ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 4, pp. 439–447. © Pleiades Publishing, Ltd., 2019.

Identification of Non-stationary Load Upon Timoshenko Beam

G. V. Fedotenkov[1,2*], D. V. Tarlakovsky[1,2], and Y. A. Vahterova[1***]**

(Submitted by A. V. Lapin)

[1] *Moscow Aviation Institute, Moscow, 125993 Russia*

[2] *Research Institute of Mechanics, Moscow State University, Moscow, 119192 Russia*

Received December 13, 2018; in final form, January 12, 2019; accepted January 20, 2019

Abstract—This paper investigates an inverse non-stationary problem of the restoration of the spatial law of a homogeneous isotropic Timoshenko beam of finite length. Hinge support conditions are used as boundary conditions. Initial conditions are assumed to be zero. It is assumed that of the beam's ends is fitted with sensors which in the course of corresponding experiment register the amount of deflection of the beam at the sensor points. The method of the solution of a direct problem is based on the principle of superposition where the deflection of the beam is associated with the space load the beam is exposed to, by means of an integral operator by the spatial coordinate and time. The kernel of such operator is so called influence function. This function is a fundamental solution of a system of differential equations of motion of the study beam. The construction of such solution represents a separate problem. The influence function is found by means of Laplace time transformation and expansion into Fourier series in a system of the problem's eigenfunctions. The solution of the inverse problem at the first stage reduces to a system of algebraic equations for vector operator whose components are time convolutions of the coefficients of expansion series for an influence function with the desired coefficients of expansion of the load in a Fourier series. At the same time, the components of the vector of the rights parts are time dependencies registered by the sensors. The resulting system is ill-conditioned [1]. The second stage serves to resolve independent Volterra integral equations of the first kind for the desired coefficients of Fourier serials for the load.

DOI: 10.1134/S1995080219040061

Keywords and phrases: *Inverse problem, Timoshenko beam, influence function, Fourier series, integral transformation, non-stationary impact, integral equations, quadrature formulas, Tikhonov regularization.*

1. INTRODUCTION

In recent times non-stationary inverse problems of mechanics of deformable solids have increasingly become important in both theoretical and applying dimensions. Problems of this class are categorized as ill-conditioned because small a large disturbance of the solution may in principal correspond to a disturbance of initial data. Notably, the initial data for problems of such kind, as a rule, are corrupted as they are found experimentally. This necessitates making use of special solution methods which will have acceptable precision also for noised initial data expressed in their corruption due to random error in measuring and computational transformations. It should be noted that problems of such type are critically important for aviation and airspace industries because the significant part of any airframe is usually made of beam elements exposed to non-stationary loads. Those include the modes of taking off and landing, various maneuvers, as well as various contingencies. The basics of resolving non-stationary inverse problems were described in the fundamental works of Hadamard [1], Markov [2], Tikhonov [3, 4], Vatulyan [5], et al. Various issues of solutions of non-stationary problems for bodies and structures (mathematical modeling of non-stationary interaction, theoretical and numerical techniques

*E-mail: greghome@mail.ru

**E-mail: tdvhome@mail.ru

***E-mail: yana-vahterova@mail.ru

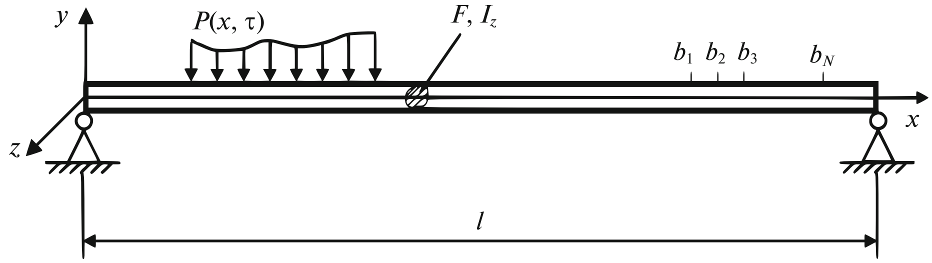

Fig. 1. Timoshenko beam subject to load.

of studying non-stationary problems, dynamics) were given in the works by Gorshkov, Tarlakovsky, et al. [6], Poruchikov [7], Rabotnov [8], Israilov [9], Gelfand, Shilov [10], Dech [11], Babakov [12], Slepnyan, Yakovlev [13], Badriev, Makarov, Paymushin [14–16], Vakhterova, Serpicheva, Fedotenkov [17]. Today non-stationary inverse problems are still poorly known, mainly because of increased dimension of non-stationary problems per unit as compared with stationary and static problems. Moreover, as in other inverse problems, here emerges a problem associated with incorrectness of mathematical statement which are resolved by means of Tikhonov regularization method with the minimization of the corresponding functional in its essence.

2. STATEMENT OF PROBLEM

Let us consider non-stationary transverse oscillations of a homogeneous Timoshenko beam of finite length l in Oxy plane of $Oxyz$ rectangular Cartesian coordinate system.

The beam's motion is described with the equations [1]:

$$\frac{\partial^2 w}{\partial t^2} = \kappa^2 c_2^2 \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \chi\right) + \frac{p}{\rho F}, \quad \frac{\partial^2 \chi}{\partial t^2} = c_p^2 \frac{\partial^2 \chi}{\partial x^2} - \frac{c_2^2 F \kappa^2}{I_z}\left(\chi - \frac{\partial w}{\partial x}\right), \tag{1}$$

where $w(x,t)$is deflection of the beam in Oxy plane, F is area of cross section, ρ is density of the material, c_2 is velocity of shear waves, c_p is velocity of bending waves, $\kappa = \sqrt{5/6}$ is coefficient of shear, χ is angle of rotation of the cross section due to shear deformation, I_z is inertia of the cross section with respect to Oz axis, t is time, $p(x,t)$ is transverse loading, b_n locations of sensors registering deflection of the beam in those points.

Let us introduce a system of non-dimensional values (dimensional parameters are primed, τ denotes non-dimensional time)

$$\eta^2 = \frac{c_p^2}{c_2^2}, \quad x = \frac{x'}{l}, \quad \tau = \frac{c_2 t}{l}, \quad w = \frac{w'}{l}, \quad p = \frac{p' l}{\rho F c_2^2},$$
$$m = \frac{m' l^2}{\rho I_z c_2^2}, \quad c_p^2 = \frac{E}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad \gamma^2 = \frac{F l^2}{I_z}.$$

Then, equations (1) in a non-dimensional notation will take on the form

$$\frac{\partial^2 w}{\partial \tau^2} = \kappa^2 \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \chi}{\partial x}\right) + p,$$
$$\frac{\partial^2 \chi}{\partial \tau^2} = \eta^2 \frac{\partial^2 \chi}{\partial x^2} - \kappa^2 \gamma^2 \left(\chi - \frac{\partial w}{\partial x}\right). \tag{2}$$

We suppose that the ends of the beams are hinge-supported. Then, the following boundary conditions are true [6]:

$$\left.\frac{\partial \chi}{\partial x}\right|_{x=0,1} = 0, \quad w|_{x=0,1} = 0. \tag{3}$$

The initial conditions are assumed to be zero:

$$w|_{\tau=0} = \left.\frac{\partial w}{\partial \tau}\right|_{\tau=0} = \chi|_{\tau=0} = \left.\frac{\partial \chi}{\partial \tau}\right|_{\tau=0} = 0. \tag{4}$$

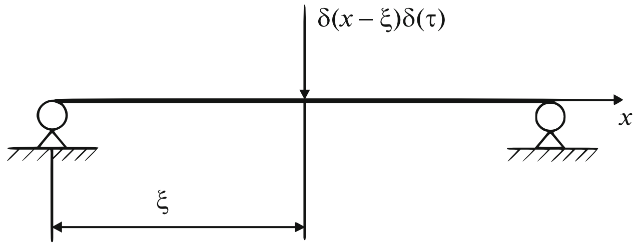

Fig. 2. Beam subjected to sudden concentrated load.

3. INFLUENCE FUNCTIONS FOR TIMOSHENKO BEAM

For the solution of direct and inverse problems it is necessary to construct influence functions $G_w(x,\xi,\tau)$ and $G_\chi(x,\xi,\tau)$.

They are the solutions of the problem (2)–(4) with replacement of the load $p(x,\tau)$ with a single sudden concentrated load $\delta(x-\xi)\,\delta(\tau)$ (Fig. 2), where $\delta(x)$ is the Dirac delta function:

$$\frac{\partial^2 G_w}{\partial \tau^2} = \kappa^2 \frac{\partial}{\partial x}\left(\frac{\partial G_w}{\partial x} - G_\chi\right) + \delta(x-\xi)\,\delta(\tau),$$
$$\frac{\partial^2 G_\chi}{\partial \tau^2} = \eta^2 \frac{\partial^2 G_\chi}{\partial x^2} - \kappa^2\gamma^2\left(G_\chi - \frac{\partial G_w}{\partial x}\right),$$
$$\left.\frac{\partial G_\chi}{\partial x}\right|_{x=0,1} = 0, \quad G_w|_{x=0,1} = 0,$$
$$G_w|_{\tau=0} = \left.\frac{\partial G_w}{\partial \tau}\right|_{\tau=0} = G_\chi|_{\tau=0} = \left.\frac{\partial G_\chi}{\partial \tau}\right|_{\tau=0} = 0. \tag{5}$$

With a knowledge of the influence functions $G_w(x,\xi,\tau)$, $G_\chi(x,\xi,\tau)$ and basing on the superposition principle [6], the solution of the problem (2)–(4) can be presented in the form

$$w = \int_0^l d\xi \int_0^\tau G_w(x,\xi,\tau-t)\,p(\xi,t)\,dt = \int_0^l G_w * p d\xi,$$
$$\chi = \int_0^l d\xi \int_0^\tau G_\chi(x,\xi,\tau-t)\,p(\xi,t)\,dt = \int_0^l G_\chi * p d\xi. \tag{6}$$

Hereinafter asterisk "*" denotes time convolution.

For construction of influence functions we will apply a Laplace time transformation to the task (5) (index L will denote Laplace transform, s is a parameter of Laplace transformation $G_w^L = G_w^L(x,\xi,s)$, $G_\chi^L = G_\chi^L(x,\xi,s)$):

$$s^2 G_w^L = \kappa^2 \frac{\partial}{\partial x}\left(\frac{\partial G_w^L}{\partial x} - G_\chi^L\right) + \delta(x-\xi),$$
$$s^2 G_\chi^L = \eta^2 \frac{\partial^2 G_\chi^L}{\partial x^2} - \kappa^2\gamma^2\left(G_\chi^L - \frac{\partial G_w^L}{\partial x}\right),$$
$$\left.\frac{\partial G_\chi^L}{\partial x}\right|_{x=0,1} = 0, \quad G_w^L\big|_{x=0,1} = 0. \tag{7}$$

Taking the boundary conditions (3) into consideration, we look for the solution of this problem in the form of trigonometric Fourier series:

$$G_w^L = \sum_{n=1}^{\infty} G_{wn}^L(\xi,\tau)\sin\lambda_n x,$$

$$G_{\chi}^{L} = \frac{G_{\chi 0}^{L}}{2} + \sum_{n=1}^{\infty} G_{\chi n}^{L}(\xi,\tau)\cos\lambda_n x, \quad \lambda_n = \pi n. \tag{8}$$

Also we present the function $\delta(x-\xi)$ in form of a series:

$$\delta(x-\xi) = \sum_{n=1}^{\infty} \delta_n(\xi)\sin\lambda_n x,$$

$$\delta_n(\xi) = \frac{2}{l}\int_0^l \delta(x-\xi)\sin\lambda_n x dx = \frac{2}{l}\sin\lambda_n\xi. \tag{9}$$

By substituting, we come to equations in coefficients of series (8). With $n=0$, we obtain

$$\left(s^2+\kappa^2\gamma^2\right)G_{\chi 0}^{L} = 0, \quad \Rightarrow G_{\chi 0}^{L} = 0, \quad G_{\chi 0}(\tau) = 0,$$

If $n>0$:

$$\begin{gathered}\left(s^2+\kappa^2\lambda_n^2\right)G_{wn}^{L} - \kappa^2\lambda_n G_{\chi n}^{L} = \delta_n(\xi), \\ -\kappa^2\gamma^2\lambda_n G_{wn}^{L} + \left(s^2+\eta^2\lambda_n^2+\kappa^2\gamma^2\right)G_{\chi n}^{L} = 0.\end{gathered} \tag{10}$$

The solution takes the form

$$G_{wn}^{L} = \frac{\Delta_{1n}}{\Delta_n}\delta_n(\xi), \quad G_{\chi n}^{L} = \frac{\Delta_{2n}}{\Delta_n}\delta_n(\xi),$$

where

$$\begin{gathered}\Delta_n = \left(s^2+\kappa^2\lambda_n^2\right)\left(s^2+\eta^2\lambda_n^2\right) + s^2\kappa^2\gamma^2, \\ \Delta_{1n}(s) = \left(s^2+\eta^2\lambda_n^2+\kappa^2\gamma^2\right), \\ \Delta_{2n}(s) = \kappa^2\gamma^2\lambda_n.\end{gathered}$$

Originals G_{wn}^{L} and $G_{\chi n}^{L}$ are obtained by mean of the second expansion theorem for Laplace transformation:

$$G_{wn} = \delta_n(\xi)\sum_{l=1}^{4}\operatorname*{res}_{s=s_{nl}}\left[\frac{\Delta_{1n}(s)}{\Delta_n(s)}\right]e^{s\tau}, \quad G_{\chi n} = \delta_n(\xi)\sum_{l=1}^{4}\operatorname*{res}_{s=s_{nl}}\left[\frac{\Delta_{2n}(s)}{\Delta_n(s)}\right]e^{s\tau},$$

were s_{nl} zero of polynomial Δ_n, $\operatorname*{res}_{s=s_{nl}} f(s)$ is residue of function $f(s)$ in point s_{nl} [12]. Root of equation Δ_n takes the form

$$\begin{gathered}s_{nl} = \pm\sqrt{\frac{-\kappa^2\lambda_n^2-\eta^2\lambda_n^2-\kappa^2\gamma^2\pm\sqrt{D_n}}{2}}, \\ D_n = \left(\kappa^2\lambda_n^2-\eta^2\lambda_n^2\right)^2 + \kappa^4\gamma^4 + 2\gamma^2\kappa^4\lambda_n^2 + 2\eta^2\gamma^2\lambda_n^2\kappa^2 > 0.\end{gathered} \tag{11}$$

The form of originals will depend on the character of zero of polynomial Δ_n. From (11) it follows that all roots are simple and partially imaginary because

$$\sqrt{D_n} < \kappa^2\lambda_n^2+\eta^2\lambda_n^2+\kappa^2\gamma^2.$$

Let us denote

$$\begin{gathered}s_{n1,n2} = \pm i\alpha_n, \quad s_{n3,n4} = \pm i\beta_n, \\ \alpha_n = \sqrt{\frac{\kappa^2\lambda_n^2+\eta^2\lambda_n^2+\kappa^2\gamma^2-\sqrt{D_n}}{2}}, \\ \beta_n = \sqrt{\frac{\kappa^2\lambda_n^2+\eta^2\lambda_n^2+\kappa^2\gamma^2+\sqrt{D_n}}{2}}.\end{gathered}$$

With all above notations and formula for residue of a function in a prime field, [18] will take the form [17]:

$$G_w(x,\xi,\tau)=\sum_{n=1}^{\infty}G_{wn}(\xi,\tau)\sin\lambda_n x,$$
$$G_{wn}(\xi,\tau)=\delta_n(\xi)\tilde{G}_{wn}(\tau),\quad \tilde{G}_{wn}(\tau)=(A_{jn}\sin\alpha_n\tau+B_{jn}\sin\beta_n\tau),$$
$$A_{jn}=\frac{\Delta_{jn}(i\alpha_n)}{\alpha_n(\alpha_n^2-\beta_n^2)},\quad B_{jn}=\frac{\Delta_{jn}(i\beta_n)}{\beta_n(\alpha_n^2-\beta_n^2)}. \tag{12}$$

4. SOLUTION OF A DIRECT NON-STATIONARY PROBLEM FOR TIMOSHENKO BEAM

We assume that a beam is subjected to a random distributed non-stationary load $p(x,\tau)$. By making use of the influence function (12) the deflection of the beam will be obtained by the first expression in the formula (6).

Let us present the function $p(x,\tau)$ in form of trigonometric Fourier series:

$$p(x,\tau)=\sum_{n=1}^{\infty}p_n(\tau)\sin\lambda_n x. \tag{13}$$

Substituting (12) and (13) into (6), we obtain

$$w(x,\tau)=\int_0^{\tau}\int_0^{l}\left[\frac{2}{l}\sum_{n=1}^{\infty}\sin(\lambda_n\xi)\tilde{G}_{wn}(\tau-t)\sin(\lambda_n x)\right]\left[\sum_{m=1}^{\infty}p_m(t)\sin(\lambda_m\xi)\right]d\xi dt. \tag{14}$$

Then, considering the orthogonality of the trigonometrical functions, the formula (14) will take the form

$$w(x,\tau)=\sum_{n=1}^{\infty}w_n(\tau)\sin\lambda_n x,\quad w_n(\tau)=\int_0^{\tau}\tilde{G}_{wn}(\tau-t)\,p_n(t)\,dt. \tag{15}$$

5. NUMERICAL IMPLEMENTATION OF SOLUTION OF DIRECT NON-STATIONARY PROBLEM FOR TIMOSHENKO BEAM

For approximated definition of deflection of beam $w(x,\tau)$ we use the formula of mean triangles.

We break down the interval of integration $[0,\tau]$ by M parts with even spacing $h=\frac{\tau}{M}$. In representation (15) we limit ourselves with fist N terms of series. Intervals in (15) are replaced with approximated quadrature formulas of mean rectangles formulas, then

$$w_h(x,\tau)\approx h\sum_{n=1}^{N}\sin\lambda_n x\sum_{m=1}^{M}\tilde{G}_{wn}(x,\tau-t_m)\,p_n(t_m),\quad t_m=h\frac{2m-1}{2}. \tag{16}$$

Examples of the solution of a direct problem with convergence estimate of formula (16) are given in [17].

6. SOLUTION OF INVERSE PROBLEM

The inverse problem is to find coefficients $p_n(\tau)$ of series (13).

We assume that at some interval of the beam there are N sensors which measure deflections of the beam $W_1(\tau)=w(b_1,\tau)$, $W_2(\tau)=w(b_2,\tau)$, ..., $W_N(\tau)=w(b_N,\tau)$ depending on time τ (Fig. 1), where $b_n=\frac{b_1-b_N}{2}+\frac{b_1-b_N}{2}\cos\left(\frac{2n-1}{2N}\pi\right)$ is a Chebyshev polynomial [19], b_1 is the coordinate of the first sensor on the beam, b_N is the coordinate of the last sensor.

Limiting ourselves with the fist members, from (15) we receive integral representations

$$W_k(\tau)=\sum_{n=1}^{N}w_n(\tau)\,a_{kn},\quad a_{kn}=\sin b_k\lambda_n,\quad k=1,..,N, \tag{17}$$

which taking expressions for $w_n(\tau)$ into consideration, comprise a system of integral equations with respect to required coefficients $p_n(\tau)$.

We write the system (17) in a vector-matrix form:

$$\mathbf{W} = \mathbf{AI}, \quad \mathbf{A} = (a_{kn})_{N\times N}, \quad \mathbf{W} = [W_k(\tau)]_{N\times 1}, \quad \mathbf{I} = [w_n(\tau)]_{N\times 1}. \tag{18}$$

By resolving this system we obtain the vector $\mathbf{I}$

$$\mathbf{I} = \mathbf{W}^*, \tag{19}$$

where $\mathbf{W}^* = \mathrm{A}^{-1}\mathbf{W} = [w_n^*(\tau)]_{N\times 1}$.

Chebyshev polynomial is used for portioning neighborhood of sensors locations to make sure that the solution of the system of linear algebraic equations (19) is correct. The vector-matrix equation (19) is equivalent to N independent Volterra integral equations of the first kind with respect to the required coefficients of serials (13):

$$\mathbf{I} = \mathbf{W}^*. \tag{20}$$

As is well known [2], if $\tilde{G}(0) = 0$, the equations (20) are incorrect according to J. Hadamard [1]. Therefore, for the solution of the problem (20) it is necessary to apply the method of Tikhonov regularization [3, 4].

7. NUMERICAL IMPLEMENTATION OF SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

For solution of equations (20) we will make use of the. formula of mean rectangles.

Let us fix some finite time point T. We break down the time interval of integration $[0, T]$ into M even parts with even spacing $h = T/M$. For each moment of time $\tau_m = hm$ we substitute the equation (20) with a numerical analogue with making use of the method of mean rectangles:

$$\begin{gathered} w_{nm}^* \approx h\sum_{k=1}^{m} G_{w,nmk}p_{nk}, \quad m = 1, ..., M, \\ w_{nm}^* = w_{nm}^*(\tau_m), \quad G_{w,nmk} = G_{w,nmk}(\tau_m - t_k), \\ p_{nk} = p_n(t_k), \quad t_k = h\frac{2k-1}{2}. \end{gathered} \tag{21}$$

As a result, we come to a system of linear algebraic equations with respect to p_{nk} which are the values of the required coefficients $p_n(\tau)$ at moments of time t_k, $k = 1, ..., M$:

$$\mathbf{G}_n\mathbf{P}_n = \mathbf{W}_n^*, \tag{22}$$

where

$$\mathbf{G}_n = (G_{w,nmk})_{M\times M} = \begin{pmatrix} G_{w,n11} & 0 & 0 & ... & 0 \\ G_{w,n21} & G_{w,n22} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ G_{w,nM1} & G_{w,nM2} & G_{w,nM3} & ... & G_{w,nMM} \end{pmatrix},$$

$$\mathbf{P} = (p_{nk})_{M\times 1}, \quad \mathbf{W}_n^* = \left(\frac{w_{nm}^*}{h}\right)_{M\times 1}.$$

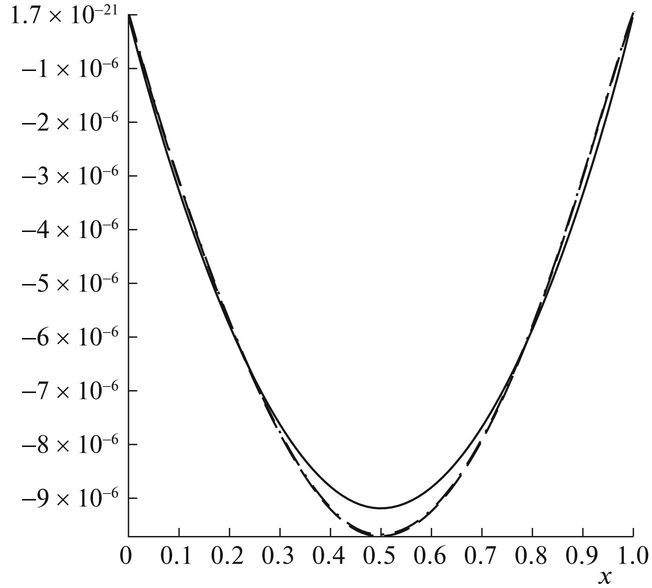

Fig. 3. $p\,(x,\tau) = x(x-l)e^{-\tau}$, $N = 2$, $\tau = 1$.

8. REGULARIZATION OF INVERSE PROBLEM

Due to incorrectness of the problem (20), the matrix $\mathbf{G}_n$ is ill-conditioned, therefore we will resolve the equation system (22) by making use of the Tikhonov regularization technique [3, 4]. Here, (22) is substituted with an equivalent problem of finding of the minimum of the Tikhonov functional:

$$\Omega_\alpha\,(\tau) = |\mathbf{G}_n\tau - \mathbf{W}_n^*|^2 + \alpha|\tau|^2.$$

It can be demonstrated [3] that the problem of minimization of the Tikhonov functional $\Omega_\alpha\,(\tau)$ reduces to the solution of an equivalent system of algebraic equations

$$\left(\mathbf{G}_n^T\mathbf{G}_n + \alpha\mathbf{E}\right)\tilde{\mathbf{P}}_n = \mathbf{G}_n^T\mathbf{W}_n^*, \tag{23}$$

where is a small positive parameter of regularization which is selected by some optimal way [3], is vector of quasi solution of the system of equations (22).

9. EXAMPLES OF SOLUTIONS OF INVERSE PROBLEM

Consider some examples of the solution of an inverse problem of identification of external non-stationary load affecting a Timoshenko beam with the following non-dimensional parameters: $\eta = 1.6$, $l = 1$, $\gamma = 346.4$, $\beta = 0.2$, $\xi = 0.4$, which correspond, for instance, to a Timoshenko beam made of steel, with the following dimensional parameters:

$$\rho = 7850\ \text{kg/m}^3, \quad E = 2\times10^{11}\ \text{Pa}, \quad \mu = 7.69\times10^{10}\ \text{Pa}, \tag{24}$$

$$\lambda = 1.15\times10^{11}\ \text{Pa}, \quad \nu = 0.3, \quad \beta = 0.2l, \quad \xi = 0.4l. \tag{25}$$

The beam is 1 m long, with rectangular cross section of $10^{-2}\times10^{-2}$. Non-dimensional time of work of sensor $T = 5$, number of time steps: $M = 100$, $b_1 = 0.4l$, $b_N = 0.9l$. Displacements $W_n\,(\tau)$ in the sensor locations are found from the solution of a direct problem with a specified external load of $p\,(x,\tau)$.

We set a small parameter as $\alpha = 10^{-5}$. By resolving a system of linear algebraic equations (23) we obtain coefficients of Fourier series which are quasi solution of the problem (22). Using the formula (13) we obtain a reconstructed load (Figs. 3 and 4). Here the solid line denotes the load $p\,(x,\tau)$ set for the solution of a direct problem by formula (15), dashed line denotes the load drawn from the solution of an inverse problem (22), dash-and-dot line denotes the load drawn from the solution of an inverse problem with insignificant noise.

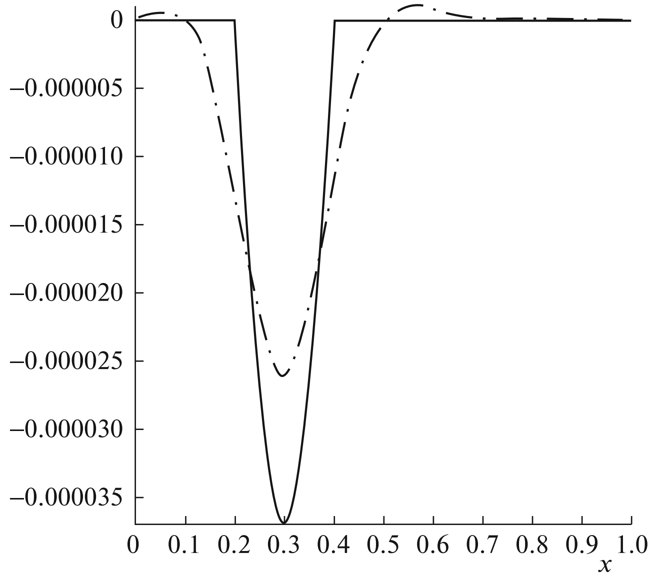

Fig. 4. $p\,(x,\tau) = 10^{-4}e^{-\tau}\sin\,(5x\pi)\left(H\left(x-\frac{1}{5}\right)-H\left(x-\frac{2}{5}\right)\right)$, $N = 20$, $\tau = 1$.

10. CONCLUSION

This paper offers a technique and describes an algorithm to resolve an inverse non-stationary problem for a Timoshenko beam for the purpose of identification of distributed non-stationary load. There are some examples of calculations given both with noised sensor readings and without them.

FUNDING

The work was carried out with the financial support of Russian Foundation for Basic Research (RFBR) (project nos. 19-58-04016, 19-08-00438).

REFERENCES

1. J. Hadamard, *Probleme de Cauchy et equations aux derives partiels hyperboliques* (Hermann, Paris, 1932).
2. E. V. Markova, *Numerical Methods of Solution of Nonclassical Volterra Linear Equations of the First Kind and their Applications* (Irkutsk, 1999) [in Russian].
3. A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-Posed Problems* (Nauka, Moscow, 1979; Winston, Washington DC, 1977).
4. A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov, and A. G. Yagoda, *Numerical Methods of Solution of Incorrect Problems* (Nauka, Moscow, 1990) [in Russian].
5. A. O. Vatulyan, *Inverse Problems in Mechanics of Deformable Solids* (Fizmatlit, Moscow, 2007) [in Russian].
6. A. G. Gorshkov, A. L. Medvedevsky, L. N. Rabinsky, and D. V. Tarlakovsky, *Waves in Continuous Media* (Fizmatlit, Moscow, 2004) [in Russian].
7. V. B. Poruchikov, *Methods of Dynamical Theory of Elasticity* (Nauka, Moscow, 1986) [in Russian].
8. Yu. N. Rabotnov, *Mechanics of Deformed Solids* (Nauka, Moscow, 1988) [in Russian].
9. M. Sh. Israilov, *Dynamic Theory of Elasticity and Diffraction of Waves* (Moscow, 1992) [in Russian].
10. I. M. Gelfand and G. E. Shilov, *Generalized Functions and Actions upon Them* (Fizmatlit, Moscow, 1959) [in Russian].
11. G. Dech, *Guidelines for Practical Application of Laplace Transformation and Z-Transformation* (Nauka, Moscow, 1971) [in Russian].
12. I. M. Babakov, *Theory of Oscillations* (Drofa, Moscow, 2004) [in Russian].
13. L. I. Slepyan and Y. S. Yakovlev, *Integral Transformations in Non-Stationary Problems* (Sudostroyenie, Leningrad, 1980) [in Russian].

14. I. B. Badriev, M. V. Makarov, and V. N. Paimushin, “Mathematical simulation of nonlinear problem of three-point composite sample bending test,” Proc. Eng. **150**, 1056–1062 (2016). http://dx.doi.org/10.1016/j.proeng.2016.07.214
15. I. B. Badriev, M. V. Makarov, and V. N. Paimushin, “Numerical investigation of physically nonlinear problem of sandwich plate bending,” Proc. Eng. **150**, 1050–1055 (2016). http://dx.doi.org/10.1016/j.proeng.2016.07.213
16. I. B. Badriev, M. V. Makarov, and V. N. Paimushin, “Solvability of a physically and geometrically nonlinear problem of the theory of sandwich plates with transversal-soft core,” Russ. Math. **59** (10), 57–60 (2015). http://dx.doi.org/10.3103/S1066369X15100072
17. G. V. Fedotenkov, E. V. Serpicheva, and Y. A. Vahterova, “Inverse problem of identification of non-stationary load for Timoshenko beam,” Izv. Tuls.Univ., Tekh. Nauki, No. 4, 82–92 (2017).
18. M. A. Lavrentiyev and B. V. Shabat, *Methods of Complex Analysis* (Nauka, Moscow, 1973) [in Russian].
19. N. S. Bakhvalov, N. P. Zhidkov, and G. M. Kobelkov, *Numerical Methods* (Binom, Moscow, 2008) [in Russian].