a_0 \in Y_{ss}$ , we have  $\text{Tr}(a_0)\bar{f} \in \hat{I}$ . Note that since the ideal  $\hat{I}$  is generated by all evaluations on  $\mathcal{A}$  of all Kemer polynomials in  $I$ , invoking Lemma 10.4, we may assume that  $f$  is a Kemer polynomial.

Let  $(d, s)$  be the Kemer index of  $A$ . Recall that  $d = \max\{d_r\}$  where  $d_r = \dim_F(A_r^{ss})$  and if  $\Psi = \{q : d_q = d\}$  then  $s = \max_{q \in \Psi} \{n_{A_q} - 1\}$ , where  $n_{A_q}$  is the nilpotency index of  $J_q$ . Let  $f(Z, X_1, \dots, X_\mu, V_1, \dots, V_s, Q)$  be a (multilinear) Kemer polynomial of  $A$  where  $Z = \{z_1, \dots, z_d\}$ ,  $X_1, \dots, X_\mu$  are small sets,  $V_1, \dots, V_s$  are big sets (the designated variables) and  $Q$  is a set of additional variables. We assume  $\mu$  is large enough so that  $f$  is an identity of any basic algebra  $A_r$  whose Kemer index is strictly smaller than  $(d, s)$  (note that since  $f$  is a Kemer polynomial of  $A$ , it is a nonidentity of  $A_q$  for some  $q \in \Psi$  and  $n_{A_q} = s$ ). Consider an evaluation of  $f$  (on  $\mathcal{A}$ ) given by  $\hat{Z} = \{\hat{z}_1, \dots, \hat{z}_d\}$ ,  $\{\hat{X}_i\}$ ,  $\{\hat{V}_j\}$ ,  $\hat{Q}$ .

In view of the embedding

$$\mathcal{A} \subseteq \hat{\mathcal{A}} \subseteq A_K = \prod_{r=1}^k (A_r)_{K}^{ss} \oplus (J_r)_K$$

each element  $\pi \in \hat{Z} \cup (\cup_i \hat{X}_i) \cup (\cup_j \hat{V}_j) \cup \hat{Q}$  can be written as  $\pi = \pi^{ss} + \pi^J$  where  $\pi^{ss} \in A_K^{ss}$  and  $\pi^J \in J_K$ . Note that in general  $\pi^{ss}$  and  $\pi^J$  are not elements of  $\mathcal{A}$ . Now, viewing the above evaluation in  $A_K$ , we have ( $t$  denotes the degree of  $f$ )

$$f(\pi_1, \dots, \pi_t) = f(\pi_1^{ss} + \pi_1^J, \dots, \pi_t^{ss} + \pi_t^J) = \sum_{i=1}^{2^t} \bar{f}_i$$

where  $\bar{f}_i = f(\pi_1^{\epsilon_1}, \dots, \pi_t^{\epsilon_t})$  and  $\pi^{\epsilon_j}$  is either  $\pi^{ss}$  or  $\pi^J$ .

Since  $A_K$  has the same Kemer index as  $A$ , the evaluation  $\bar{f}_i$  (on  $A_K$ ) vanishes unless all small sets,  $Z$ ,  $\{X_i\}_i$  get semisimple values (and precisely one variable of each big set  $\{V_1, \dots, V_s\}$  gets a radical value). In particular we have

$$\bar{f} = f(\hat{Z}, \{\hat{X}_i\}, \{\hat{V}_j\}, \hat{Q}) = \sum_{r=1}^k f(\hat{z}_1^{ss}, \dots, \hat{z}_d^{ss}, \{\hat{X}_i\}, \{\hat{V}_j\}, \hat{Q}),$$



where  $\hat{z}_i = \hat{z}_i^{ss} + \hat{z}_i^J$ . Our task is then to show  $\text{Tr}(a_0)f(\hat{z}_1^{ss}, \dots, \hat{z}_d^{ss}, \{\hat{X}_i\}, \{\hat{V}_j\}, \hat{Q}) \in \hat{I}$ . To this end let us simplify the notation. We let  $a_1 = \hat{z}_1^{ss}, \dots, a_d = \hat{z}_d^{ss}$ ,  $b_1 = \hat{z}_1^J, \dots, b_d = \hat{z}_d^J$  (that is  $\hat{z}_i = a_i + b_i$ ,  $i = 1, \dots, d$ ) and since the tuples  $\{\hat{X}_i\}, \{\hat{V}_j\}, \hat{Q}$  will not play any role in the proof we denote them by  $\Lambda$ . Thus  $\bar{f} = f(a_1, \dots, a_d, \Lambda)$ .

Claim: for any  $r$  the following holds:

$$\text{Tr}(a_0^{(r)})f(a_1^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}) = \sum_{i=1}^d f(a_1^{(r)}, \dots, a_{i-1}^{(r)}, a_0^{(r)} a_i^{(r)}, a_{i+1}^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}),$$

where  $a_i = (a_i^{(1)}, \dots, a_i^{(k)}) = (p_1(a_i), \dots, p_k(a_i))$  and  $\Lambda^{(r)} = p_r(\Lambda)$ .

Since the variables  $z_1, \dots, z_d$  alternate in  $f$ , the value  $f(a_1^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)})$  is zero unless the elements  $a_1^{(r)}, \dots, a_d^{(r)}$  are linearly independent over  $K$ . On the other hand, since  $A_r$  is basic,  $d_r = \dim_F(A_r^{ss}) = \dim_K((A_r)_K^{ss}) \leq d$  and so  $f(a_1^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}) \neq 0$  only if  $d_r = d$  and the set  $\{a_1^{(r)}, \dots, a_d^{(r)}\}$  is a basis of  $(A_r)_K^{ss}$  over  $K$ .

Write  $a_0^{(r)} a_i^{(r)} = \sum_{j=1}^d \gamma_{i,j} a_j$  where  $\gamma_{i,j} \in K$ .

We compute:

$$\sum_{i=1}^d f(a_1^{(r)}, \dots, a_{i-1}^{(r)}, a_0^{(r)} a_i^{(r)}, a_{i+1}^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}) = \sum_{i=1}^d \sum_{j=1}^d \gamma_{i,j} f(a_1^{(r)}, \dots, a_{i-1}^{(r)}, a_j^{(r)}, a_{i+1}^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}).$$

As  $f(a_1^{(r)}, \dots, a_{i-1}^{(r)}, a_j^{(r)}, a_{i+1}^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)})$  is zero when  $a_j^{(r)} \neq a_i^{(r)}$ , we obtain

$$\sum_{i=1}^d \gamma_{i,i} f(a_1^{(r)}, \dots, a_{i-1}^{(r)}, a_i^{(r)}, a_{i+1}^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)}) = \text{Tr}(a_0^{(r)})f(a_1^{(r)}, \dots, a_d^{(r)}, \Lambda^{(r)})$$

proving the claim.

Having established the claim for  $r = 1, \dots, k$  we conclude

$$\text{Tr}(a_0) \cdot f(a_1, \dots, a_d, \Lambda) = \sum_{i=1}^d f(a_1, \dots, a_{i-1}, a_0 a_i, a_{i+1}, \dots, a_d, \Lambda).$$

In order to complete the proof we show that  $f(a_1, \dots, a_{i-1}, a_0 a_i, a_{i+1}, \dots, a_d, \Lambda) \in \hat{I}$ ,  $i = 1, \dots, d$ . The argument is similar to the one above. Recall that  $a_0 \in Y_{ss} \subseteq \hat{\mathcal{A}}$ . Furthermore, by the construction of  $\hat{\mathcal{A}}$ , there is an element  $\hat{z}_0 \in \mathcal{A}$  such that  $\hat{z}_0 = a_0 + b_0$ , where  $b_0 \in J_K$ . This implies

$$f(a_1, \dots, a_{i-1}, a_0 a_i, a_{i+1}, \dots, a_d, \Lambda) = f(\hat{z}_1, \dots, \hat{z}_{i-1}, \hat{z}_0 \hat{z}_i, \hat{z}_{i+1}, \dots, \hat{z}_d, \Lambda)$$

which is clearly in  $\hat{I}$ . The proposition is now proved.  $\square$

Proposition 10.5 and Corollary 10.3 yield

**Corollary 10.6.** *Let  $A$  be a finite dimensional  $F$ -algebra and let  $\mathcal{A}$  be an affine relatively free algebra. Let  $I$  be a  $T$ -ideal generated by some Kemer polynomials of  $A$  and let  $\hat{I}$  be the ideal of  $\mathcal{A}$  consisting of all evaluations of  $I$  on  $\mathcal{A}$ . Then  $\mathcal{A}/\hat{I}$  is representable.*

Let us explain how the last result fits in our plan for proving Kemer’s theorem. We have started the proof by showing the existence of a finite dimensional  $F$ -algebra  $A$  whose ideal of identities is contained in

$\text{Id}(W)$ . Our goal is to replace the algebra  $A$  by a representable algebra  $A'$ , with  $\text{Id}(A) \subseteq \text{Id}(A') \subseteq \Gamma$ , which has the same Kemer index as  $\Gamma$ , and moreover shares with  $\Gamma$  the same Kemer polynomials. This may be considered as an “approximation” (i.e. not necessarily PI equivalence) of  $\Gamma$  by a representable algebra. Let us sketch briefly the construction of  $A'$ . We take the  $T$ -ideal  $I$  generated by Kemer polynomials of  $A$  which are contained in  $\Gamma$ . If  $A$  and  $\Gamma$  have different Kemer indices, then all Kemer polynomials of  $A$  are contained in  $\Gamma$  and hence the Kemer index of  $\mathcal{A}/\hat{I}$  is strictly smaller than the Kemer index of  $A$ . A finite number of such steps yield a representable algebra  $A_1$  with the same Kemer index as  $\Gamma$  and  $\text{Id}(A_1) \subseteq \Gamma$ . Once we have reached the Kemer index of  $\Gamma$  (from above in the lexicographic ordering), we consider the  $T$ -ideal  $I$  generated by all Kemer polynomials of  $A_1$  which are contained in  $\Gamma$ . As above, we let  $\hat{I}$  be ideal of  $\mathcal{A}_1$  (an affine relatively free algebra of  $A_1$ ) consisting of all evaluations of  $I$  on  $\mathcal{A}_1$  and conclude our construction by putting  $A' = \mathcal{A}_1/\hat{I}$ . Before we present the details of the proof, let us explain why we insist in modding out ideals of a relatively free algebra  $\mathcal{A}$  (and not of  $A$  for instance).

It is clear that if  $B$  is any algebra and  $I$  is a  $T$ -ideal, modding out from  $B$  the ideal  $\hat{I}$  consisting of all evaluations of  $I$  on  $B$ , yields an algebra whose  $T$ -ideal of identities contains  $I$ . However, in general, we don't know whether other polynomials “become” identities. For instance if  $B = M_n(F)$ , then taking any  $T$ -ideal  $I \not\subseteq \text{Id}(B)$  gives  $B/\hat{I} = 0$ . A key property of the relatively free algebra  $\mathcal{A}$  is that  $\text{Id}(\mathcal{A}/\hat{I}) = \text{Id}(A) + I$  (see Lemma 8.4).

## 11. $\Gamma$ -Phoenix property

Suppose  $\Gamma$  is a  $T$ -ideal containing a Capelli polynomial. We know that this is equivalent to saying that  $\Gamma$  is a  $T$ -ideal of an affine PI algebra and also equivalent to  $\Gamma$  containing the  $T$ -ideal of a finite dimensional algebra  $A$ . If we denote by  $p_\Gamma$  and  $p_A$  the Kemer index of  $\Gamma$  and  $A$  respectively, then  $p_\Gamma \leq p_A$ . Our goal in this section is to show that it is possible to replace  $A$  by another finite dimensional algebra  $B$ , with  $\text{Id}(A) \subseteq \text{Id}(B) \subseteq \Gamma$ , which is “closer” to  $\Gamma$  in the sense that its Kemer index and Kemer polynomials are exactly as those of  $\Gamma$ . This will allow us to deduce the Phoenix property for Kemer polynomials of  $\Gamma$  from (the already established) Phoenix property for Kemer polynomials of basic algebras (see Corollary 7.15).

Let us recall our notation once again. Let  $A$  be a finite dimensional algebra which is a direct product of basic algebras  $A_1 \times \cdots \times A_s$ . Let  $p_A$  and  $p_i$  denote the Kemer index of  $A$  and  $A_i$ ,  $i = 1, \dots, s$  respectively. We let  $\mu_i$  the minimal number of small sets in its Kemer polynomials. Finally, write  $\mu_0$  for the maximum of  $\{\mu_1, \dots, \mu_s\}$ .  $\mathcal{A}$  denotes an affine relatively free algebra that corresponds to  $A$ .

In the next proposition  $\text{Id}(A) \subseteq \Gamma = \text{Id}(W)$  where  $W$  is an affine PI algebra over  $F$  and  $A$  is a finite dimensional algebra over  $F$ .

**Proposition 11.1.** *There exists a representable algebra  $B$  with the following properties:*

- (1)  $\text{Id}(B) \subseteq \Gamma$ .
- (2) The Kemer index  $p_B$  of  $B$  coincides with  $p_\Gamma$ .
- (3)  $\Gamma$  and  $B$  have the same Kemer polynomials corresponding to every  $\mu$  which is  $\geq \mu_0$ .

**Note 11.2.** Any representable  $F$ -algebra is PI equivalent to a finite dimensional  $F$ -algebra and hence we may assume  $B$  is finite dimensional over  $F$ .

**Proof of proposition.** Our first goal is to construct a representable algebra  $B$  with  $\text{Id}(B) \subseteq \Gamma$  and  $p_B = p_\Gamma$ . To this end, we assume  $p_A > p_\Gamma$ . It follows that there exists  $\mu_0$ , such that any Kemer polynomial of  $A$  with at least  $\mu_0$  small sets is in  $\Gamma$  (indeed, if there is no such  $\mu_0$ , then for any  $\mu_0$ , there is a Kemer polynomial of  $A$  with  $\mu_0$  small sets of cardinality  $d$  and  $s$  big sets of cardinality  $d + 1$  (where  $p_A = (d, s)$ ) which is not in  $\Gamma$ . This says, by definition of the Kemer index, that  $(d, s) \leq p_\Gamma$ , contrary to our assumption). Let  $I$  be

the  $T$ -ideal generated by all Kemer polynomials of  $A$  with at least  $\mu_0$  small sets and let  $\hat{I}$  be the ideal of  $\mathcal{A}$  consisting of all evaluations of polynomials in  $I$  on  $\mathcal{A}$ . Due to [Corollary 10.6](#) and [Lemma 8.4](#) we know  $\mathcal{A}/\hat{I}$  is representable and  $\text{Id}(\mathcal{A}/\hat{I}) = \text{Id}(A) + I \subseteq \Gamma$ . Let us show that  $p_{\mathcal{A}/\hat{I}} < p_A$ . Clearly,  $p_{\mathcal{A}/\hat{I}} \leq p_A = p_A$ . Suppose  $p_{\mathcal{A}/\hat{I}} = p_A$  and let  $f$  be a Kemer polynomial of  $\mathcal{A}/\hat{I}$  with at least  $\mu_0$  small sets. Since  $\text{Id}(\mathcal{A}/\hat{I}) \supseteq \text{Id}(A)$ , the polynomial  $f$  is a nonidentity of  $A$ . It follows that  $f$  is a Kemer polynomial of  $A$  and hence is in  $I$ . We obtain that  $f \in \text{Id}(\mathcal{A}/\hat{I})$  contradicting our assumption on  $f$ . It is clear that repeating the process above (a finite number of times) we obtain a representable algebra  $B$  with  $\text{Id}(B) \subseteq \Gamma$  and  $p_B = p_\Gamma$ .

In order to complete the proof of the proposition let us assume we have a finite dimensional algebra  $A$  with  $\text{Id}(A) \subseteq \Gamma$  and  $p_A = p_\Gamma$ . We need to construct a representable algebra  $B$  with  $\text{Id}(A) \subseteq \text{Id}(B) \subseteq \Gamma$  (and hence  $p_B = p_\Gamma$ ) such that  $B$  and  $\Gamma$  have the same Kemer polynomials (with at least  $\mu_0$  small sets). Let  $I$  be the  $T$ -ideal generated by all Kemer polynomials of  $A$  which are contained in  $\Gamma$  and let  $\hat{I}$  the corresponding ideal of  $\mathcal{A}$ . Note that in this final step, it is necessarily not true that all Kemer polynomials of  $A$  are contained in  $\Gamma$  since any Kemer polynomial of  $\Gamma$  is a Kemer polynomial of  $A$ . Consider the algebra  $B = \mathcal{A}/\hat{I}$ . We know  $B$  is representable and has the same Kemer polynomials as  $\Gamma$ .  $\square$

**Theorem 11.3** ( *$\Gamma$ -Phoenix property of Kemer polynomials*). *Let  $\Gamma$  be a  $T$ -ideal as above and let  $f$  be a Kemer polynomial of  $\Gamma$ . Then it satisfies the  $\Gamma$ -Phoenix property.*

**Proof.** Let  $\langle f \rangle$  be the  $T$ -ideal generated by  $f$  and let  $h \in \langle f \rangle$  be a polynomial not in  $\Gamma$ . We need to show there is  $f' \in \langle h \rangle$  which is Kemer of  $\Gamma$ . By the proposition there is a finite dimensional algebra  $A$  with  $\text{Id}(A) \subseteq \Gamma$  whose Kemer polynomials are precisely those of  $\Gamma$ . Hence  $f$  is a Kemer polynomial of  $A$  and assuming (as we may) that  $A = A_1 \times \dots \times A_s$  where  $A_i$  are basic, the polynomial  $f$  is Kemer of  $A_i$  for some  $i$ . But more than that,  $f$  is Kemer with respect to each basic algebra  $A_j$  as long as  $f \notin \text{Id}(A_j)$ . Note that for any such  $j$ ,  $p_j = p_A = p_\Gamma$ . Now the polynomial  $h$  is not in  $\Gamma$  and hence is not in  $\text{Id}(A)$ . It follows that  $h \notin \text{Id}(A_{j_0})$  for some  $j_0$  showing that  $f \notin \text{Id}(A_{j_0})$ . As mentioned above,  $f$  must be Kemer for  $A_{j_0}$  and so we may apply the Phoenix property for Kemer polynomials of the basic algebra  $A_{j_0}$  (see [Corollary 7.15](#)). This says that there is  $f' \in \langle h \rangle$  which is Kemer for  $A_{j_0}$  and hence Kemer for  $A$ . Applying once again [Proposition 11.1](#) we have that  $f'$  is Kemer of  $\Gamma$ . The theorem is now proved.  $\square$

## 12. Technical tools

### 12.1. Zubrilin–Razmyslov traces

We have already seen the usefulness of traces for the purpose of representability (via the Cayley–Hamilton theorem). The theme of this section is to get a version of this theorem to a more general PI setting. We start by introducing the analogue of “characteristic values” (this terminology will be clearer below).

**Definition 12.1.** Let  $f(x_1, \dots, x_n, \Lambda)$  be a polynomial and let  $z$  be any variable where  $z \neq x_i, i = 1, \dots, n$ . We define the polynomial

$$\delta_k^z|_{x_1, \dots, x_n}(f) = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, \dots, i_k}$$

where  $f_{i_1, \dots, i_k}$  is the polynomial obtained from  $f$  by substituting  $zx_{i_j}$  in  $x_{i_j}$  for  $j = 1, \dots, k$ . We often write  $f_{i_1, \dots, i_k} = f|_{zx_{i_1} \rightarrow x_{i_1}, \dots, zx_{i_k} \rightarrow x_{i_k}}$  (Zubrilin’s notation).

For  $k = 0$ , we set  $\delta_0^z|_{x_1, \dots, x_n}(f) = f$ .

Notice that the operators  $\delta_k^z$ ’s are  $F$ -linear. They depend on the variables  $x_1, \dots, x_n$ , however since we always refer to the same variables  $x_1, \dots, x_n$  we adopt the abbreviated notation  $\delta_k^z$ .

**Remark 12.2.** Notice that this definition makes sense also in case  $f$  does not depend on some of the variables  $x_1, \dots, x_n$  and in particular, for polynomials  $f$  which are free of these variables.

For instance the polynomial  $\delta_4^z|_{x_1, \dots, x_6}(x_1y_1x_2y_2) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 6} f_{i_1, \dots, i_4}$  consists of 15 monomials arising from  $(i_1, i_2, i_3, i_4)$  with  $1 \leq i_1 < i_2 < i_3 < i_4 \leq 6$ . This yields

$$\delta_4^z|_{x_1, \dots, x_6}(x_1y_1x_2y_2) = 6zx_1y_1zx_2y_2 + 4zx_1y_1x_2y_2 + 4x_1y_1zx_2y_2 + x_1y_1x_2y_2.$$

**Note 12.3.** We may consider the operation  $\delta_k^z$  in the following setting. Let  $F\langle x_1, \dots, x_n, \Sigma \rangle$  be an affine free algebra where  $\Sigma$  is a finite set of variables. Let  $z = p(\Sigma)$  be any polynomial which is free of the variables  $x_1, \dots, x_n$ . We define the operator  $\delta_k^p$  on  $F\langle x_1, \dots, x_n, \Sigma \rangle$  by

$$\delta_k^{p(\Sigma)}|_{x_1, \dots, x_n}(f) = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, \dots, i_k}$$

where  $f_{i_1, \dots, i_k}$  is the polynomial obtained from  $f$  by substituting  $p(\Sigma)x_{i_j}$  in  $x_{i_j}$  for  $j = 1, \dots, k$ .

**Remark 12.4.** Another way to view  $\delta_k^z|_{x_1, \dots, x_n}(f)$ , in case  $f$  is independent of the variable  $z$  and multilinear in  $x_1, \dots, x_n$ , is by taking the homogeneous component of degree  $k$  in  $z$  in the polynomial  $f((1+z)x_1, \dots, (1+z)x_n, \Lambda)$ .

**Lemma 12.5.** *Notation as above. Suppose that  $z = p(\Sigma)$  is independent of the variables  $x_1, \dots, x_n$ . The following hold.*

- (1) *If  $f$  is multilinear and alternating on  $x_1, \dots, x_n$  then so does  $\delta_k^z(f)$ .*
- (2)  *$\delta_k^z$  and  $\delta_{k'}^z$  commute.*

**Proof.** (1) Suppose  $k > 1$  and  $1 \leq i \neq j \leq n$ . Write:

$$\begin{aligned} \delta_k^z(f) &= \sum_{S \in A_k} f|_{\forall t \in S: z x_t \rightarrow x_t} + \sum_{S \in A_{k-1}} f|_{z x_i \rightarrow x_i, \forall t \in S: z x_t \rightarrow x_t} + \\ &+ \sum_{S \in A_{k-1}} f|_{z x_j \rightarrow x_j, \forall t \in S: z x_t \rightarrow x_t} + \sum_{S \in A_{k-2}} f|_{z x_i \rightarrow x_i, z x_j \rightarrow x_j, \forall t \in S: z x_t \rightarrow x_t} \end{aligned} \tag{1}$$

where  $A_k$  contains all the subsets of  $\{1, \dots, n\}$  of size  $k$  not containing  $i$  nor  $j$ . We need to show that  $\delta_k^z(f)|_{x_i=x_j} = 0$ . We do so by showing that the first sum, the last sum and the sum of the two middle sums are zero separately:

$$\begin{aligned} \sum_{S \in A_k} (f|_{\forall t \in S: z x_t \rightarrow x_t})|_{x_i=x_j} &= \left( \sum_{S \in A_k} (f|_{x_i=x_j})|_{\forall t \in S: z x_t \rightarrow x_t} \right)|_{x_i=x_j} = 0 \\ \sum_{S \in A_{k-2}} (f|_{z x_i \rightarrow x_i, z x_j \rightarrow x_j, \forall t \in S: z x_t \rightarrow x_t})|_{x_i=x_j} &= \left( \sum_{S \in A_{k-2}} (f|_{z x_i \rightarrow x_i, z x_i \rightarrow x_j})|_{\forall t \in S: z x_t \rightarrow x_t} \right)|_{x_i=x_j} = 0 \end{aligned}$$

and

$$\left( \sum_{S \in A_{k-1}} (f|_{z x_i \rightarrow x_i, \forall t \in S: z x_t \rightarrow x_t}) + \sum_{S \in A_{k-1}} (f|_{z x_j \rightarrow x_j, \forall t \in S: z x_t \rightarrow x_t}) \right)|_{x_i \rightarrow x_j} =$$

$$\begin{aligned} & \sum_{S \in A_{k-1}} f|_{z x_i \rightarrow x_i, x_i \rightarrow x_j \ \forall t \in S: z x_t \rightarrow x_t} - \left( \sum_{S \in A_{k-1}} (f|_{x_i \leftrightarrow x_j})|_{z x_j \rightarrow x_j, \ \forall t \in S: z x_t \rightarrow x_t} \right) |_{x_i \rightarrow x_j} = \\ & \sum_{S \in A_{k-1}} f|_{z x_i \rightarrow x_i, x_i \rightarrow x_j \ \forall t \in S: z x_t \rightarrow x_t} - \left( \sum_{S \in A_{k-1}} f|_{x_i \rightarrow x_j, z x_j \rightarrow x_i, \ \forall t \in S: z x_t \rightarrow x_t} \right) |_{x_i \rightarrow x_j} = \\ & \sum_{S \in A_{k-1}} f|_{z x_i \rightarrow x_i, x_i \rightarrow x_j \ \forall t \in S: z x_t \rightarrow x_t} - \left( \sum_{S \in A_{k-1}} f|_{z x_i \rightarrow x_i, x_i \rightarrow x_j, \ \forall t \in S: z x_t \rightarrow x_t} \right) = 0 \end{aligned}$$

All in all we get 0. We still need to consider the cases  $k = 0$  and  $k = 1$ . The case  $k = 0$  is trivial since  $\delta_0^z(f) = f$ . In case  $k = 1$  we get the same proof as for  $k > 1$  with the only difference that the last sum in (1) does not appear here.

(2) Denote by  $L_k$  the set containing all the subsets of  $\{1, \dots, n\}$  of size  $k$ . Then

$$\delta_k^z(f) = \sum_{S \in L_k} f|_{\forall i \in S: z x_i \rightarrow x_i}$$

and

$$\delta_{k'}^z(\delta_k^z(f)) = \sum_{S_2 \in L_{k'}} \sum_{S_1 \in L_k} (f|_{\forall i \in S_1: z x_i \rightarrow x_i})|_{\forall j \in S_2: z x_j \rightarrow x_j} = \sum_{S_1 \in L_k, S_2 \in L_{k'}} f|_{\forall i \in S_1 \cap S_2: z^2 x_i \rightarrow x_i, \ \forall i \in S_1 \oplus S_2: z x_i \rightarrow x_i}$$

where  $S_1 \oplus S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$ . This proves the claim. Notice that the second equality holds because  $z$  is independent of  $\{x_1, \dots, x_n\}$ .  $\square$

Suppose  $W$  is an affine algebra with Kemer index  $(n, r)$ . We let  $\mu$  be the minimal number of small sets in its Kemer polynomials. Let  $f = f(X, \Lambda)$  be a Kemer polynomial of  $W$  ( $X$  and  $\Lambda$  are sets of variables) and suppose that (already)  $\mu$  small sets and all big sets of variables are contained in  $\Lambda$ .

**Lemma 12.6.** *Suppose  $f(x_1, \dots, x_{n+1}, \Lambda)$  is such a Kemer polynomial which in addition is multilinear on  $x_1, \dots, x_{n+1}$  and alternates on  $x_1, \dots, x_n$ . Then*

$$\sum_{t=0}^n (-1)^t \delta_t^z (f|_{z^{n-t} x_{n+1} \rightarrow x_{n+1}}) \in \text{Id}(W).$$

**Proof.** It easy to check that if  $g(x_1, \dots, x_{n+1}, E)$ ,  $E$  any set of variables, is multilinear on  $x_1, \dots, x_{n+1}$  and alternates on  $x_1, \dots, x_n$ , then

$$\tilde{g} = g(x_1, \dots, x_{n+1}, E) - \sum_{k=1}^n g(x_1, \dots, x_{k-1}, x_{n+1}, x_{k+1}, \dots, x_n, x_k, E)$$

alternates on  $x_1, \dots, x_{n+1}$ . Therefore, if we replace  $g$  by  $\delta_t^z(f)$  we obtain that

$$\widetilde{\delta_t^z(f)} = (\delta_t^z(f)) - \sum_{k=1}^n (\delta_t^z(f))|_{x_k \leftrightarrow x_{n+1}}$$

has  $\mu$  alternating small sets and  $r + 1$  alternating big sets. Thus,  $\widetilde{\delta_t^z(f)}$  is an identity of  $W$ .

Substituting  $z^{n-t}x_{n+1}$  in  $x_{n+1}$  yields:

$$\begin{aligned} (\delta_t^z(f))|_{z^{n-t}x_{n+1} \rightarrow x_{n+1}} &\equiv \sum_{k=1}^n ((\delta_t^z(f))|_{x_k \leftrightarrow x_{n+1}})|_{z^{n-t}x_{n+1} \rightarrow x_{n+1}} \pmod{\text{Id}(W)} \\ &= \sum_{k=1}^n (\delta_t^z(f))|_{x_k \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_k}. \end{aligned}$$

Since the operator  $\delta_t^z$  “has no effect” on  $x_{n+1}$ , one can see easily that the operation of  $\delta_t^z$  commutes with the substitution  $z^{n-t}x_{n+1} \rightarrow x_{n+1}$ , that is  $(\delta_t^z(f))|_{z^{n-t}x_{n+1} \rightarrow x_{n+1}} = \delta_t^z(f|_{z^{n-t}x_{n+1} \rightarrow x_{n+1}})$  and so we have

$$\delta_t^z(f|_{z^{n-t}x_{n+1} \rightarrow x_{n+1}}) \equiv \sum_{k=1}^n (\delta_t^z(f))|_{x_k \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_k} \pmod{\text{Id}(W)}.$$

Therefore, the lemma will be proved if we show

$$\sum_{t=0}^n \sum_{k=1}^n (-1)^t (\delta_t^z(f))|_{x_k \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_k} \in \text{Id}(W)$$

and after changing the order of summation, it is clear that it is enough to show that for every  $k$

$$\sum_{t=0}^n (-1)^t (\delta_t^z(f))|_{x_k \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_k} = 0.$$

For simplicity (and in fact, without loss of generality) we show the statement in case  $k = 1$ . The following equality holds

$$\begin{aligned} (\delta_t^z(f))|_{x_1 \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_1} &= \underbrace{\sum_{1 < i_1 < \dots < i_t \leq n} (f_{i_1, \dots, i_t})|_{x_1 \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_1}}_{g_t} \\ &+ \underbrace{\sum_{1 = i_1 < i_2 < \dots < i_t \leq n} (f_{i_1, \dots, i_t})|_{x_1 \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_1}}_{h_t}. \end{aligned}$$

Thus,

$$\begin{aligned} g_t &= \sum_{1 < i_1 < \dots < i_t \leq n} f|_{x_1 \leftarrow z^{n-t}x_{n+1}, x_{i_1} \leftarrow zx_{i_1}, \dots, x_{i_t} \leftarrow zx_{i_t}, x_{n+1} \leftarrow x_1} \\ h_t &= \sum_{1 < i_2 < \dots < i_t \leq n} f|_{x_1 \leftarrow z^{n-t+1}x_{n+1}, x_{i_2} \leftarrow zx_{i_2}, \dots, x_{i_t} \leftarrow zx_{i_t}, x_{n+1} \leftarrow x_1}. \end{aligned}$$

Observe that  $g_n = h_0 = 0$  and  $h_{t+1} = g_t$  for  $t = 0, \dots, n - 1$  and so

$$\sum_{t=0}^n (-1)^t (\delta_t^z(f))|_{x_1 \rightarrow x_{n+1}, z^{n-t}x_{n+1} \rightarrow x_1} = \sum_{t=0}^n (-1)^t (h_t + g_t) = 0. \quad \square$$

We will use the above lemma in the following setting.

**Corollary 12.7.** Let  $f(x_1, \dots, x_{n+1}, \Lambda) = \sum_{\sigma} f_{1,\sigma} x_{n+1} f_{2,\sigma}$  be a Kemer polynomial which is multilinear on  $x_1, \dots, x_{n+1}$  and alternating on  $x_1, \dots, x_n$ . Suppose the variable  $z$  is not one of the variables  $x_1, \dots, x_n$ . Then

$$\sum_{t=0}^n (-1)^t \delta_t^z \left( \sum_{\sigma} f_{1,\sigma} z^{n-t+1} f_{2,\sigma} \right) \in \text{Id}(W).$$

**Proof.** Use the previous lemma to obtain:

$$\sum_{t=0}^n (-1)^t \delta_t^z \left( \sum_{\sigma} f_{1,\sigma} z^{n-t} x_{n+1} f_{2,\sigma} \right) \in \text{Id}(W).$$

Now substitute  $x_{n+1} \leftarrow z$ .  $\square$

We will be interested in the following special case.

**Corollary 12.8.** Let  $f(x_1, \dots, x_{n+1}, \Lambda) = \sum_{\sigma} f_{1,\sigma} x_{n+1} f_{2,\sigma}$  as in the corollary above. Let  $F\langle x_1, \dots, x_n, \Sigma \rangle$  be an affine free algebra where  $\Sigma$  is any set of variables disjoint to the set  $\{x_1, \dots, x_n\}$  and let  $p = p(\Sigma) \in F\langle x_1, \dots, x_n, \Sigma \rangle$ . Then

$$\sum_{t=0}^n (-1)^t \delta_t^p \left( \sum_{\sigma} f_{1,\sigma} p^{n-t+1} f_{2,\sigma} \right) \in \text{Id}(W) \cap F\langle x_1, \dots, x_n, \Sigma \rangle.$$

### 12.2. Interpretation lemma

**Lemma 12.9.** Let  $A$  be an  $F$ -algebra and let  $L$  be an ideal of  $A$ . Suppose the polynomial ring  $R = F[t_1, \dots, t_n]$  acts  $F$ -linearly on  $L$  such that  $f \cdot (ax) = (f \cdot a)x$  and  $f \cdot (xa) = x(f \cdot a)$ , for all  $x \in A$ ,  $a \in L$  and  $f \in R$  (that is the action of  $R$  on  $L$  commutes with the  $A$ -bimodule structure on  $L$ ). Then the natural map  $A \rightarrow A' = R \otimes_F A / (t_i \otimes a - 1 \otimes (t_i \cdot a)) | a \in L, i = 1, \dots, n$  is an embedding.

**Proof.** Consider the exact sequence  $0 \rightarrow \ker \nu \rightarrow R \otimes_F L \rightarrow R \otimes_R L \cong L \rightarrow 0$  where  $\nu : R \otimes_F L \rightarrow R \otimes_R L$  is the natural map of  $F$ -algebras. Clearly  $\ker \nu = \langle t_i \otimes a - 1 \otimes (t_i \cdot a) | a \in L, i = 1, \dots, n \rangle_{R \otimes_F L}$ . However, the ideal  $\ker \nu \subseteq R \otimes_F L$  is invariant under the action of  $R \otimes_F A$  and hence if we denote by  $V$  the vector space that supplements  $L$  in  $A$ , we have

$$\begin{aligned} A' &= R \otimes_F A / \ker \nu \cong R \otimes_F (L \oplus V) / \ker \nu \\ &= (R \otimes_F L \oplus R \otimes_F V) / \ker \nu \cong (R \otimes_F L / \ker \nu) \oplus R \otimes_F V \cong (L \oplus R \otimes_F V). \end{aligned}$$

This shows the natural map  $A \rightarrow A'$  is an embedding and the lemma is proved.  $\square$

### 13. Representable spaces

Let us summarize what we have and what remains to be done. We are assuming that  $\Gamma$  is the  $T$ -ideal of an affine algebra  $W$  with Kemer index  $p = (n, r)$  and  $S_p$  is the  $T$ -ideal generated by its Kemer polynomials (with at least  $\mu$  small sets). The idea is to proceed by induction on the Kemer index of  $\Gamma$ . We assume the main theorem holds for all affine algebras with Kemer index smaller than  $p$  and prove it for  $\Gamma$  (note that  $p = (0, 0)$  if and only if  $W = 0$ ). Consider the  $T$ -ideal  $\Gamma' = \Gamma + S_p$ . Clearly, its Kemer index  $p'$  is smaller than  $p$  (for otherwise, any Kemer polynomial of  $\Gamma'$  is a Kemer polynomial of  $\Gamma$  and hence in  $S_p$ ) and hence there is a representable algebra  $A'$  (or a finite dimensional algebra over a field extension) with  $\text{Id}(A') = \Gamma'$ .

The ingredient we are still missing is the existence of a representable algebra  $B_p$  satisfying all the identities of  $W$  and such that any polynomial in  $S_p$  (which is not in  $\Gamma$ ) is a nonidentity of  $B_p$ . Then it will be easy to conclude that the representable algebra  $A' \times B_p$  is PI equivalent to  $W$ .

Let  $\mathcal{W}_0 = F\langle \Sigma \rangle / \text{Id}_{F\langle \Sigma \rangle}(W)$  be an affine relatively free algebra of  $W$ . Here,  $\text{Id}_R(W)$  denotes the ideal of an algebra  $R$  generated by all evaluations of  $\text{Id}(W)$  on  $R$ . Denote by  $X_p$  a set of  $n(\mu + 1) + (n + 1)r$  variables (i.e. precisely the number of variables needed to support the  $\mu + 1$  small sets and  $r$  big sets in a Kemer polynomial). Suppose the set  $X_p$  is disjoint to the variables  $\Sigma$  (that generate  $\mathcal{W}_0$ ) and denote by  $\mathcal{W} = F\langle X_p, \Sigma \rangle / \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  the affine relatively free algebra of  $W$  generated by  $\Sigma$  and  $X_p$ .

Let us fix (for the rest of this section) a decomposition of  $X_p$  into  $\mu + 1$  sets of variables  $X_1, \dots, X_{\mu+1}$ , each containing exactly  $n$  elements, and  $r$  additional sets  $X_{\mu+2}, \dots, X_{\mu+1+r}$ , each containing  $n + 1$  variables.

Let  $f$  be a Kemer polynomial of  $W$  with at least  $\mu + 1$  small sets.

Before getting into the definitions and the precise construction of the representable algebra  $B_p$ , let us give here a short outline of the construction. We consider the affine relatively free algebra  $\mathcal{W} = F\langle X_p, \Sigma \rangle / \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  of  $W$  generated by  $\Sigma$  and  $X_p$ . The set  $X_p$  was already “fragmented” into  $\mu + 1$  sets of variables  $X_1, \dots, X_{\mu+1}$ , each containing exactly  $n$  elements, and  $r$  additional sets  $X_{\mu+2}, \dots, X_{\mu+1+r}$ , each containing  $n + 1$  variables, in particular, sufficiently many small sets and big sets to support nonvanishing evaluations of any Kemer polynomial  $f$  on  $\mathcal{W} = F\langle X_p, \Sigma \rangle / \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  in such a way that precisely  $\mu + 1$  small sets of such  $f$  take values precisely in the  $\mu + 1$  small sets  $X_1, \dots, X_{\mu+1}$  (modulo  $\text{Id}_{F\langle X_p, \Sigma \rangle}(W)$ ) and all big sets of  $f$  are evaluated (in one to one correspondence) on the sets  $X_{\mu+2}, \dots, X_{\mu+1+r}$ . It is not difficult to show that any Kemer polynomial  $f$  has a nonzero evaluation of that kind. These are by definition, the *admissible evaluations*. All such (nonzero evaluations) evaluations span a vector space  $V$  in  $\mathcal{W}$ . Our goal is to mod out ideals of  $\mathcal{W}$  such that at the end we obtain a representable algebra  $B_p$ , and yet the space  $V$  embeds in  $B_p$ . This will show that the Kemer polynomials are nonidentities of  $B_p$ . What are the ideals we mod out by? We consider a Shirshov base in  $\mathcal{W} = F\langle X_p, \Sigma \rangle / \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  represented by monomials  $z$  which either contain elements of  $X_p$  or not. If  $z$  is such a monomial (i.e.  $z + \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  is an element of the Shirshov base) that contains a variable of  $X_p$ , then by modding out with the ideal  $I$  of  $\mathcal{W}$  generated (or in fact consisting) of elements of  $F\langle X_p, \Sigma \rangle$  in which at least one element of  $X_p$  appears twice, then the element  $z + \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  is nilpotent modulo  $I$  and hence integral. On the other hand it is easy to see that the space  $V$  intersects trivially the ideal  $I$ . Most of the efforts are devoted to construct ideals such that by modding out them successively, each element of the Shirshov base which is free of elements of  $X_p$ , one at a time, becomes integral and yet the space  $V$  embeds.

**Definition 13.1.** An evaluation of  $f$  on  $F\langle X_p, \Sigma \rangle$  is admissible if the following conditions are satisfied.

- (1) Precisely  $\mu + 1$  small sets of  $f$ , say  $\check{X}_1, \dots, \check{X}_{\mu+1}$ , are evaluated bijectively on the sets  $X_1, \dots, X_{\mu+1}$ .
- (2) All big sets of  $f$  are evaluated bijectively on the sets  $X_{\mu+2}, \dots, X_{\mu+1+r}$ .
- (3) The rest of the variables of  $f$  are evaluated on  $F\langle \Sigma \rangle$ .

An evaluation of  $f$  on  $\mathcal{W}$  is admissible if it is represented by an admissible evaluation on  $F\langle X_p, \Sigma \rangle$ .

We denote by  $\mathcal{S}_p$  the  $F$ -span (in  $\mathcal{W}$ ) of all admissible evaluations of all Kemer polynomials of  $W$ .

Our goal in this section is to prove that  $\mathcal{S}_p$  is a representable space of  $\mathcal{W}$ . Here is the precise definition.

**Definition 13.2.** Let  $W$  be a PI  $F$ -algebra and let  $S$  be an  $F$ -subspace of  $W$ . We say that  $S$  is an representable space of  $W$  if there exist a representable algebra  $B$  with  $\text{Id}(B) \supseteq \text{Id}(W)$  and a homomorphism

$$\phi : W \rightarrow B$$

such that  $\phi$  maps  $S$  isomorphically into  $B$ .



**Remark 13.3.** Our main difficulty in the construction of  $B$  is that on one hand it should be not “too big” so that it is representable and on the other hand not “too small” so that  $\mathcal{S}_p$  embeds.

The compromise is achieved by “forcing” a Shirshov base of  $\mathcal{W}$  to be integral over some commutative Noetherian  $F$ -algebra.

Let  $Y$  be a Shirshov base of  $\mathcal{W}$  consisting of elements which are represented by monomials on the set  $\Sigma \sqcup X_p$ . Denote by  $Y_0 = \{b_1 + \text{Id}_{F\langle X_p, \Sigma \rangle}(W), \dots, b_t + \text{Id}_{F\langle X_p, \Sigma \rangle}(W)\}$  the elements of the Shirshov base where the representing monomials are independent of  $X_p$  and let  $Y_1 = Y \setminus Y_0$  be the remaining elements of the Shirshov base.

Consider the algebra  $\mathcal{U} = \mathcal{W}/I$  where  $I$  is generated by elements of the form  $xwx + \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  and  $x^2 + \text{Id}_{F\langle X_p, \Sigma \rangle}(W)$  where  $x \in X_p$  and  $w \in F\langle X_p, \Sigma \rangle$ . Denote by  $\phi : \mathcal{W} \rightarrow \mathcal{U}$  the natural map. It is clear from the definition of  $\mathcal{S}_p$  that  $\mathcal{S}_p \cap I = 0$  and so, by abuse of notation, we write  $\mathcal{S}_p$  also for the (isomorphic) image  $\phi(\mathcal{S}_p)$  in  $\mathcal{U}$ . Note that the elements of  $\bar{Y}_1 = \phi(Y_1)$  are nilpotent and hence integral over  $F$ . Our goal is then to “force”  $\phi(Y_0)$ , the remaining elements of the Shirshov base, to become integral over a suitable Noetherian ring, yielding a representable algebra and yet an algebra where the space  $\mathcal{S}_p$  embeds.

As defined above, the elements  $b_i, i = 1, \dots, t$ , are monomials in the affine free algebra  $F\langle X_p, \Sigma \rangle$ . In the sequel, these elements will be considered in different quotients of  $F\langle X_p, \Sigma \rangle$ . By abuse of notations we denote them by  $\bar{b}_i, i = 1, \dots, t$ . This should not confuse the reader.

Let  $\bar{Y}_0 = \phi(Y_0) = \{\bar{b}_1, \dots, \bar{b}_t\}$  and define by induction for  $i = 1, \dots, t$

$$B_p^{(i)} = \frac{F[\theta_1^{(i)}, \dots, \theta_n^{(i)}] \otimes B_p^{(i-1)}}{J_i}, \quad B_p^{(0)} = \mathcal{U}$$

where  $F[\theta_1^{(i)}, \dots, \theta_n^{(i)}]$  is a polynomial  $F$ -algebra and  $J_i$  is the ideal generated by the element

$$1 \otimes \bar{b}_i^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(i)} \otimes \bar{b}_i^{n-k+1}.$$

(We remind the reader that  $n$  is the size of a small set in Kemer polynomials.)

In other words,

$$B_p^{(i)} = \frac{R_i \otimes F\langle X_p, \Sigma \rangle}{J^{(i)} + I + \text{Id}_{R_i\langle X_p, \Sigma \rangle}(W)}$$

where  $R_i = F[\theta_1^{(1)}, \dots, \theta_n^{(1)}, \dots, \theta_1^{(i)}, \dots, \theta_n^{(i)}]$  and  $J^{(i)}$  is generated by

$$1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1}, \quad j = 1, \dots, i.$$

Denote by  $\mathcal{S}_p^{(i)}$  the ideal generated by the projection of  $\mathcal{S}_p$  into  $B_p^{(i)}$ , and by  $\mathcal{J}^{(i)}$  the ideal  $J^{(i)} + I + \text{Id}_{R_i\langle X_p, \Sigma \rangle}(W)$ .

**Lemma 13.4.** *Let  $X$  be any subset of  $X_p$  of cardinality  $m$  and  $z$  be any monomial consisting of variables of  $\Sigma$ . Then the following actions on  $R_i\langle X_p, \Sigma \rangle$  preserve the ideal  $\mathcal{J}^{(i)}$  (note that here we do not insist in the decomposition of  $X_p$  into the subsets  $X_i$  we fixed above).*

- (1) *The Zubrilin–Razmyslov traces  $\delta_k^z = \delta_k^z|_X$ , where  $k = 0, \dots, m$ .*

(2) The alternation of the set  $X$  which is defined by

$$\text{Alt}_X(f) = \sum_{\sigma \in S_X} (-1)^\sigma f|_{\forall x \in X: \sigma(x) \rightarrow x}.$$

(3) Zero substitution on elements of  $X$ . That is

$$F_X(f) = f|_{\forall x \in X: 0 \rightarrow x}.$$

**Proof.** Since the actions above are linear, it is sufficient to check that they preserve the ideals  $\text{Id}_{R_i \langle X_p, \Sigma \rangle}(W)$ ,  $I$  and  $J^{(i)}$ .

Since  $\text{Id}_{R_i \langle X_p, \Sigma \rangle}(W)$  is a  $T$ -ideal it is clearly preserved by each of the actions defined in (1)–(3). Next we turn to the ideal  $J^{(i)}$ , starting with (3). Suppose  $g, h \in R_i \langle X_p, \Sigma \rangle$  are monomials. Then for any  $j = 1, \dots, i$  and any nonnegative integer  $k$ ,  $F_X(gb_j^k h) = 0$  if some variable of  $X$  is in  $gh$  whereas  $F_X(gb_j^k h) = gb_j^k h$  if no variable of  $X_p$  is in  $gh$ . Hence

$$F_X \left( 1 \otimes g \left( 1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1} \right) 1 \otimes h \right)$$

is zero or  $1 \otimes g \left( 1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1} \right) 1 \otimes h$  which are elements of  $J^{(i)}$ .

Let us show  $\text{Alt}|_X$  preserves  $J^{(i)}$ . Since the element  $1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1}$  does not contain any variable of  $X$ , we have for  $j = 1, \dots, i$

$$\text{Alt}|_X \left( 1 \otimes g \left( 1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1} \right) 1 \otimes h \right)$$

is a sum of polynomials of the type

$$1 \otimes g_\sigma \left( 1 \otimes b_j^{n+1} + \sum_{k=1}^n (-1)^k \theta_k^{(j)} \otimes b_j^{n-k+1} \right) 1 \otimes h_\sigma \in J^{(i)}.$$

Finally, a similar reasoning shows that  $\delta_k^z$  also preserves  $J^{(i)}$ .

We now turn to show the invariance of the ideal  $I$ . Consider elements  $1 \otimes g, 1 \otimes h, 1 \otimes w \in R_i \otimes F \langle X_p, \Sigma \rangle$  where  $g, h, w$  are monomials. Let  $x_0 \in X$ . As we mentioned above the action of  $F_X$  on  $1 \otimes g$  is either zero or  $1 \otimes g$ . Hence, since  $I$  is generated by elements in  $\mathcal{W}$  represented by monomials in  $R_i \langle X_p, \Sigma \rangle$ , it is evident that  $I$  is preserved by  $F_X$ . The result of acting with  $\text{Alt}_X$  or  $\delta_k^z$  on an element of the form  $1 \otimes gx_0^2 h$  or  $1 \otimes gx_0 wx_0 h$ , is a sum of monomials each having some element of  $X_p$  appearing at least twice. Therefore,  $I$  is preserved by  $\delta_k^z$  and  $\text{Alt}_X$ .  $\square$

For future reference we record the conclusion here.

**Corollary 13.5.** *The operations considered above (on  $R_i \otimes F \langle X_p, \Sigma \rangle$ ), namely  $\delta_{k, X_l}^{b_i}, \text{Alt}_{X_l}, l = 1, \dots, \mu + r + 1$  and  $F_X$ , where  $X = \{x\}$  and  $x \in X_p$ , determine well defined actions on  $B_p^{(i)}$ , ( $i = 1, \dots, t$ ).*

**Lemma 13.6.** *The following hold:*

- (1)  $B_p = B_p^{(t)}$  is representable.
- (2)  $S_p$  is mapped isomorphically into  $B_p^{(i)}$  for  $i = 1, \dots, t$ .

**Proof.** By construction, the algebra  $B_p$  has an integral Shirshov base over the polynomial algebra  $R_t$  and hence is representable by [Lemma 9.3](#) and [Theorem 9.4](#). This proves the first part of the lemma. The second part however requires more work.

We proceed by induction on  $i$ . Since  $\mathcal{S}_p$  embeds in  $\mathcal{U}$  we have the result for  $i = 0$ . Let  $i > 0$  and suppose (2) holds for  $i - 1$ , namely  $\mathcal{S}_p$  embeds in  $B_p^{(i-1)}$ . We denote the image in  $B_p^{(i-1)}$  again by  $\mathcal{S}_p$ .

Let  $f = f(X_p, \Sigma)$  be an admissible evaluation of a Kemer polynomial of  $W$ . Suppose  $\bar{f} \in J_i$ . We need to show  $\bar{f}$  is zero in  $B_p^{(i-1)}$ .

By the definition of  $J_i$ ,  $1 \otimes \bar{f}$  has the form

$$1 \otimes \bar{f} = \sum_j v_j \otimes \bar{g}_j \cdot \underbrace{\left( 1 \otimes \bar{b}^{n+1} + \sum_{k=1}^n (-1)^k \theta_k \otimes \bar{b}^{n-k+1} \right)}_B 1 \otimes \bar{h}_j \tag{2}$$

where  $g_j, h_j \in R_{i-1} \otimes F \langle X_p, \Sigma \rangle$  are monomials in the variables of  $\Sigma \sqcup X_p$  (with coefficients in  $R_{i-1}$ ) and  $v_j \in F[\theta_1, \dots, \theta_n]$ . Here,  $b = b_i, \theta_1 = \theta_1^{(i)}, \dots, \theta_n = \theta_n^{(i)}$  and  $B = (1 \otimes b^{n+1} + \sum_{k=1}^n (-1)^k \theta_k \otimes b^{n-k+1})$ . Note that with this notation the equality in (2) takes place in  $F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)}$ .

We claim that we may assume that for every  $j$  in the summation above the elements  $g_j$  and  $h_j$  are such that all variables of  $X_p$  appear exactly once in either  $g_j$  or  $h_j$  (but not in both). Indeed, suppose  $A_x$  is the set of all indexes  $j$  such that  $g_j h_j$  contains the variable  $x \in X$ . Applying  $F_{\{x\}}$  to  $\bar{f} - \sum_{j \in A_x} (v_j \otimes \bar{g}_j) \cdot \bar{B} \cdot (1 \otimes \bar{h}_j)$  yields zero whereas the action of  $F_{\{x\}}$  on the remaining terms is trivial and so can be ignored in the sum above. Repeating this process to all the variables of  $X_p$  we obtain that (we may assume) each variable of  $X_p$  appears in every  $g_j h_j$ . The claim is now proved since modding by  $I$  guaranties that if there is a variable of  $X_p$  that appears more than once in some  $(1 \otimes g_j) \cdot B \cdot (1 \otimes h_j)$ , then  $(1 \otimes \bar{g}_j) \bar{B} (1 \otimes \bar{h}_j) = 0$ .

Now we wish to apply  $\text{Alt}_X$  for  $X = X_1, \dots, X_{\mu+r+1}$  on the two sides of (2). To this end consider for each  $j$  the polynomial

$$M_j = (v_j \otimes g_j) B (1 \otimes h_j) \in F[\theta_1, \dots, \theta_n] \otimes (R_{i-1} \otimes F \langle X_p, \Sigma \rangle)$$

representing

$$\bar{M}_j = (v_j \otimes \bar{g}_j) \bar{B} (1 \otimes \bar{h}_j) \in F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)}.$$

Alternating all variables of  $X_p$  (with respect to the fixed decomposition into small and big sets) in  $M_j$  we obtain

$$L_j = \sum_{\sigma} M_{j,\sigma}$$

and clearly  $\bar{L}_j = \sum_{\sigma} \bar{M}_{j,\sigma} \in F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)}$  (to simplify notation we include the sign arising from alternation in  $M_{j,\sigma}$ ).

Now, by [Corollary 13.5](#) we know that alternation of elements of  $X_p$  is well defined on  $B_p^{(i-1)}$ , so we obtain

$$\bar{L}_j = \sum_{\sigma} \bar{M}_{j,\sigma} = 1 \otimes \alpha \bar{f}$$

where  $\alpha = (n!)^{\mu+1} ((n+1)!)^r$ .

Now, we wish to apply the interpretation lemma ([Lemma 12.9](#)) on the algebra  $F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)}$  with the ideal  $\mathcal{S}_p^{(i-1)} \subseteq B_p^{(i-1)}$  (see notation above). Indeed, we will provide an interpretation of  $\theta_1, \dots, \theta_n$  via a suitable action on the ideal  $\mathcal{S}_p^{(i-1)}$  such that the image of  $\bar{L}_j$  (modulo the interpretation) vanishes and

hence also the image of  $\sum_j \bar{L}_j$ , whereas the elements of  $B_p^{(i-1)}$  are mapped isomorphically. This will imply that  $\bar{f} = 0$  in  $B_p^{(i-1)}$ .

Recall once again (Corollary 13.5) that the operation  $\delta_k^b$  is well defined on  $B_p^{(i-1)}$  ( $b$  is free of variables of  $X_p$ ) so we may set

$$\theta_k(q) = \delta_k^b(q), q \in B_p^{(i-1)}.$$

However as shown in the lemma below, only the restriction of this action on the ideal  $\mathcal{S}_p^{(i-1)}$  commutes with the  $B_p^{(i-1)}$ -bimodule structure.

**Lemma 13.7.** *The following hold:*

(1) For every  $h \in \mathcal{S}_p$ , any  $k$  and any  $x, y \in B_p^{(i-1)}$

$$\delta_k^{\bar{b}}(xh) = x\delta_k^{\bar{b}}(h), \delta_k^{\bar{b}}(hy) = \delta_k^{\bar{b}}(h)y$$

(2) for any  $k, s = 1, \dots, n$  and  $h \in \mathcal{S}_p$

$$\delta_k^{\bar{b}}\delta_s^{\bar{b}}(h) = \delta_s^{\bar{b}}\delta_k^{\bar{b}}(h).$$

**Proof.** Since  $b$  is a monomial having its variables in  $\Sigma$ , Lemma 12.5 implies that  $\delta_k^b$  and  $\delta_s^b$  commute, so the same holds for  $\delta_k^{\bar{b}}$  and  $\delta_s^{\bar{b}}$ .

To complete the proof of the lemma it suffices to show that if  $h \in \mathcal{S}_p$  and  $g$  is a monomial in  $B_p^{(i-1)}$  then  $\delta_k^{\bar{b}}(gh) = g\delta_k^{\bar{b}}(h)$  and  $\delta_k^{\bar{b}}(hg) = \delta_k^{\bar{b}}(h)g$ . Indeed, if  $g$  contains an element of  $X_p$ , then  $hg = 0 = gh$  and also  $g\delta_k^{\bar{b}}(h) = 0 = \delta_k^{\bar{b}}(h)g$  verifying the condition. If  $g$  is a monomial in  $B_p^{(i-1)}$  free of variables of  $X_p$ , then clearly  $\delta_k^{\bar{b}}(hg) = \delta_k^{\bar{b}}(h)g$  and  $\delta_k^{\bar{b}}(gh) = g\delta_k^{\bar{b}}(h)$ . This completes the proof of Lemma 13.7.  $\square$

We apply the interpretation lemma on the element  $\bar{L}_j = \sum_{\sigma} \bar{M}_{j,\sigma} \in F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)}$ .

We obtain

$$\begin{aligned} \sum_{\sigma} \bar{M}_{j,\sigma} &= \sum_{\sigma} (v_j \otimes \bar{g}_{j,\sigma}) \bar{B}(1 \otimes \bar{h}_{j,\sigma}) = \\ \sum_{\sigma} (v_j \otimes \bar{g}_{j,\sigma}) &\left( 1 \otimes \bar{b}^{n+1} + \sum_{k=1}^n (-1)^k \theta_k \otimes \bar{b}^{n-k+1} \right) (1 \otimes \bar{h}_{j,\sigma}) = \end{aligned}$$

putting  $\theta_0 = 1$  we have

$$\begin{aligned} \sum_{\sigma} (v_j \otimes \bar{g}_{j,\sigma}) &\left( \sum_{k=0}^n (-1)^k \theta_k \otimes \bar{b}^{n-k+1} \right) (1 \otimes \bar{h}_{j,\sigma}) = \\ \sum_{k=0}^n (-1)^k v_j \theta_k &\otimes \sum_{\sigma} \bar{g}_{j,\sigma} \bar{b}^{n-k+1} \bar{h}_{j,\sigma}. \end{aligned}$$

Now, since  $\sum_{\sigma} \bar{g}_{j,\sigma} \bar{b}^{n-k+1} \bar{h}_{j,\sigma} \in \mathcal{S}_p^{(i-1)}$ , we may apply the interpretation

$$\theta_k \otimes q = 1 \otimes \delta_k^{\bar{b}}(q), k = 0, \dots, n$$

(note that  $\delta_0$  is the identity map) and obtain

$$v_j \otimes \sum_{k=0}^n (-1)^k \delta_k^{\bar{b}} \left( \sum_{\sigma} \bar{g}_{j,\sigma} \bar{b}^{n-k+1} \bar{h}_{j,\sigma} \right)$$

which vanishes by [Corollary 12.8](#).

As mentioned above, by the interpretation lemma the map

$$B_p^{(i-1)} \rightarrow F[\theta_1, \dots, \theta_n] \otimes B_p^{(i-1)} / (\theta_k \otimes q - 1 \otimes \delta_k^{\bar{b}}(q) | q \in \mathcal{S}_p^{(i-1)}, i = 1, \dots, n)$$

is an embedding and hence  $\bar{f} = 0$  in  $B_p^{(i-1)}$ . This completes the proof of [Lemma 13.6](#).  $\square$

Thus we have a map  $\phi : \mathcal{W} \rightarrow B_p$  where  $B_p$  is a representable algebra,  $\text{Id}(B_p) \supseteq \text{Id}(W)$  and such that the space  $\mathcal{S}_p$  is mapped isomorphically. Consequently we have the following corollary.

**Corollary 13.8.** *Let  $f$  be any Kemer polynomial of the algebra  $W$  (at least  $\mu+1$  small sets). Then  $f \notin \text{Id}(B_p)$ .*

**Proof.** Since  $\mathcal{W}_0$  is an affine relatively free algebra of  $W$ , there exists an evaluation of  $f$  on  $\mathcal{W}_0$  which is not zero. It follows that  $f$  has a nonzero *admissible* evaluation  $\bar{f}$  on  $\mathcal{W}$  with  $\bar{f} \in \mathcal{S}_p$  and hence  $\bar{f} \notin \ker(\phi)$ . This proves the Corollary.  $\square$

### 14. Representability – the proof

We have all ingredients needed to prove the main theorem ([Theorem 1.1](#)).

**Proof.** The proof is by induction on the Kemer index  $p$  associated to a  $T$ -ideal  $\Gamma$  (containing a Capelli polynomial). If  $p = (0, 0)$  then  $\Gamma = F\langle X \rangle$  and so  $W = 0$ . Suppose the theorem is true for any affine algebra with Kemer index smaller than  $p$ . Denote by  $S_p$  the  $T$ -ideal generated by all Kemer polynomials corresponding to  $\Gamma$ , and let  $\Gamma' = \Gamma + S_p$ . It is clear that the Kemer index of  $\Gamma'$  is strictly smaller than  $p$ . Hence, by the inductive hypothesis there is a representable algebra  $A'$  having  $\Gamma'$  as its  $T$ -ideal of identities.

Let  $B_p$  be the representable algebra constructed in the previous section. We will show  $\Gamma = \text{Id}(A' \times B_p)$ .

It is clear that  $\Gamma \subset \text{Id}(A' \times B_p)$  since  $\Gamma$  is contained in  $\Gamma'$  and by construction  $\Gamma \subseteq \text{Id}(B_p)$ . Suppose there is  $f \notin \Gamma$  with  $f \in \text{Id}(A' \times B_p) = \text{Id}(A') \cap \text{Id}(B_p)$ . Since  $f \in \text{Id}(A') = \Gamma'$ , we may assume  $f \in S_p$ . Using the Phoenix property, see [Theorem 11.3](#), we obtain a Kemer polynomial  $f'$  (with at least  $\mu + 1$  small sets) such that  $f' \in (f)$ . But by [Corollary 13.8](#),  $f \notin \text{Id}(B_p)$  and this contradicts our previous assumption on  $f$ . This completes the proof.  $\square$

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