# Differential calculus over $\mathbb{N}$-graded commutative rings

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**Abstract**

The Chevalley–Eilenberg differential calculus and differential operators over $\mathbb{N}$-graded commutative rings are constructed. This is a straightforward generalization of the differential calculus over commutative rings, and it is the most general case of the differential calculus over rings that is not the non-commutative geometry. Since any $\mathbb{N}$-graded ring possesses the associated $\mathbb{Z}_2$-graded structure, this also is the case of the graded differential calculus over Grassmann algebras and the supergeometry and field theory on graded manifolds.

## Contents

# 1 Introduction

This work addresses the differential calculus over $\mathbb{N}$-graded commutative rings.

This is a straightforward generalization of the differential calculus over commutative rings (Section 2), and it is the most general case of the differential calculus over rings that is not the non-commutative geometry (Section 9).

Throughout the work, all algebras are associative, unless they are Lie and graded Lie algebras. By a ring is meant a unital algebra, i.e., it contains a unit element $\mathbf{1} \neq 0$. Hereafter, $\mathcal{K}$ denotes a commutative ring without a divisor of zero (i.e., an integral domain), and $\mathbb{N}$ is a set of natural numbers, including 0.

DEFINITION 1.1: A direct sum of $\mathcal{K}$-modules

$$P = \bigoplus_{i\in\mathbb{N}} P^i \tag{1.1}$$

is called the $\mathbb{N}$-graded $\mathcal{K}$-module (the internally graded module in the terminology of [27]). Its element $p$ is said to be homogeneous of degree $|p|$ if $p \in P^{|p|}$. □

DEFINITION 1.2: A $\mathcal{K}$-ring $\Omega$ is said to be $\mathbb{N}$-graded if it is an $\mathbb{N}$-graded $\mathcal{K}$-module

$$\Omega = \Omega^* = \bigoplus_k \Omega^k, \qquad k \in \mathbb{N}, \tag{1.2}$$

so that, for homogeneous elements $\alpha \in \Omega^{|\alpha|}$ of degree $|\alpha|$, their product is a homogeneous element $\alpha \cdot \alpha' \in \Omega^{|\alpha|+|\alpha'|}$ of degree $|\alpha| + |\alpha'|$. In particular, it follows that $\Omega^0$ is a $\mathcal{K}$-ring, while $\Omega^{k>0}$ are $\Omega^0$-bimodules and, accordingly, $\Omega^*$ is well. □

It should be emphasized that a $\mathcal{K}$-ring $\Omega$ can admit different $\mathbb{N}$-graded structures $\Omega^*$ (1.2) (Theorem 6.1).

For instance, any ring $\mathcal{A}$ is the $\mathbb{N}$-graded ring $\mathcal{A}^*$ where $\mathcal{A}^0 = \mathcal{A}$ and $\mathcal{A}^{>0} = 0$.

We mainly consider $\mathbb{N}$-graded commutative rings.

DEFINITION 1.3: An $\mathbb{N}$-graded ring is said to be graded commutative if

$$\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha, \qquad \alpha, \beta \in \Omega^*. \tag{1.3}$$

In this case, $\Omega^0$ is a commutative $\mathcal{K}$-ring, and $\Omega^*$ is an $\Omega^0$-ring. $\Box$

Any commutative $\mathbb{N}$-graded ring $\mathcal{A}^*$ can be regarded as an even $\mathbb{N}$-graded commutative ring $\Lambda^*$ such that $\Lambda^{2i} = \mathcal{A}^i$, $\Lambda^{2i+1} = 0$. Obviously, an even $\mathbb{N}$-graded commutative ring is a commutative $\mathcal{K}$-ring. Therefore, studying the differential calculus over $\mathbb{N}$-graded rings (Section 6), we refer to that over commutative rings (Sections 2 – 4)

DEFINITION 1.4: An $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\Omega^*$ is said to be generated in degree 2 if $\Omega^0 = \mathcal{K}$, $\Omega^1 = 0$, and $\Omega^* = \mathcal{P}[\Omega^2]$ is a polynomial $\mathcal{K}$-ring of a $\mathcal{K}$-module $\Omega^2$ (Example 2.3). It is an even $\mathbb{N}$-graded commutative ring $\Omega^*$ where $\Omega^{2i+1} = 0$, $i \in \mathbb{N}$. $\Box$

A polynomial $\mathcal{K}$-ring $\mathcal{P}[Q]$ of a $\mathcal{K}$-module $Q$ in Example 2.3 exemplifies a commutative $\mathbb{N}$-graded ring. If $\mathcal{K}$ is a field and $Q$ is a free $\mathcal{K}$-module of finite rank, a polynomial $\mathcal{K}$-ring $\mathcal{P}[Q]$ is finitely generated in degree 2 by virtue of Definition 6.1. If $\mathcal{K}$ is a field, all $\mathbb{N}$-graded structures of this ring are mutually isomorphic in accordance with Theorem 6.1.

An $\mathbb{N}$-graded commutative ring $\Omega^*$ possesses an associated $\mathbb{Z}_2$-graded commutative structure

$$\Omega = \Omega_0 \oplus \Omega_1, \qquad \Omega_0 = \bigoplus_k \Omega^{2k}, \qquad \Omega_0 = \bigoplus_k \Omega^{2k+1}, \qquad k \in \mathbb{N}, \quad (1.4)$$

$$\alpha \cdot \beta = (-1)^{[\alpha][\beta]} \beta \cdot \alpha, \qquad \alpha, \beta \in \Omega_*, \quad (1.5)$$

where the symbol [.] stands for the $\mathbb{Z}_2$-degree (Definition 5.3).

In view of this fact, we also consider the differential calculus over $\mathbb{Z}_2$-graded commutative rings (Section 5.2), but focus our consideration on Grassmann algebras (Definition 5.4). They are the case of $\mathbb{N}$-graded commutative rings of the following type.

DEFINITION 1.5: An $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\Omega = \Omega^*$ is called the Grassmann-graded $\mathcal{K}$-ring if it is finitely generated in degree 1 (Definition 6.1), i.e., the following hold:

- $\Omega^0 = \mathcal{K}$,
- $\Omega^1$ is a free $\mathcal{K}$-module of finite rank,
- $\Omega^*$ is generated by $\Omega^1$, namely, if $R$ is an ideal generated by $\Omega^1$, then there are $\mathcal{K}$-module isomorphism $\Omega/R = \mathcal{K}$, $R/R^2 = \Omega^1$. $\Box$

Let us note that, a Grassmann-graded $\mathcal{K}$-ring $\Omega^*$ seen as a $\mathbb{Z}_2$-graded commutative ring $\Omega$ can admit different Grassmann-graded structures $\Omega^*$ and $\Omega'^*$. However, since it is finitely generated in degree 1 (Definition 6.1), all these structures mutually are isomorphic in accordance with Theorem 6.1 if $\mathcal{K}$ is a field.

We follow the conventional technique of the differential calculus over commutative rings, including formalism of differential operators on modules over commutative rings, the Chevalley–Eilenberg differential calculus over commutative rings, and the apparat of connections on modules and rings (Section 2)

[15, 18, 22, 39]. In particular, this is a case of the conventional differential calculus on smooth manifolds (Section 4).

One can generalize the Chevalley–Eilenberg differential calculus to a case of an arbitrary ring (Section 9) [11, 16, 39]. However, an extension of the notion of a differential operator to modules over a non-commutative ring meets difficulties [16, 39]. A key point is that a multiplication in a non-commutative ring is not a zero-order differential operator.

One overcomes this difficulty in a case of $\mathbb{Z}_2$-graded and $\mathbb{N}$-graded commutative rings by means of a reformulation of the notion of differential operators (Remark 5.5). As a result, the differential calculus technique has been extended to $\mathbb{Z}_2$-graded commutative rings (Section 5.2) [16, 39, 40].

Since any $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ possesses a structure of a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$, and commutation relations (1.3) of its elements depend of their $\mathbb{Z}_2$-gradation degree, but not the $\mathbb{N}$-gradation one, the differential calculus on $\mathbb{N}$-graded modules over $\mathbb{N}$-graded commutative rings is defined just as that over $\mathbb{Z}_2$-graded commutative rings (Section 5.2).

However, it should be emphasized that an $\mathbb{N}$-graded differential operator is an $\mathbb{N}$-graded $\mathcal{K}$-module homomorphism which obeys the conditions (5.13), i.e., it is a sum of homogeneous morphisms of fixed $\mathbb{N}$-degrees, but not the $\mathbb{Z}_2$ ones. Therefore, any $\mathbb{N}$-graded differential operator also is a $\mathbb{Z}_2$-graded differential operator, but the converse might not be true (Remark 6.1).

Let us note that the differential calculus over $\mathbb{Z}_2$-graded commutative rings, namely, Grassmann algebras (Definition 5.4) provides the mathematical formulation of Lagrangian formalism of even and odd variables on $\mathbb{Z}_2$-graded manifolds and bundles [4, 17, 40, 43].

There are different approaches to formulating graded manifolds [2, 8, 12, 37, 52]. We follow their definition in terms of local-ringed space (Section 3.1) and consider local-ringed spaces whose stalks are Grassmann-graded rings (Section 6). Since Grassmann-graded rings also are Grassmann algebras, we follow formalism of $\mathbb{Z}_2$-graded manifolds in Section 5.3.

Let $Z$ be an $n$-dimensional real smooth manifold. Let $\mathcal{A}$ be a real Grassmann-graded ring. By virtue of Theorem 5.2, it is isomorphic to the exterior algebra $\wedge W$ of a real vector space $W = \mathcal{A}^1$. Therefore, we come to Definition 6.2 of an $\mathbb{N}$-graded manifold as a simple $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A}_E)$ modelled over some vector bundle $E \to Z$ (Definition 5.6). In Section 5.3 on $\mathbb{Z}_2$-graded manifolds, we have restricted our consideration to simple graded manifolds, and this Section, in fact, presents formalism of $\mathbb{N}$-graded manifolds (Remark 6.1).

## 2 Differential calculus over commutative rings

Any commutative ring can be seen as an even graded commutative ring whose components $\Omega^{k>0}$ are empty. Conversely, every even graded commutative ring is commutative.

Therefore we start with the differential calculus over commutative rings. As was mentioned above, this is a case of the conventional differential calculus on smooth manifolds (Section 4). This technique fails to be extended straightforwardly to non-commutative rings (Section 9), unless this is a case of graded commutative rings (Sections 5 – 6).

## 2.1 Commutative algebra

In this Section, the relevant basics on modules over commutative rings are summarized [24, 25, 27, 29].

An algebra $\mathcal{A}$ is defined to be an additive group which additionally is provided with distributive multiplication. As was mentioned above, all algebras throughout are associative, unless they are Lie and graded Lie algebras. By a ring is meant a unital algebra, i.e., it contains a unit element $\mathbf{1} \neq 0$. Non-zero elements of a ring $\mathcal{A}$ form the multiplicative monoid $M\mathcal{A} \subset \mathcal{A}$ of $\mathcal{A}$. If it is a group, $\mathcal{A}$ is called the division ring. A field is a commutative division ring.

A ring $\mathcal{A}$ is said to be the domain (the integral domain or the entire ring in commutative algebra) if it has no a divisor of zero, i.e., $ab = 0$, $a, b \in \mathcal{A}$, implies either $a = 0$ or $b = 0$. For instance, a division ring is a domain, and a field is an integral domain. A polynomial $\mathcal{A}$-ring over an integral domain $\mathcal{A}$ (Example 2.3) is an integral domain.

A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be the left (resp. right) ideal if it is a subgroup of an additive group $\mathcal{A}$ and $ab \in \mathcal{I}$ (resp. $ba \in \mathcal{I}$) for all $a \in \mathcal{A}$, $b \in \mathcal{I}$. If $\mathcal{I}$ is both a left and right ideal, it is called the two-sided ideal. For instance, any ideal of a commutative algebra is two-sided. An ideal is a subalgebra, but a proper ideal (i.e., $\mathcal{I} \neq \mathcal{A}$) of a ring is not a subring. A proper ideal of a ring is said to be maximal if it does not belong to another proper ideal. A proper ideal $\mathcal{I}$ of a ring $\mathcal{A}$ is called completely prime (prime in commutative algebra) if $ab \in \mathcal{I}$ implies either $a \in \mathcal{I}$ or $b \in \mathcal{I}$. Any maximal ideal of a commutative ring is prime.

Given a two-sided ideal $\mathcal{I} \subset \mathcal{A}$, an additive factor group $\mathcal{A}/\mathcal{I}$ is an algebra, called the factor algebra. If $\mathcal{A}$ is a ring, then $\mathcal{A}/\mathcal{I}$ is so. If $\mathcal{A}$ is a commutative ring and $\mathcal{I}$ is its prime ideal, the factor ring $\mathcal{A}/\mathcal{I}$ is an entire ring, and it is a field if $\mathcal{I}$ is a maximal ideal.

DEFINITION 2.1: A ring $\mathcal{A}$ is called local if it has a unique maximal two-sided ideal. This ideal consists of all non-invertible elements of $\mathcal{A}$. $\Box$

**Remark 2.1:** Local rings conventionally are defined in commutative algebra [24, 25]. This notion has been extended to $\mathbb{Z}_2$-graded commutative rings too [2]. Any division ring, in particular, a field is local. Its unique maximal ideal contains only zero. Grassmann-graded rings in Definition 1.5 and Grassmann algebras in Definition 5.4 are local. $\Box$

**Remark 2.2:** One can associate a local ring to any commutative ring $\mathcal{A}$ as follows. Let $S \subset M\mathcal{A}$ be a multiplicative subset of $\mathcal{A}$, i.e., a submonoid of the multiplicative monoid $M\mathcal{A}$ of $\mathcal{A}$. Let us say that two pairs $(a, s)$ and $(a', s')$, $a, a' \in \mathcal{A}$, $s, s' \in S$, are equivalent if there exists an element $s'' \in S$ such that

$$s''(s'a - sa') = 0.$$

We abbreviate with $a/s$ the equivalence classes of $(a, s)$. A set $S^{-1}\mathcal{A}$ of these equivalence classes is a local commutative ring with respect to operations

$$s/a + s'/a' = (s'a + sa')/(ss'), \qquad (a/s) \cdot (a'/s') = (aa')/(ss').$$

There is a homomorphism

$$\Phi_S : \mathcal{A} \ni \mapsto a/\mathbf{1} \in S^{-1}\mathcal{A} \tag{2.1}$$

such that any element of $\Phi_S(S)$ is invertible in $S^{-1}\mathcal{A}$. If a ring $\mathcal{A}$ has no divisor of zero and $S$ does not contain a zero element, then $\Phi_S$ (2.1) is a monomorphism. In particular, if $S = M\mathcal{A}$, the ring $S^{-1}\mathcal{A}$ is a field, called the field of quotients or the fraction field of $\mathcal{A}$. If $\mathcal{A}$ is a field, its fraction field coincides with $\mathcal{K}$. □

Given an algebra $\mathcal{A}$, an additive group $P$ is said to be the left (resp. right) $\mathcal{A}$-module if it is provided with distributive multiplication $\mathcal{A} \times P \to P$ by elements of $\mathcal{A}$ such that $(ab)p = a(bp)$ (resp. $(ab)p = b(ap)$) for all $a, b \in \mathcal{A}$ and $p \in P$. If $\mathcal{A}$ is a ring, one additionally assumes that $\mathbf{1}p = p = p\mathbf{1}$ for all $p \in P$. Left and right module structures are usually written by means of left and right multiplications $(a, p) \to ap$ and $(a, p) \to pa$, respectively. If $P$ is both a left module over an algebra $\mathcal{A}$ and a right module over an algebra $\mathcal{A}'$, it is called the $(\mathcal{A} - \mathcal{A}')$-bimodule (the $\mathcal{A}$-bimodule if $\mathcal{A} = \mathcal{A}'$). If $\mathcal{A}$ is a commutative algebra, an $\mathcal{A}$-bimodule $P$ is said to be commutative if $ap = pa$ for all $a \in \mathcal{A}$ and $p \in P$. Any left or right module over a commutative algebra $\mathcal{A}$ can be brought into a commutative bimodule. Therefore, unless otherwise stated (Section 2.2), any $\mathcal{A}$-module over a commutative algebra is a commutative $\mathcal{A}$-bimodule, which is called the $\mathcal{A}$-module if there is no danger of confusion.

A module over a field is called the vector space. If an algebra $\mathcal{A}$ is a commutative bimodule over a ring $\mathcal{K}$ (i.e., a commutative $\mathcal{K}$-bimodule), it is said to be the $\mathcal{K}$-algebra. Any algebra can be seen as a $\mathbb{Z}$-algebra.

Hereafter, by $\mathcal{A}$ in this Section is meant a commutative ring.

The following are standard constructions of new modules from the old ones.

• A direct sum $P_1 \oplus P_2$ of $\mathcal{A}$-modules $P_1$ and $P_2$ is an additive group $P_1 \times P_2$ provided with an $\mathcal{A}$-module structure

$$a(p_1, p_2) = (ap_1, ap_2), \qquad p_{1,2} \in P_{1,2}, \qquad a \in \mathcal{A}.$$

Let $\{P_i\}_{i \in I}$ be a set of $\mathcal{A}$-modules. Their direct sum $\oplus P_i$ consists of elements $(\ldots, p_i, \ldots)$ of the Cartesian product $\prod P_i$ such that $p_i \neq 0$ at most for a finite number of indices $i \in I$.

• A tensor product $P \otimes Q$ of $\mathcal{A}$-modules $P$ and $Q$ is an additive group which is generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying relations

$$(p + p') \otimes q = p \otimes q + p' \otimes q, \quad p \otimes (q + q') = p \otimes q + p \otimes q',$$
$$pa \otimes q = p \otimes aq, \qquad p \in P, \qquad q \in Q, \qquad a \in \mathcal{A}.$$

It is provided with an $\mathcal{A}$-module structure

$$a(p \otimes q) = (ap) \otimes q = p \otimes (qa) = (p \otimes q)a.$$

If a ring $\mathcal{A}$ is treated as an $\mathcal{A}$-module, a tensor product $\mathcal{A} \otimes_{\mathcal{A}} Q$ is canonically isomorphic to $Q$ via the assignment

$$\mathcal{A} \otimes_{\mathcal{A}} Q \ni a \otimes q \leftrightarrow aq \in Q.$$

**Example 2.3:** Let $Q$ be an $\mathcal{A}$-module. We denote $Q^{\otimes k} = \overset{k}{\otimes} Q$. Let us consider an $\mathbb{N}$-graded module

$$\otimes Q = \mathcal{A} \oplus Q \oplus \cdots \oplus Q^{\otimes k} \oplus \cdots . \tag{2.2}$$

It is an $\mathbb{N}$-graded $\mathcal{A}$-algebra with respect to a tensor product $\otimes$. It is called the tensor algebra of an $\mathcal{A}$-module $Q$. Its quotient $\wedge Q$ with respect to an ideal generated by elements $q \otimes q' + q' \otimes q$, $q, q' \in Q$, is an $\mathbb{N}$-graded commutative algebra, called the exterior algebra of an $\mathcal{A}$-module $Q$. The quotient $\mathcal{P}[Q] = \vee Q$ of $\otimes Q$ (2.2) with respect to an ideal generated by elements $pq \otimes q' - q' \otimes q$, $q, q' \in Q$, is called the polynomial $\mathcal{A}$-ring of an $\mathcal{A}$-module $Q$. This is an even $\mathbb{N}$-graded commutative ring. □

• Given a submodule $Q$ of an $\mathcal{A}$-module $P$, the quotient $P/Q$ of an additive group $P$ with respect to its subgroup $Q$ also is provided with an $\mathcal{A}$-module structure. It is called the factor module.

• A set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of $\mathcal{A}$-linear morphisms of an $\mathcal{A}$-module $P$ to an $\mathcal{A}$-module $Q$ is naturally an $\mathcal{A}$-module. An $\mathcal{A}$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of an $\mathcal{A}$-module $P$. There is a natural monomorphism $P \to P^{**}$.

An $\mathcal{A}$-module $P$ is called free if it admits a basis, i.e., a linearly independent subset $I \subset P$ spanning $P$ such that each element of $P$ has a unique expression as a linear combination of elements of $I$ with a finite number of non-zero coefficients from a ring $\mathcal{A}$. Any module over a division ring, e.g., a vector space is free. Every module is isomorphic to a quotient of a free module. A module is said to be finitely generated (or of finite rank) if it is a quotient of a free module with a finite basis.

One says that a module $P$ is projective if it is a direct summand of a free module, i.e., there exists a module $Q$ such that $P \oplus Q$ is a free module. A module $P$ is projective iff $P = \mathbf{p}S$ where $S$ is a free module and $\mathbf{p}$ is a projector of $S$, i.e., $\mathbf{p}^2 = \mathbf{p}$.

THEOREM 2.1: If $P$ is a projective module of finite rank, then its dual $P^*$ is so, and $P^{**}$ is isomorphic to $P$. □

THEOREM 2.2: Any projective module over a local commutative ring is free. □

The forthcoming constructions are extended to a case of modules over graded commutative rings (Remark 2.5).

A composition of module morphisms

$$P \xrightarrow{i} Q \xrightarrow{j} T$$

is said to be exact at $Q$ if $\operatorname{Ker} j = \operatorname{Im} i$. A composition of module morphisms

$$0 \to P \xrightarrow{i} Q \xrightarrow{j} T \to 0 \tag{2.3}$$

is called the short exact sequence if it is exact at all the terms $P$, $Q$, and $T$. This condition implies that: (i) $i$ is a monomorphism, (ii) $\operatorname{Ker} j = \operatorname{Im} i$, and (iii) $j$ is an epimorphism onto a factor module $T = Q/P$.

THEOREM 2.3: Given an exact sequence of modules (2.3) and another $\mathcal{A}$-module $R$, the sequence of modules

$$0 \to \operatorname{Hom}_{\mathcal{A}}(T, R) \xrightarrow{j^*} \operatorname{Hom}_{\mathcal{A}}(Q, R) \xrightarrow{i^*} \operatorname{Hom}(P, R)$$

is exact at the first and second terms, i.e., $j^*$ is a monomorphism, but $i^*$ need not be an epimorphism. $\square$

One says that the exact sequence (2.3) is split if there exists a monomorphism $s : T \to Q$ such that $j \circ s = \operatorname{Id} T$ or, equivalently,

$$Q = i(P) \oplus s(T) \cong P \oplus T.$$

THEOREM 2.4: The exact sequence (2.3) always is split if $T$ is a projective module. $\square$

**Remark 2.4:** A directed set $I$ is a set with an order relation $<$ which satisfies the following three conditions:
(i) $i < i$, for all $i \in I$;
(ii) if $i < j$ and $j < k$, then $i < k$;
(iii) for any $i, j \in I$, there exists $k \in I$ such that $i < k$ and $j < k$.
It may happen that $i \neq j$, but $i < j$ and $j < i$ simultaneously. $\square$

A family of $\mathcal{A}$-modules $\{P_i\}_{i \in I}$, indexed by a directed set $I$, is called the direct system if, for any pair $i < j$, there exists a morphism $r^i_j : P_i \to P_j$ such that

$$r^i_i = \operatorname{Id} P_i, \qquad r^i_j \circ r^j_k = r^i_k, \qquad i < j < k.$$

A direct system of modules admits a direct limit. This is an $\mathcal{A}$-module $P_\infty$ together with morphisms $r^i_\infty : P_i \to P_\infty$ such that $r^i_\infty = r^j_\infty \circ r^i_j$ for all $i < j$. A module $P_\infty$ consists of elements of a direct sum $\oplus P_i$ modulo the identification of elements of $P_i$ with their images in $P_j$ for all $i < j$. In particular, a direct system

$$P_0 \longrightarrow P_1 \cdots \longrightarrow P_i \xrightarrow{r^i_{i+1}} \cdots, \qquad I = \mathbb{N},$$

indexed by $\mathbb{N}$, is called the direct sequence.

**Remark 2.5:** It should be noted that direct limits also exist in the categories of commutative and graded commutative algebras and rings, but not in categories whose objects are non-commutative groups. $\square$

A morphism of a direct system $\{P_i, r^i_j\}_I$ to a direct system $\{Q_{i'}, \rho^{i'}_{j'}\}_{I'}$ consists of an order preserving map $f : I \to I'$ and $\mathcal{A}$-module morphisms $F_i : P_i \to Q_{f(i)}$ which obey compatibility conditions

$$\rho^{f(i)}_{f(j)} \circ F_i = F_j \circ r^i_j. \tag{2.4}$$

If $P_\infty$ and $Q_\infty$ are direct limits of these direct systems, there exists a unique $\mathcal{A}$-module morphism $F_\infty : P_\infty \to Q_\infty$ such that

$$\rho^{f(i)}_\infty \circ F_i = F_\infty \circ r^i_\infty, \qquad i \in I.$$

PROPOSITION 2.5: A construction of a direct limit morphism preserve monomorphisms, epimorphisms and, consequently, isomorphisms. Namely, if all $F_i : P_i \to Q_{f(i)}$ are monomorphisms (resp. epimorphisms and isomorphisms), so is $F_\infty : P_\infty \to Q_\infty$. □

**Example 2.6:** In particular, let $\{P_i, r^i_j\}_I$ be a direct system of $\mathcal{A}$-modules and $Q$ an $\mathcal{A}$-module together with $\mathcal{A}$-module morphisms $F_i : P_i \to Q$ which obey the compatibility conditions $F_i = F_j \circ r^i_j$ (2.4). Then there exists an $\mathcal{A}$-module morphism $F_\infty : P_\infty \to Q$ such that $F_i = F_\infty \circ r^i_\infty$ for any $i \in I$. If all $F_i$ are monomorphisms or epimorphisms, so is $F_\infty$. □

THEOREM 2.6: Direct limits commute with direct sums and tensor products of modules. Namely, let $\{P_i\}$ and $\{Q_i\}$ be two direct systems of $\mathcal{A}$-modules which are indexed by the same directed set $I$, and let $P_\infty$ and $Q_\infty$ be their direct limits. Then direct limits of direct systems $\{P_i \oplus Q_i\}$ and $\{P_i \otimes Q_i\}$ are $P_\infty \oplus Q_\infty$ and $P_\infty \otimes Q_\infty$, respectively. □

THEOREM 2.7: Let short exact sequences

$$0 \to P_i \xrightarrow{F_i} Q_i \xrightarrow{\Phi_i} T_i \to 0 \tag{2.5}$$

for all $i \in I$ define a short exact sequence of direct systems of modules $\{P_i\}_I$, $\{Q_i\}_I$, and $\{T_i\}_I$ which are indexed by the same directed set $I$. Then there exists a short exact sequence of their direct limits

$$0 \to P_\infty \xrightarrow{F_\infty} Q_\infty \xrightarrow{\Phi_\infty} T_\infty \to 0. \tag{2.6}$$

□

In particular, a direct limit of factor modules $Q_i/P_i$ is a factor module $Q_\infty/P_\infty$. By virtue of Theorem 2.6, if all the exact sequences (2.5) are split, the exact sequence (2.6) is well.

In a case of inverse systems of modules, we restrict our consideration to inverse sequences

$$P^0 \longleftarrow P^1 \longleftarrow \cdots P^i \overset{\pi^{i+1}_i}{\longleftarrow} \cdots .$$

Its inverse limit is a module $P^\infty$ together with morphisms $\pi_i^\infty : P^\infty \to P^i$ so that $\pi_i^\infty = \pi_i^j \circ \pi_j^\infty$ for all $i < j$. It consists of elements $(\ldots, p^i, \ldots)$, $p^i \in P^i$, of the Cartesian product $\prod P^i$ such that $p^i = \pi_i^j(p^j)$ for all $i < j$.

A morphism of an inverse system $\{P_i, \pi_j^i\}$ to an inverse system $\{Q_i, \varrho_j^i\}$ consists of $\mathcal{A}$-module morphisms $F_i : P_i \to Q_i$ which obey compatibility conditions

$$F_j \circ \pi_j^i = \varrho_j^i \circ F_i. \tag{2.7}$$

If $P_\infty$ and $Q_\infty$ are inverse limits of these inverse systems, there exists a unique $\mathcal{A}$-module morphism $F_\infty : P_\infty \to Q_\infty$ such that

$$F_j \circ \pi_j^\infty = \varrho_j^\infty \circ F_\infty.$$

Proposition 2.8: Inverse limits preserve monomorphisms, but not epimorphisms. □

**Example 2.7:** In particular, let $\{P_i, \pi_j^i\}$ be an inverse system of $\mathcal{A}$-modules and $Q$ an $\mathcal{A}$-module together with $\mathcal{A}$-module morphisms $F_i : Q \to P_i$ which obey compatibility conditions $F_j = \pi_j^i \circ F_i$. Then there exists a unique morphism $F_\infty : Q \to P_\infty$ such that $F_j = \pi_j^\infty \circ F_\infty$. □

**Example 2.8:** Let $\{P_i, \pi_j^i\}$ be an inverse system of $\mathcal{A}$-modules and $Q$ an $\mathcal{A}$-module. Given a term $P_r$, let $\Phi_r : P_r \to Q$ be an $\mathcal{A}$-module morphism. It yields the pull-back morphisms

$$\pi_r^{r+k*}\Phi_r = \Phi_r \circ \pi_r^{r+k} : P_{r+k} \to Q \tag{2.8}$$

which obviously obey the compatibility conditions (2.7). Then there exists a unique morphism $\Phi_\infty : P_\infty \to Q$ such that $\Phi_\infty = \Phi_r \circ \pi_r^\infty$. □

Theorem 2.9: If a sequence

$$0 \to P^i \xrightarrow{F^i} Q^i \xrightarrow{\Phi^i} T^i, \qquad i \in \mathbb{N},$$

of inverse systems of $\mathcal{A}$-modules $\{P^i\}$, $\{Q^i\}$ and $\{T^i\}$ is exact, so is a sequence of inverse limits

$$0 \to P^\infty \xrightarrow{F^\infty} Q^\infty \xrightarrow{\Phi^\infty} T^\infty.$$

□

In contrast with direct limits (Remark 2.5), the inverse ones exist in the category of groups which need not be commutative.

**Example 2.9:** Let $\{P_i\}$ be a direct sequence of $\mathcal{A}$-modules. Given an $\mathcal{A}$-module $Q$, modules $\mathrm{Hom}_{\mathcal{A}}(P_i, Q)$ constitute an inverse sequence such that its inverse limit is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P_\infty, Q)$. □

**Example 2.10:** Let $\{P_i\}$ be an inverse sequence of $\mathcal{A}$-modules. Given an $\mathcal{A}$-module $Q$, modules $\mathrm{Hom}_{\mathcal{A}}(P_i, Q)$ constitute a direct sequence such that its direct limit is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P_\infty, Q)$. □

## 2.2 Differential operators on modules and rings

This Section addresses the notion of (linear) differential operators on modules over commutative rings [15, 18, 22, 39].

As was mentioned above, $\mathcal{K}$ throughout is a commutative ring without a divisor of zero. Let $\mathcal{A}$ be a commutative $\mathcal{K}$-ring, and let $P$ and $Q$ be $\mathcal{A}$-modules. A $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P. \tag{2.9}$$

For the sake of convenience, we will refer to the second one as an $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a \Phi = a\Phi - \Phi \bullet a, \qquad a \in \mathcal{A}. \tag{2.10}$$

Definition 2.2: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called the $s$-order $Q$-valued differential operator on $P$ if

$$(\delta_{a_0} \circ \cdots \circ \delta_{a_s})\Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. □

A set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (2.9). Of course, an $s$-order differential operator also is of $(s+1)$-order.

In particular, zero-order differential operators obey a condition

$$\delta_a \Delta(p) = a\Delta(p) - \Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

and, consequently, they coincide with $\mathcal{A}$-module morphisms $P \to Q$. A first-order differential operator $\Delta$ satisfies a condition

$$(\delta_b \circ \delta_a)\Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}. \tag{2.11}$$

The following fact reduces the study of $Q$-valued differential operators on an $\mathcal{A}$-module $P$ to that of $Q$-valued differential operators on a ring $\mathcal{A}$ seen as an $\mathcal{A}$-module.

Proposition 2.10: Let us consider an $\mathcal{A}$-module morphism

$$h_s : \mathrm{Diff}_s(\mathcal{A}, Q) \to Q, \qquad h_s(\Delta) = \Delta(\mathbf{1}). \tag{2.12}$$

Any $s$-order $Q$-valued differential operator $\Delta \in \mathrm{Diff}_s(P,Q)$ on $P$ uniquely factorizes as

$$\Delta : P \xrightarrow{\mathfrak{f}_\Delta} \mathrm{Diff}_s(\mathcal{A}, Q) \xrightarrow{h_s} Q$$

through the morphism $h_s$ (2.12) and some homomorphism

$$\mathfrak{f}_\Delta : P \to \mathrm{Diff}_s(\mathcal{A}, Q), \qquad (\mathfrak{f}_\Delta p)(a) = \Delta(ap), \qquad a \in \mathcal{A},$$

of an $\mathcal{A}$-module $P$ to an $\mathcal{A}^\bullet$-module $\mathrm{Diff}_s(\mathcal{A},Q)$ [22]. The assignment $\Delta\to\mathfrak{f}_\Delta$ defines an isomorphism

$$\mathrm{Diff}_s(P,Q)=\mathrm{Hom}_{\mathcal{A}-\mathcal{A}^\bullet}(P,\mathrm{Diff}_s(\mathcal{A},Q)).$$

□

Therefore, let $P=\mathcal{A}$. Any zero-order $Q$-valued differential operator $\Delta$ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an $\mathcal{A}$-module isomorphism $\mathrm{Diff}_0(\mathcal{A},Q)=Q$ via the association

$$Q\ni q\to\Delta_q\in\mathrm{Diff}_0(\mathcal{A},Q),$$

where $\Delta_q$ is given by an equality $\Delta_q(\mathbf{1})=q$.

A first-order $Q$-valued differential operator $\Delta$ on $\mathcal{A}$ fulfils a condition

$$\Delta(ab)=b\Delta(a)+a\Delta(b)-ba\Delta(\mathbf{1}),\qquad a,b\in\mathcal{A}.$$

Definition 2.3: It is called the $Q$-valued derivation of $\mathcal{A}$ if $\Delta(\mathbf{1})=0$, i.e., the Leibniz rule

$$\Delta(ab)=\Delta(a)b+a\Delta(b),\qquad a,b\in\mathcal{A},\tag{2.13}$$

holds. □

One obtains at once that any first-order differential operator on $\mathcal{A}$ falls into a sum

$$\Delta(a)=a\Delta(\mathbf{1})+[\Delta(a)-a\Delta(\mathbf{1})]$$

of a zero-order differential operator $a\Delta(\mathbf{1})$ and a derivation $\Delta(a)-a\Delta(\mathbf{1})$. If $\partial$ is a derivation of $\mathcal{A}$, then $a\partial$ is well for any $a\in\mathcal{A}$. Hence, derivations of $\mathcal{A}$ constitute an $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A},Q)$, called the derivation module. There is an $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A},Q)=Q\oplus\mathfrak{d}(\mathcal{A},Q).\tag{2.14}$$

If $Q=\mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A}=\mathfrak{d}\mathcal{A}(\mathcal{A},\mathcal{A})$ of $\mathcal{A}$ also is a Lie algebra over a ring $\mathcal{K}$ with respect to a Lie bracket

$$[u,u']=u\circ u'-u'\circ u,\qquad u,u'\in\mathfrak{d}\mathcal{A}.\tag{2.15}$$

Accordingly, the decomposition (2.14) takes a form

$$\mathrm{Diff}_1(\mathcal{A})=\mathcal{A}\oplus\mathfrak{d}\mathcal{A}.\tag{2.16}$$

Since, an $s$-order differential operator also is of $(s+1)$-order, we have a direct sequence

$$\mathrm{Diff}_0(P,Q)\xrightarrow{\mathrm{in}}\mathrm{Diff}_1(P,Q)\cdots\xrightarrow{\mathrm{in}}\mathrm{Diff}_r(P,Q)\longrightarrow\cdots\tag{2.17}$$

of $Q$-valued differential operators on an $\mathcal{A}$-module $P$. Its direct limit is an $\mathcal{A}-\mathcal{A}^\bullet$-module $\mathrm{Diff}_\infty(P,Q)$ of all $Q$-valued differential operators on $P$.

**Example 2.11:** Let $Q$ be a free $\mathcal{K}$-module of finite rank and $\mathcal{P}[Q]$ a polynomial ring of $Q$ (Example 2.3). Any differential operator on $\mathcal{P}[Q]$ is a composition of derivations. Every derivation of $\mathcal{P}[Q]$ is defined by its action in $Q$. Let $\{q^i\}$ be a basis for $Q$. Let us consider derivations

$$\partial_i(q^j)=\delta_i^j, \qquad \partial_i\circ\partial_j=\partial_j\circ\partial_i. \tag{2.18}$$

Then any derivation of $\mathcal{P}[Q]$ takes a form

$$u=u^i\partial_i, \qquad u_i\in\mathcal{P}[Q]. \tag{2.19}$$

Derivations (2.19) constitute a free $\mathcal{P}[Q]$-module $\mathfrak{d}\mathcal{P}[Q]$ of finite rank. It is also a Lie algebra over $\mathcal{K}$ with respect to the Lie bracket (2.15). □

## 2.3 Jets of modules and rings

An $s$-order differential operator on an $\mathcal{A}$-module $P$ is represented by a zero-order differential operator on a module of $s$-order jets of $P$ (Theorems 2.11 and 2.12). We also use modules of jets in order to define differential forms over a ring (Remark 2.12) and connections on modules and rings (Section 2.5), and ringed spaces (Section 3.1). Afterwards, we however can leave jets of modules. Firstly, the cochain complex (2.41) of differential forms over a $\mathcal{K}$-ring $\mathcal{A}$ coincides with the minimal Chevalley–Eilenberg differential calculus (2.59) over $\mathcal{A}$ (Theorem 2.13). Secondly, Definition 2.7 of connections on an $\mathcal{A}$-module $P$ leads to an equivalent to Definition 2.8 which does not involve jets. Thirdly, jets of projective modules of finite rank over a ring $C^\infty(X)$ of smooth real functions on a manifold $X$ are jets of sections of vector bundles over $X$ (corollary (iv) of Theorem 4.4).

Given an $\mathcal{A}$-module $P$, let us consider a tensor product $\mathcal{A}\otimes_\mathcal{K}P$ of $\mathcal{K}$-modules $\mathcal{A}$ and $P$. We put

$$\delta^b(a\otimes p)=(ba)\otimes p-a\otimes(bp), \qquad p\in P, \qquad a,b\in\mathcal{A}. \tag{2.20}$$

Let us denote by $\mu^{k+1}$ the submodule of $\mathcal{A}\otimes_\mathcal{K}P$ generated by elements of the type

$$\delta^{b_0}\circ\cdots\circ\delta^{b_k}(a\otimes p)=a\delta^{b_0}\circ\cdots\circ\delta^{b_k}(\mathbf{1}\otimes p).$$

DEFINITION 2.4: A $k$-order jet module $\mathcal{J}^k(P)$ of a module $P$ is defined as the quotient of a $\mathcal{K}$-module $\mathcal{A}\otimes_\mathcal{K}P$ by $\mu^{k+1}$. We denote its elements $c\otimes_k p$. □

In particular, a first-order jet module $\mathcal{J}^1(P)$ consists of elements $c\otimes_1 p$ modulo the relations

$$\delta^a\circ\delta^b(\mathbf{1}\otimes_1 p)=ab\otimes_1 p-b\otimes_1(ap)-a\otimes_1(bp)+\mathbf{1}\otimes_1(abp)=0. \tag{2.21}$$

A $\mathcal{K}$-module $\mathcal{J}^k(P)$ is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$b(a \otimes_k p) = ba \otimes_k p, \qquad b \bullet (a \otimes_k p) = a \otimes_k (bp). \tag{2.22}$$

There exists a module morphism

$$J^k : P \ni p \to \mathbf{1} \otimes_k p \in \mathcal{J}^k(P) \tag{2.23}$$

of an $\mathcal{A}$-module $P$ to an $\mathcal{A}^\bullet$-module $\mathcal{J}^k(P)$ such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by elements $J^k p$, $p \in P$.

The above mentioned relation between differential operators on modules and jets of modules is stated by the following theorem [15, 22].

THEOREM 2.11: Any $k$-order $Q$-valued differential operator $\Delta$ on an $\mathcal{A}$-module $P$ uniquely factorizes as

$$\Delta : P \xrightarrow{J^k} \mathcal{J}^k(P) \xrightarrow{\mathfrak{f}^\Delta} Q \tag{2.24}$$

through the morphism $J^k$ (2.23) and some $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta$ : $\mathcal{J}^k(P) \to Q$. The correspondence $\Delta \to \mathfrak{f}^\Delta$ defines an $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_k(P, Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^k(P), Q). \tag{2.25}$$

$\square$

Due to natural monomorphisms $\mu^r \to \mu^s$ for all $r > s$, there are $\mathcal{A}$-module epimorphisms of jet modules

$$\pi^{i+1}_i : \mathcal{J}^{i+1}(P) \to \mathcal{J}^i(P). \tag{2.26}$$

In particular,

$$\pi^1_0 : \mathcal{J}^1(P) \ni a \otimes_1 p \to ap \in P. \tag{2.27}$$

Thus, there is an inverse sequence

$$P \xleftarrow{\pi^1_0} \mathcal{J}^1(P) \cdots \xleftarrow{\pi^r_{r-1}} \mathcal{J}^r(P) \longleftarrow \cdots \tag{2.28}$$

of jet modules. Its inverse limit $\mathcal{J}^\infty(P)$ is an $\mathcal{A}$-module together with $\mathcal{A}$-module morphisms

$$\pi^\infty_r : \mathcal{J}^\infty(P) \to \mathcal{J}^r(P), \qquad \pi^\infty_{r<s} = \pi^s_r \circ \pi^\infty_s.$$

In particular, let us consider a module $P$ together with the morphisms $J^r$ (2.23) which obey compatibility conditions $J^r(p) = \pi^{r+k}_r \circ J^{r+k}(p)$, $p \in P$. Then it follows from Example 2.7 that there exists an $\mathcal{A}$-module morphism

$$J^\infty : P \ni p \to (p, J^1 p, \ldots, J^r p, \ldots) \in \mathcal{J}^\infty(P) \tag{2.29}$$

so that $J^r(p) = \pi^\infty_r \circ J^\infty(p)$.

The inverse sequence (2.28) yields a direct sequence

$$\mathrm{Hom}_{\mathcal{A}}(P,Q) \xrightarrow{\pi_0^{1*}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^1(P),Q) \cdots \xrightarrow{\pi_{r-1}^{r}{}^*} \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^r(P),Q) \cdots, \tag{2.30}$$

where

$$\pi_{r-1}^{r}{}^* : \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^{r-1}(P),Q) \ni \Delta \to \Delta\circ\pi_{r-1}^r \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^r(P),Q)$$

is the pull-back $\mathcal{A}$-module morphism (2.8). Its direct limit is an $\mathcal{A}$-module $\mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^\infty(P),Q)$ (Example 2.10).

THEOREM 2.12: We have the isomorphisms (2.25) of the direct systems (2.17) and (2.30) which leads to an $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_\infty(P,Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^\infty(P),Q) \tag{2.31}$$

of their direct limits in accordance with Proposition 2.5. □

*Proof:* Any element $\Delta_\infty = \Delta \in \mathrm{Diff}_\infty(P,Q)$ factorizes as

$$\Delta_\infty : P \xrightarrow{J^\infty} \mathcal{J}^\infty(P) \xrightarrow{\mathfrak{f}^\Delta_\infty} Q \tag{2.32}$$

through the morphism $J^\infty$ (2.29) and an $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta_\infty = \mathfrak{f}^\Delta\circ\pi^\infty_k$ (Example 2.8) so that the diagram

$$\begin{tikzcd} & \mathcal{J}^\infty(P) \arrow[d] \arrow[dr, "\mathfrak{f}^\Delta_\infty"] & \\ P \arrow[ur, "J^\infty"] \arrow[r, "J^k"'] & \mathcal{J}^k(P) \arrow[r, "\mathfrak{f}^\Delta"'] & Q \end{tikzcd}$$

is commutative. □

Let us consider jet modules $\mathcal{J}^s = \mathcal{J}^s(\mathcal{A})$ of a ring $\mathcal{A}$ itself. In particular, the first-order jet module $\mathcal{J}^1$ consists of the elements $a\otimes_1 b$, $a,b\in\mathcal{A}$, subject to the relations

$$ab\otimes_1 \mathbf{1} - b\otimes_1 a - a\otimes_1 b + \mathbf{1}\otimes_1 (ab) = 0. \tag{2.33}$$

The $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (2.22) on $\mathcal{J}^1$ read

$$c(a\otimes_1 b) = (ca)\otimes_1 b, \qquad c\bullet(a\otimes_1 b) = a\otimes_1 (cb) = (a\otimes_1 b)c.$$

Besides the monomorphism (2.23):

$$J^1 : \mathcal{A}\ni a \to \mathbf{1}\otimes_1 a \in \mathcal{J}^1,$$

there is an $\mathcal{A}$-module monomorphism

$$i_1 : \mathcal{A}\ni a\to a\otimes_1 \mathbf{1}\in\mathcal{J}^1.$$

With these monomorphisms, we have a canonical $\mathcal{A}$-module splitting

$$\begin{aligned}
&\mathcal{J}^1 = i_1(\mathcal{A}) \oplus \mathcal{O}^1, \qquad (2.34)\\
&aJ^1(b) = a \otimes_1 b = ab \otimes_1 \mathbf{1} + a(\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}),
\end{aligned}$$

where an $\mathcal{A}$-module $\mathcal{O}^1$ is generated by elements $\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}$ for all $b \in \mathcal{A}$. Let us consider the corresponding $\mathcal{A}$-module epimorphism

$$h^1 : \mathcal{J}^1 \ni \mathbf{1} \otimes_1 b \to \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1 \qquad (2.35)$$

and the composition

$$d^1 = h^1 \circ J^1 : \mathcal{A} \ni b \to \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1, \qquad (2.36)$$

which is a $\mathcal{K}$-module morphism. This is an $\mathcal{O}^1$-valued derivation of a $\mathcal{K}$-ring $\mathcal{A}$ which obeys the Leibniz rule

$$d^1(ab) = \mathbf{1} \otimes_1 ab - ab \otimes_1 \mathbf{1} + a \otimes_1 b - a \otimes_1 b = ad^1 b + (d^1 a)b.$$

It follows from the relation (2.33) that

$$ad^1 b = (d^1 b)a \qquad (2.37)$$

for all $a, b \in \mathcal{A}$. Thus, seen as an $\mathcal{A}$-module, $\mathcal{O}^1$ is generated by elements $d^1 a$ for all $a \in \mathcal{A}$.

Let $\mathcal{O}^{1*} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}^1, \mathcal{A})$ be the dual of an $\mathcal{A}$-module $\mathcal{O}^1$. In view of the splittings (2.16) and (2.34), the isomorphism (2.25) reduces to the duality relation

$$\mathfrak{d}\mathcal{A} = \mathcal{O}^{1*}, \qquad (2.38)$$

$$\mathfrak{d}\mathcal{A} \ni u \leftrightarrow f_u \in \mathcal{O}^{1*}, \qquad f_u(d^1 a) = u(a), \qquad a \in \mathcal{A}. \qquad (2.39)$$

However, a monomorphism $\mathcal{O}^1 \to \mathcal{O}^{1**} = \mathfrak{d}\mathcal{A}^*$ need not be an isomorphism.

**Remark 2.12:** In view of the relation (2.38), one thinks of elements of an $\mathcal{A}$-module $\mathcal{O}^1$ as being differential forms over a ring $\mathcal{A}$. □

Let us consider the exterior algebra $\mathcal{O}^* = \wedge \mathcal{O}^1$ of an $\mathcal{A}$-module $\mathcal{O}^1$ (Example 2.3). There exist the higher degree generalizations

$$\begin{aligned}
&h^k : \mathcal{J}^1(\mathcal{O}^{k-1}) \to \mathcal{O}^k,\\
&d^k = h^k \circ J^1 : \mathcal{O}^{k-1} \to \mathcal{O}^k \qquad (2.40)
\end{aligned}$$

of the morphisms (2.35) and (2.36). The operators (2.40) are nilpotent, i.e., $d^k \circ d^{k-1} = 0$. They form a cochain complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d^1} \mathcal{O}^1 \xrightarrow{d^2} \cdots \mathcal{O}^k \xrightarrow{d^{k+1}} \cdots, \qquad (2.41)$$

which is the minimal Chevalley–Eilenberg differential calculus (2.59) over a $\mathcal{K}$-ring $\mathcal{A}$ (Theorem 2.13).

## 2.4 Chevalley–Eilenberg differential calculus

We start with a general notion of the differential graded ring which is not necessarily commutative.

DEFINITION 2.5: An $\mathbb{N}$-graded ring $\Omega^*$ (Definition 1.2) is called the differential graded ring (henceforth, DGR) if it is a cochain complex of $\mathcal{K}$-modules

$$0 \to \mathcal{K} \longrightarrow \Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \Omega^k \xrightarrow{\delta} \cdots \tag{2.42}$$

with respect to a coboundary operators $\delta$ which obeys the graded Leibniz rule

$$\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{|\alpha|}\alpha \cdot \delta\beta. \tag{2.43}$$

□

In particular, $\delta : \Omega^0 \to \Omega^1$ is a $\Omega^1$-valued derivation of a $\mathcal{K}$-ring $\Omega^0$ (Definition 2.3).

The cochain complex (2.42) is called the de Rham complex of a differential graded ring $(\Omega^*, \delta)$. It also is said to be the differential graded calculus over a $\mathcal{K}$-ring $\Omega^0$. Cohomology $H^*(\Omega^*)$ of the complex (2.42) is called the de Rham cohomology of a differential graded ring $(\Omega^*, \delta)$.

Given a differential graded ring $(\Omega^*, \delta)$, one considers its minimal differential graded subring $(\overline{\Omega}^*, \delta)$ which contains $\Omega^0$. Seen as a $(\Omega^0 - \Omega^0)$-ring, it is generated by elements $\delta a$, $a \in \mathcal{A}$, and consists of monomials $\alpha = a_0\delta a_1 \cdots \delta a_k$, $a_i \in \Omega^0$, whose product obeys the juxtaposition rule

$$(a_0\delta a_1) \cdot (b_0\delta b_1) = a_0\delta(a_1 b_0) \cdot \delta b_1 - a_0 a_1 \delta b_0 \cdot \delta b_1$$

in accordance with the equality (2.43).

DEFINITION 2.6: A complex $(\overline{\Omega}^*, \delta)$ is called the minimal differential graded calculus over $\Omega^0$. □

Let us show that any commutative $\mathcal{K}$-ring $\mathcal{A}$ defines the differential graded calculus (2.45), called the Chevalley–Eilenberg differential calculus over $\mathcal{A}$.

Since the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra, let us consider the extended Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$ (8.4):

$$0 \to \mathcal{K} \xrightarrow{\text{in}} C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}], \tag{2.44}$$

of the Lie algebra $\mathfrak{d}\mathcal{A}$ with coefficients in a ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module [16, 39]. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$:

$$0 \to \mathcal{K} \xrightarrow{\text{in}} \mathcal{A} \xrightarrow{d} \mathcal{O}^1[\mathfrak{d}\mathcal{A}] \xrightarrow{d} \cdots, \tag{2.45}$$

of $\mathcal{A}$-multilinear skew-symmetric maps

$$\mathcal{O}^k[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\overset{k}{\times} \mathfrak{d}\mathcal{A}, \mathcal{A}) \ni \phi^k : \overset{k}{\times} \mathfrak{d}\mathcal{A} \to \mathcal{A} \tag{2.46}$$

with respect to the Chevalley–Eilenberg coboundary operator (8.5):

$$d\phi(u_0,\ldots,u_k)=\sum_{i=0}^{k}(-1)^i u_i(\phi(u_0,\ldots,\widehat{u}_i,\ldots,u_k))+ \qquad (2.47)$$
$$\sum_{i<j}(-1)^{i+j}\phi([u_i,u_j],u_0,\ldots,\widehat{u}_i,\ldots,\widehat{u}_j,\ldots,u_k).$$

Indeed, a direct verification shows that if $\phi$ (2.46) is an $\mathcal{A}$-multilinear map, $d\phi$ (2.47) also is well.

In particular,

$$(da)(u)=u(a), \qquad a\in\mathcal{O}^0[\mathfrak{d}\mathcal{A}]=\mathcal{A}, \qquad (2.48)$$
$$(d\phi)(u_0,u_1)=u_0(\phi(u_1))-u_1(\phi(u_0))-\phi([u_0,u_1]), \qquad (2.49)$$
$$\phi\in\mathcal{O}^1[\mathfrak{d}\mathcal{A}]=\mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A}).$$

It follows that $d(\mathbf{1})=0$, i.e., $d$ is an $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$-valued derivation of $\mathcal{A}$ (Definition 2.3).

Let us define an $\mathbb{N}$-graded module

$$\mathcal{O}^*[\mathfrak{d}\mathcal{A}]=\underset{i\in\mathbb{N}}{\oplus}\mathcal{O}^i[\mathfrak{d}\mathcal{A}]. \qquad (2.50)$$

It is provided with the structure of an $\mathbb{N}$-graded $\mathcal{A}$-ring with respect to a product

$$\phi\wedge\phi'(u_1,...,u_{r+s})= \qquad (2.51)$$
$$\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}),$$
$$\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}], \qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}], \qquad u_k\in\mathfrak{d}\mathcal{A},$$

where $\mathrm{sgn}^{\cdots}_{\cdots}$ denotes the sign of a permutation. This product obeys relations

$$d(\phi\wedge\phi')=d(\phi)\wedge\phi'+(-1)^{|\phi|}\phi\wedge d(\phi'), \quad \phi,\phi'\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \qquad (2.52)$$
$$\phi\wedge\phi'=(-1)^{|\phi||\phi'|}\phi'\wedge\phi. \qquad (2.53)$$

By virtue of the first one, $(\mathcal{O}^*[\mathfrak{d}\mathcal{A}],d)$ is a differential graded ring (Definition 2.5), called the Chevalley–Eilenberg differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$ [16, 39]. The relation (2.53) shows that $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is a graded commutative ring (Definition 1.3).

Since $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]=\mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A})=\mathfrak{d}\mathcal{A}^*$ and, consequently, $\mathfrak{d}\mathcal{A}\subset\mathfrak{d}\mathcal{A}^{**}=\mathcal{O}^1[\mathfrak{d}\mathcal{A}]^*$, we have the interior product

$$u\rfloor\phi=\phi(u), \qquad u\in\mathfrak{d}\mathcal{A}, \qquad \phi\in\mathcal{O}^1[\mathfrak{d}\mathcal{A}]. \qquad (2.54)$$

It is extended as

$$(u\rfloor\phi)(u_1,\ldots,u_{k-1})=k\phi(u,u_1,\ldots,u_{k-1}), \quad u\in\mathfrak{d}\mathcal{A}, \quad \phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \qquad (2.55)$$

to a differential graded ring $(\mathcal{O}^*[\mathfrak{d}\mathcal{A}], d)$, and obeys a relation

$$u\rfloor(\phi\wedge\sigma) = u\rfloor\phi\wedge\sigma + (-1)^{|\phi|}\phi\wedge u\rfloor\sigma. \tag{2.56}$$

With the interior product (2.55), one defines a derivation

$$\mathbf{L}_u(\phi) = d(u\rfloor\phi) + u\rfloor d\phi, \quad \phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \tag{2.57}$$
$$\mathbf{L}_u(\phi\wedge\sigma) = \mathbf{L}_u(\phi)\wedge\sigma + \phi\wedge\mathbf{L}_u\sigma, \tag{2.58}$$

of an $\mathbb{N}$-graded ring $(\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ for any $u\in\mathfrak{d}\mathcal{A}$. Then one can think of elements of $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ as being differential forms over $\mathcal{A}$.

The minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over a ring $\mathcal{A}$ consists of the monomials $a_0da_1\wedge\cdots\wedge da_k$, $a_i\in\mathcal{A}$ (Definition 2.6).

THEOREM 2.13: The de Rham complex

$$0\to\mathcal{K}\longrightarrow\mathcal{A}\xrightarrow{d}\mathcal{O}^1\mathcal{A}\xrightarrow{d}\cdots\mathcal{O}^k\mathcal{A}\xrightarrow{d}\cdots \tag{2.59}$$

of $\mathcal{O}^*\mathcal{A}$ is exactly the cochain complex (2.41). □

*Proof:* Comparing the equalities (2.39) and (2.48) shows that $d^1 = d$ on an $\mathcal{A}$-module

$$\mathcal{O}^1\mathcal{A} = \mathcal{O}^1\subseteq\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathfrak{d}\mathcal{A}^*,$$

generated by elements $d^1a$, $a\in\mathcal{A}$. The complex (2.59) is called the de Rham complex of a $\mathcal{K}$-ring $\mathcal{A}$, and its cohomology $H^*(\mathcal{A})$ is said to be the de Rham cohomology of $\mathcal{A}$. □

**Example 2.13:** Let $\mathcal{P}[Q]$ be a polynomial ring of a free $\mathcal{K}$-module $Q$ of finite rank in Example 2.11. Since $\mathfrak{d}\mathcal{P}[Q]$ is a free $\mathcal{P}[Q]$-module of finite rank, its $\mathcal{P}[Q]$-dual $\mathcal{O}^1[\mathfrak{d}\mathcal{P}[Q]]$ is a free $\mathcal{P}[Q]$-module of finite rank possessing the dual basis $\{dq^i\}$ such that the relation (2.54) takes a form $\partial_i\rfloor da^j = \delta^j_i$. It follows that the Chevalley–Eilenberg differential calculus (2.50) of $\mathcal{P}[Q]$ consists of monomials

$$\phi = \phi_{i_1\ldots i_r}dq^{i_1}\wedge\cdots\wedge dq^{i_r}, \qquad \phi_{i_1\ldots i_r}\in\mathcal{P}[Q]. \tag{2.60}$$

Thus, it is the minimal Chevalley–Eilenberg differential calculus (2.59):

$$0\to\mathcal{K}\longrightarrow\mathcal{P}[Q]\xrightarrow{d}\mathcal{O}^1\xrightarrow{d}\cdots\mathcal{O}^r\xrightarrow{d}\cdots, \tag{2.61}$$

where the Chevalley–Eilenberg coboundary operator reads

$$d\phi = \partial_k(\phi_{i_1\ldots i_r})dq^k\wedge dq^{i_1}\wedge\cdots\wedge dq^{i_r}. \tag{2.62}$$

□

## 2.5 Connections on modules and rings

We employ the jets of modules in Section 2.3 in order to introduce connections on modules over commutative rings [28, 39].

It is readily observed that a first-order jet module $\mathcal{J}^1(P)$ of an $\mathcal{A}$-module $P$ is isomorphic to a tensor product

$$\mathcal{J}^1(P) = \mathcal{J}^1 \otimes P, \qquad (a \otimes_1 bp) \leftrightarrow (a \otimes_1 b) \otimes p.$$

of an $\mathcal{A}^\bullet$-module $\mathcal{J}^1$ and an $\mathcal{A}$-module $P$. Then the isomorphism (2.34) leads to the splitting

$$\begin{aligned} &\mathcal{J}^1(P) = (\mathcal{A} \oplus \mathcal{O}^1) \otimes P = (\mathcal{A} \otimes P) \oplus (\mathcal{O}^1 \otimes P), \\ &a \otimes_1 bp \leftrightarrow (ab + ad^1(b)) \otimes p. \end{aligned} \tag{2.63}$$

Applying the epimorphism $\pi^1_0$ (2.27) to this splitting, one obtains a short exact sequence of $\mathcal{A}$- and $\mathcal{A}^\bullet$-modules

$$\begin{aligned} &0 \longrightarrow \mathcal{O}^1 \otimes P \to \mathcal{J}^1(P) \xrightarrow{\pi^1_0} P \longrightarrow 0, \\ &(a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to (c \otimes_1 \mathbf{1} + a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to cp. \end{aligned} \tag{2.64}$$

This exact sequence canonically is split by an $\mathcal{A}^\bullet$-module morphism

$$P \ni ap \to \mathbf{1} \otimes ap = a \otimes p + d^1(a) \otimes p \in \mathcal{J}^1(P).$$

However, it need not be split by an $\mathcal{A}$-module morphism, unless $P$ is a projective $\mathcal{A}$-module.

Definition 2.7: A connection on an $\mathcal{A}$-module $P$ is defined as an $\mathcal{A}$-module morphism

$$\Gamma : P \to \mathcal{J}^1(P), \qquad \Gamma(ap) = a\Gamma(p), \tag{2.65}$$

which splits the exact sequence (2.64) or, equivalently, the exact sequence

$$0 \to \mathcal{O}^1 \otimes P \to (\mathcal{A} \oplus \mathcal{O}^1) \otimes P \to P \to 0. \tag{2.66}$$

□

If the splitting $\Gamma$ (2.65) exists, it reads

$$J^1 p = \Gamma(p) + \nabla(p),$$

where $\nabla$ is the complementary morphism

$$\nabla : P \to \mathcal{O}^1 \otimes P, \qquad \nabla(p) = \mathbf{1} \otimes_1 p - \Gamma(p). \tag{2.67}$$

Though this complementary morphism in fact is a covariant differential on a module $P$, it is traditionally called the connection on a module. It satisfies the Leibniz rule

$$\nabla(ap) = d^1 a \otimes p + a\nabla(p), \tag{2.68}$$

i.e., $\nabla$ is an $(\mathcal{O}^1 \otimes P)$-valued first-order differential operator on $P$. Thus, we come to the following equivalent definition of a connection [21].

DEFINITION 2.8: A connection on an $\mathcal{A}$-module $P$ is the $\mathcal{K}$-module morphism $\nabla$ (2.67) which obeys the Leibniz rule (2.68). Sometimes, it is called the Koszul connection. $\Box$

The morphism $\nabla$ (2.67) naturally can be extended to a morphism

$$\nabla : \mathcal{O}^1 \otimes P \to \mathcal{O}^2 \otimes P.$$

Then we have a morphism

$$R = \nabla^2 : P \to \mathcal{O}^2 \otimes P,$$

called the curvature of a connection $\nabla$ on a module $P$.

In view of the isomorphism (2.38), any connection in Definition 2.8 determines a connection in the following sense.

DEFINITION 2.9: A connection on an $\mathcal{A}$-module $P$ is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathrm{Diff}_1(P, P) \tag{2.69}$$

such that first-order differential operators $\nabla_u$ obey the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{2.70}$$

$\Box$

Definitions 2.8 and 2.9 are equivalent if $\mathcal{O}^1 = \mathfrak{d}\mathcal{A}^* = \mathcal{O}^{1**}$. For instance, this is the case of a projective module $\mathcal{O}^1$ of finite rank (Theorem 2.1).

A curvature of the connection (2.69) is defined as a zero-order differential operator

$$R(u, u') = [\nabla_u, \nabla_{u'}] - \nabla_{[u,u']} \tag{2.71}$$

on a module $P$ for all $u, u' \in \mathfrak{d}\mathcal{A}$.

Let $P$ be a commutative $\mathcal{A}$-ring and $\mathfrak{d}P$ the derivation module of $P$ as a $\mathcal{K}$-ring. Definition 2.9 is modified as follows.

DEFINITION 2.10: A connection on an $\mathcal{A}$-ring $P$ is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathfrak{d}P, \tag{2.72}$$

which is a connection on $P$ as an $\mathcal{A}$-module, i.e., obeys the Leinbniz rule (2.70). $\Box$

Two such connections $\nabla_u$ and $\nabla'_u$ differ from each other in a derivation of an $\mathcal{A}$-ring $P$, i.e., which vanishes on $\mathcal{A} \subset P$. A curvature of the connection (2.72) is given by the formula (2.71).

# 3 Local-ringed spaces

Local-ringed spaces are sheaves of local rings. For instance, smooth manifolds, represented by sheaves of real smooth functions, constitute a subcategory of the category of local-ringed spaces (Section 4).

A sheaf $\mathfrak{R}$ on a topological space $X$ is said to be the ringed space if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a commutative ring [16, 39, 48]. A ringed space often is denoted by a pair $(X, \mathfrak{R})$ of a topological space $X$ and a sheaf $\mathfrak{R}$ of rings on $X$ which are called the body and the structure sheaf of a ringed space, respectively.

In comparison with morphisms of sheaves on the same topological space in Section 8.3, morphisms of ringed spaces are defined to be particular morphisms of sheaves on different topological spaces as follows.

**Example 3.1:** Let $\varphi : X \to X'$ be a continuous map. Given a sheaf $S$ on $X$, its direct image $\varphi_* S$ on $X'$ is generated by the presheaf of assignments

$$X' \supset U' \to S(\varphi^{-1}(U'))$$

for any open subset $U' \subset X'$. Conversely, given a sheaf $S'$ on $X'$, its inverse image $\varphi^* S'$ on $X$ is defined as the pull-back onto $X$ of a topological fibre bundle $S'$ over $X'$, i.e., $\varphi^* S'_x = S_{\varphi(x)}$. This sheaf is generated by the presheaf which associates to any open $V \subset X$ the direct limit of modules $S'(U)$ over all open subsets $U \subset X'$ such that $V \subset f^{-1}(U)$. □

**Example 3.2:** Let $i : X \to X'$ be a closed subspace of $X'$. Then $i_* S$ is a unique sheaf on $X'$ such that

$$i_* S|_X = S, \qquad i_* S|_{X' \setminus X} = 0.$$

Indeed, if $x' \in X \subset X'$, then $i_* S(U') = S(U' \cap X)$ for any open neighborhood $U$ of this point. If $x' \notin X$, there exists its neighborhood $U'$ such that $U' \cap X$ is empty, i.e., $i_* S(U') = 0$. A sheaf $i_* S$ is called the trivial extension of a sheaf $S$. □

By a morphism of ringed spaces $(X, \mathfrak{R}) \to (X', \mathfrak{R}')$ is meant a pair $(\varphi, \Phi)$ of a continuous map $\varphi : X \to X'$ and a sheaf morphism $\Phi : \mathfrak{R}' \to \varphi_* \mathfrak{R}$ or, equivalently, a sheaf morphisms $\varphi^* \mathfrak{R}' \to \mathfrak{R}$ [48]. Restricted to each stalk, a sheaf morphism $\Phi$ is assumed to be a ring homomorphism. A morphism of ringed spaces is said to be:

- a monomorphism if $\varphi$ is an injection and $\Phi$ is an epimorphism,
- an epimorphism if $\varphi$ is a surjection, while $\Phi$ is a monomorphism.

DEFINITION 3.1: A ringed space is said to be the local-ringed space (the geometric space in the terminology of [48]) if it is a sheaf of local rings. □

A key point of the study of local-ringed space is that any projective module over a local ring is free (Theorem 2.2).

**Example 3.3:** A sheaf $C^0_X$ of germs of continuous real functions on a topological space $X$ (Example 8.1) is a local-ringed space. Its stalk $C^0_x$, $x \in X$, contains a unique maximal ideal of germs of functions vanishing at $x$. □

## 3.1 Differential calculus over local-ringed spaces

Let $(X, \mathfrak{R})$ be a local-ringed space. By a sheaf $\mathfrak{d}\mathfrak{R}$ of derivations of the sheaf $\mathfrak{R}$ is meant a subsheaf of endomorphisms of $\mathfrak{R}$ such that any section $u$ of $\mathfrak{d}\mathfrak{R}$ over an open subset $U \subset X$ is a derivation of a ring $\mathfrak{R}(U)$. It should be emphasized that, since the monomorphism (8.6) is not necessarily an isomorphism, a derivation of a ring $\mathfrak{R}(U)$ need not be a section of a sheaf $\mathfrak{d}\mathfrak{R}|_U$. Namely, it may happen that, given open sets $U' \subset U$, there is no restriction morphism

$$\mathfrak{d}(\mathfrak{R}(U)) \to \mathfrak{d}(\mathfrak{R}(U')).$$

Given a local-ringed space $(X, \mathfrak{R})$, a sheaf $P$ on $X$ is called the sheaf of $\mathfrak{R}$-modules if every stalk $P_x$, $x \in X$, is an $\mathfrak{R}_x$-module or, equivalently, if $P(U)$ is an $\mathfrak{R}(U)$-module for any open subset $U \subset X$. A sheaf of $\mathfrak{R}$-modules $P$ is said to be locally free if there exists an open neighborhood $U$ of every point $x \in X$ such that $P(U)$ is a free $\mathfrak{R}(U)$-module. If all these free modules are of finite rank (resp. of the same finite rank), one says that $P$ is of finite type (resp. of constant rank). The structure module of a locally free sheaf is called the locally free module.

Let $(X, \mathfrak{R})$ be a local-ringed space and $\mathfrak{P}$ a sheaf of $\mathfrak{R}$-modules on $X$. For any open subset $U \subset X$, let us consider a jet module $\mathcal{J}^1(\mathfrak{P}(U))$ of a module $\mathfrak{P}(U)$. It consists of elements of $\mathfrak{R}(U) \otimes \mathfrak{P}(U)$ modulo the pointwise relations (2.21). Hence, there is the restriction morphism

$$\mathcal{J}^1(\mathfrak{P}(U)) \to \mathcal{J}^1(\mathfrak{P}(V))$$

for any open subsets $V \subset U$, and the jet modules $\mathcal{J}^1(\mathfrak{P}(U))$ constitute a presheaf. This presheaf defines the sheaf $\mathfrak{J}^1\mathfrak{P}$ of jets of $\mathfrak{P}$ (or simply the jet sheaf). The jet sheaf $\mathfrak{J}^1\mathfrak{R}$ of a sheaf $\mathfrak{R}$ of local rings is introduced in a similar way. Since the relations (2.21) and (2.33) on a ring $\mathfrak{R}(U)$ and modules $\mathfrak{P}(U)$, $\mathcal{J}^1(\mathfrak{P}(U))$, $\mathcal{J}^1(\mathfrak{R}(U))$ are pointwise relations for any open subset $U \subset X$, they commute with the restriction morphisms. Therefore, the direct limits of the quotients modulo these relations exist [29]. Then we have the sheaf $\mathcal{O}^1\mathfrak{R}$ of one-forms over a sheaf $\mathfrak{R}$, the sheaf isomorphism

$$\mathfrak{J}^1(\mathfrak{P}) = (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P},$$

and the exact sequences of sheaves

$$0 \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P} \to \mathfrak{J}^1(\mathfrak{P}) \to \mathfrak{P} \to 0, \tag{3.1}$$

$$0 \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P} \to (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P} \to \mathfrak{P} \to 0. \tag{3.2}$$

They reflect the quotient (2.35), the isomorphism (2.63) and the exact sequences of modules (2.64), (2.66), respectively.

**Remark 3.4:** It should be emphasized that, because of the inequality (8.6), the duality relation (2.38) is not extended to the sheaves $\mathfrak{d}\mathfrak{R}$ and $\mathcal{O}^1\mathfrak{R}$ in general, unless $\mathfrak{d}\mathfrak{R}$ and $\mathcal{O}^1\mathfrak{R}$ are locally free sheaves of finite rank. If $\mathfrak{P}$ is a locally free sheaf of finite rank, so is $\mathfrak{J}^1\mathfrak{P}$. □

Following Definitions 2.7 and 2.8 of a connection on modules, we come to the following notion of a connection on sheaves.

DEFINITION 3.2: Given a local-ringed space $(X, \mathfrak{R})$ and a sheaf $\mathfrak{P}$ of $\mathfrak{R}$-modules on $X$, a connection on a sheaf $\mathfrak{P}$ is defined as a splitting of the exact sequence (3.1) or, equivalently, the exact sequence (3.2). $\Box$

Theorem 8.4 leads to the following compatibility of the notion of a connection on sheaves with that of a connection on modules.

PROPOSITION 3.1: If there exists a connection on a sheaf $\mathfrak{P}$ in Definition 3.2, then there exists a connection on a module $\mathfrak{P}(U)$ for any open subset $U \subset X$. Conversely, if for any open subsets $V \subset U \subset X$ there are connections on modules $\mathfrak{P}(U)$ and $\mathfrak{P}(V)$ related by the restriction morphism, then the sheaf $\mathfrak{P}$ admits a connection. $\Box$

As an immediate consequence of Proposition 3.1, we find that the exact sequence of sheaves (3.2) is split iff there exists a sheaf morphism

$$\nabla : \mathfrak{P} \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}, \tag{3.3}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s), \qquad f \in \mathcal{A}(U), \qquad s \in \mathfrak{P}(U),$$

for any open subset $U \in X$. It leads to the following equivalent definition of a connection on sheaves in the spirit of Definition 2.8.

DEFINITION 3.3: The sheaf morphism (3.3) is a connection on a sheaf $\mathfrak{P}$. $\Box$

Similarly to the case of connections on modules, a curvature of the connection (3.3) on a sheaf $\mathfrak{P}$ is given by the expression

$$R = \nabla^2 : \mathfrak{P} \to \mathcal{O}^2_X \otimes \mathfrak{P}. \tag{3.4}$$

The exact sequence (3.2) need not be split. One can obtain the following criteria of the existence of a connection on a sheaf.

Let $\mathfrak{P}$ be a locally free sheaf of $\mathfrak{R}$-modules. Then we have the exact sequence of sheaves

$$0 \to \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}) \to \mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P}) \to \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P}) \to 0$$

and the corresponding exact sequence (8.15) of the cohomology groups

$$\begin{aligned} 0 \to H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P})) \to H^0(X; \mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P})) \to \\ H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P})) \to H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P})) \to \cdots . \end{aligned}$$

The identity morphism $\mathrm{Id} : \mathfrak{P} \to \mathfrak{P}$ belongs to $H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P}))$. Its image in $H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}))$ is called the Atiyah class. If this class vanishes, there exists an element of $\mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P}))$ whose image is $\mathrm{Id}\,\mathfrak{P}$, i.e., a splitting of the exact sequence (3.2).

## 3.2 Affine schemes

One can associate to any commutative ring $\mathcal{A}$ a local-ringed space as follows.

Let $\mathcal{I}$ be a prime ideal of $\mathcal{A}$. Then $\mathcal{A} \setminus \mathcal{I}$ is a multiplicative subset of $\mathcal{A}$ (Remark 2.2) and $(\mathcal{A} \setminus \mathcal{I})^{-1}\mathcal{A}$ is a local ring.

Let $\operatorname{Spec}\mathcal{A}$ be a set of prime ideals of a ring $\mathcal{A}$. It is called the spectrum of $\mathcal{A}$. Let us assign to each ideal $\mathcal{I}$ of $\mathcal{A}$ a set

$$V(\mathcal{I}) = \{\, x \in \operatorname{Spec}\mathcal{A} \,:\, \mathcal{I} \subseteq x \,\}. \tag{3.5}$$

These sets possess the properties

$$V(\{0\}) = \operatorname{Spec}\mathcal{A}, \qquad V(\mathcal{A}) = \emptyset,$$
$$\bigcap_i V(\mathcal{I}_i) = V(\sum_i \mathcal{I}_i), \qquad V(\mathcal{I}) \cup V(\mathcal{I}') = V(\mathcal{I}\mathcal{I}').$$

In view of these properties, one can regard the sets (3.5) as closed sets of some topology on a spectrum $\operatorname{Spec}\mathcal{A}$. It is called the Zariski topology. A base for this topology consists of sets

$$U(a) = \{\, x \in \operatorname{Spec}\mathcal{A} \,:\, a \notin x \,\} = \operatorname{Spec}\mathcal{A} \setminus V(\mathcal{A}a) \tag{3.6}$$

as $a$ runs through $\mathcal{A}$. In particular, a set of closed points is nothing but the set $\operatorname{Specm}\mathcal{A}$ of all maximal ideals of $\mathcal{A}$. Endowed with the relative Zariski topology, it is called the maximal spectrum of $\mathcal{A}$. A ring morphism $\zeta : \mathcal{A} \to \mathcal{A}'$ yields a continuous map

$$\zeta^{\natural} : \operatorname{Spec}\mathcal{A}' \ni x' \mapsto \zeta^{-1}(x') \in \operatorname{Spec}\mathcal{A}. \tag{3.7}$$

In particular, let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the quotient morphism with respect to an ideal $\mathcal{I}$ of $\mathcal{A}$. Then $\zeta^{\natural}$ (3.7) is a homeomorphism of $\operatorname{Spec}(\mathcal{A}/\mathcal{I})$ onto a closed subspace $V(\mathcal{I}) \subset \operatorname{Spec}\mathcal{A}$.

Given a commutative ring $\mathcal{A}$ and its spectrum $\operatorname{Spec}\mathcal{A}$, one can define a sheaf $\mathfrak{A}$ on $\operatorname{Spec}\mathcal{A}$ whose stalk at a point $x \in \operatorname{Spec}\mathcal{A}$ is a local ring $\mathfrak{A}_x = (\mathcal{A} \setminus x)^{-1}\mathcal{A}$. A structure ring $\mathfrak{A}(\operatorname{Spec}\mathcal{A})$ of global sections of a sheaf $\mathfrak{A}$ is exactly a ring $\mathcal{A}$ itself. The local-ringed space $(\operatorname{Spec}\mathcal{A}, \mathfrak{A})$ is called an affine scheme [25, 44, 51]. A local-ringed space $(X, \mathfrak{R})$ which is locally isomorphic to an affine scheme is called a scheme. Let us recall the following standard notions.

(i) If a scheme $(X, \mathfrak{R})$ has an affine open cover $\{U_i = \operatorname{Spec}\mathcal{A}_i\}$ such that every $\mathcal{A}_i$ is a Noetherian ring (i.e., any ideal of $\mathcal{A}_i$ is finitely generated), $(X, \mathfrak{R})$ is said to be locally Noetherian. A locally Noetherian scheme is called Noetherian if its body $X$ is quasi-compact.

(ii) A scheme $(X, \mathfrak{R})$ is called reduced if the stalk of $\mathfrak{R}$ at each point of $X$ has no nilpotent elements.

(iii) A scheme $(X, \mathfrak{R})$ is said to be irreducible if its body $X$ is not a union of two proper closed subsets.

(iv) A scheme is called integral if it is reduced and irreducible.

A morphism of schemes, by definition, is a morphism between them as local-ringed spaces. Given a morphism of schemes

$$\varphi : (X, \mathfrak{R}) \to (X', \mathfrak{R}'), \tag{3.8}$$

$(X,\mathfrak{R})$ is said to be a scheme over $(X',\mathfrak{R}')$. The morphism $\varphi$ (3.8) is called separated if the range of the diagonal morphism

$$X \to X \underset{X'}{\times} X$$

is closed. In this case, one says that $(X,\mathfrak{R})$ is separated over $X'$. A scheme $(X,\mathfrak{R})$ is called separated if it is a separated over $\operatorname{Spec}\mathbb{Z}$. All affine schemes are separated.

A morphism of schemes

$$(X,\mathfrak{R}) \to (X' = \operatorname{Spec}\mathcal{A},\mathfrak{A})$$

is said to be locally of finite type (resp. of finite type) if $(X,\mathfrak{R})$ has an open affine cover (resp. a finite open affine cover) $\{U_i = \operatorname{Spec}\mathcal{A}_i\}$ such that each $\mathcal{A}_i$ is a finitely generated $\mathcal{A}$-algebra. A general morphism of schemes $\varphi$ (3.8) is said to be locally of finite type (resp. of finite type) if there is an open affine cover $\{V_i\}$ of $X'$ such that every restriction of $\varphi$ to $\varphi^{-1}(V_i)$ is locally of finite type (resp. of finite type). In this case, one says that $(X,\mathfrak{R})$ is locally of finite type (resp. of finite type) over $X'$.

**Example 3.5:** Let $\mathcal{K}$ be a field. Its $\operatorname{Spec}\mathcal{K}$ consists only of one point with $\mathcal{K}$ as the stalk of the structure sheaf. A scheme of finite type over $\operatorname{Spec}\mathcal{K}$ is called an algebraic scheme over $\mathcal{K}$. $\Box$

Let $(X,\mathfrak{R})$ be a ringed space. A sheaf of $\mathfrak{R}$-modules $S$ is said to be quasi-coherent if for each point $x$ of $X$ there exists a neighborhood $U$ of $x$ and an exact sequence

$$M \longrightarrow N \longrightarrow S|_U \to 0,$$

where $M$ and $N$ are free $\mathfrak{R}|_U$-modules. Let $S$ be a locally free sheaf of $\mathfrak{R}$-modules of finite type. It is said to be of finite presentation if, locally, there exists an exact sequence

$$\mathfrak{R}^m \longrightarrow \mathfrak{R}^n \longrightarrow S \to 0,$$

where $m$ and $n$ are positive integers (which need not be globally constant). A locally free sheaf of finite type is called coherent if the kernel of any homomorphism $\mathfrak{R}|_U^n \to S|_U$, where $n$ is an arbitrary positive integer and $U$ is an open set, is of finite type. Obviously, if $S$ is coherent, then it is of finite presentation, and is quasi-coherent. If $\mathfrak{R}$ itself is coherent as a sheaf of $\mathfrak{R}$-modules, then it is called the coherent sheaf of rings. In this case, every sheaf of $\mathfrak{R}$-modules of finite presentation is coherent. For instance, the structure sheaf of a locally Noetherian scheme is a coherent sheaf of rings.

Let $(X = \operatorname{Spec}\mathcal{A},\mathfrak{R} = \mathfrak{A})$ be an affine scheme. Then every quasi-coherent sheaf $S$ of $\mathfrak{A}$-modules on $X$ is generated by its global sections. The correspondence $S \to S(X)$ defines an equivalence between the category of quasi-coherent sheaves on $X$ and the category of $\mathcal{A}$-modules. If $\mathcal{A}$ is Noetherian, then the coherent sheaves and the finite $\mathcal{A}$-modules correspond to each other under this equivalence.

Let $(X, \mathfrak{R})$ be a separated scheme and $\mathfrak{U} = \{U_i\}$ an affine open cover of $X$. For each quasi-coherent sheaf $S$ of $\mathfrak{R}$-modules, the cohomology $H^*(X; S)$ of $X$ with coefficients in $S$ is canonically isomorphic to the cohomology $H^*(\mathfrak{U}; S(U))$. One defines the cohomological dimension $\mathrm{cd}(X)$ of a scheme $(X, \mathfrak{R})$ as the largest integer $r$ such that $H^m(X; S) \neq 0$ for a quasi-coherent sheaf of $\mathfrak{R}$-modules on $X$. For instance, if $X$ is an affine scheme, then $\mathrm{cd}(X) = 0$. The converse is true under the assumption that $X$ is Noetherian.

## 3.3 Affine varieties

Let $\mathcal{K}$ throughout this Section be an algebraically closed field, i.e., any polynomial of non-zero degree with coefficients in $\mathcal{K}$ has a root in $\mathcal{K}$. The reason is that, dealing with non-linear algebraic equations, one can not expect a simple clear-cut theory, without assuming that a field is algebraically closed. If $\mathcal{K}$ fails to be algebraically closed, one can extend it in an appropriate way.

Let a commutative $\mathcal{K}$-ring $\mathcal{A}$ be finitely generated, and let it possess no nilpotent elements. Then it is isomorphic to the quotient of some polynomial $\mathcal{K}$-ring which is the coordinate ring (3.9) of a certain affine variety (Theorem 3.2).

A subset of an $n$-dimensional affine space $\mathcal{K}^n$ is called an affine variety if it is a set of zeros (common roots) of some set of polynomials of $n$ variables with coefficients in $\mathcal{K}$ [25, 44]. Unless otherwise stated, the dimension $n$ holds fixed. Let $\mathcal{K}[x]$ be a ring of polynomials of $n$ variables with coefficients in a field $\mathcal{K}$ (Example 2.3). Given an affine variety $\mathcal{V}$, a set $I(\mathcal{V})$ of polynomials in $\mathcal{K}[x]$ which vanish at every point of $\mathcal{V}$ is an ideal of $\mathcal{K}[x]$ called the characteristic ideal of $\mathcal{V}$. Herewith, $\mathcal{V} = \mathcal{V}'$ if and only if $I(\mathcal{V}) = I(\mathcal{V}')$. Therefore, an affine variety $\mathcal{V}$ can be given by the generating set of its characteristic ideal $I(\mathcal{V})$, i.e., by a finite system $f_i = 0$ of polynomials $f_i \in \mathcal{K}[x]$.

An affine variety which is a subset of another affine variety is called a subvariety. An affine variety is said to be irreducible if it is not the union of two proper subvarities. A maximal irreducible subvariety of an affine variety is called its irreducible component. Note that any affine variety can be written uniquely as the union of a finite number of irreducible components. An affine variety $\mathcal{V}$ is irreducible iff $I(\mathcal{V})$ is a prime ideal. Moreover, let $\mathcal{I}$ be a prime ideal of $\mathcal{K}[x]$ and $\mathcal{V}$ be an affine variety in $\mathcal{K}^n$ of zeros of elements of $\mathcal{I}$. Then $I(\mathcal{V}) = \mathcal{I}$. This fact states one-to-one correspondence between the prime ideals of $\mathcal{K}[x]$ and the irreducible affine varieties in $\mathcal{K}^n$. In particular, maximal ideals correspond to points of $\mathcal{K}^n$.

The intersection and the union of subvarieties of an affine variety $\mathcal{V}$ are also subvarieties. Thus, subvarieties can be taken as a system of closed sets of a topology on $\mathcal{V}$ which is called the Zariski topology on the affine variety $\mathcal{V}$. Unless otherwise stated, an affine variety is provided with this topology.

Given an affine variety $\mathcal{V}$, the factor ring

$$\mathcal{K}_{\mathcal{V}} = \mathcal{K}[x]/I(\mathcal{V}) \tag{3.9}$$

is called the coordinate ring of $\mathcal{V}$. If an affine variety $\mathcal{V}$ is irreducible, the ring $\mathcal{K}_{\mathcal{V}}$ (3.9) has no divisor of zero.

THEOREM 3.2: Let a $\mathcal{K}$-ring $\mathcal{A}$ be finitely generated, and let it possess no nilpotent elements. Given a set $(a_1, \ldots, a_n)$ of generating elements of $\mathcal{A}$, let us consider the epimorphism $\phi : \mathcal{K}[x] \to \mathcal{A}$ defined by the equalities $\phi(x_i) = a_i$. Zeros of polynomials in $\operatorname{Ker}\phi$ make up an affine variety whose coordinate ring $\mathcal{K}_{\mathcal{V}}$ (3.9) is exactly $\mathcal{A}$. $\Box$

Let $R_{\mathcal{V}}$ be the fraction field of a coordinate ring $\mathcal{K}_{\mathcal{V}}$ (Remark 2.2). It is called the function field of $\mathcal{V}$. There is the monomorphism $\mathcal{K}_{\mathcal{V}} \to R_{\mathcal{V}}$ (2.1). The function field $R_{\mathcal{V}}$ is finitely generated over $\mathcal{K}$, and its transcendence degree is called the dimension of the irreducible affine variety $\mathcal{V}$.

**Example 3.6:** Let $\mathcal{A}$ in Theorem 3.2 be a polynomial $\mathcal{K}$-ring $\mathcal{K}[x]$. Then $\operatorname{Ker}\phi = 0$, and it yields an affine variety $\mathcal{V} = \{0\}$ whose coordinate ring (3.9) is $\mathcal{K}_{\mathcal{V}} = \mathcal{K}[x]$. $\Box$

Let $\mathcal{W}$ be an irreducible subvariety of $\mathcal{V}$ and $I(\mathcal{W})$ a subset of $\mathcal{K}_{\mathcal{V}}$ consisting of elements which vanish on $\mathcal{W}$. Then $I(\mathcal{W})$ is a prime ideal of $\mathcal{K}_{\mathcal{V}}$. Let us consider a multiplicative subset $\mathcal{K}_{\mathcal{V}} \setminus I(\mathcal{W})$ and a subring

$$R_{\mathcal{W}} = (\mathcal{K}_{\mathcal{V}} \setminus I(\mathcal{W}))^{-1}\mathcal{K}_{\mathcal{V}}$$

of $R_{\mathcal{V}}$ (Remark 2.2). It is said to be a local ring of a subvariety $\mathcal{W}$. Functions $f \in R_{\mathcal{W}} \subset R_{\mathcal{V}}$ are called regular at $\mathcal{W}$. For a given function $f \in R_{\mathcal{V}}$, a set of points of $\mathcal{V}$ where $f$ is regular is Zariski open. Given an open subset $U \subset \mathcal{V}$, let us denote $R_U$ a ring of regular functions on $U$. Assigning $R_U$ to each open set $U$, one can define a sheaf of rings $\mathfrak{R}_{\mathcal{V}}$ of germs of regular functions on $\mathcal{V}$. Its stalk at a point $x \in \mathcal{V}$ coincides with a local ring $R_x$. A sheaf $\mathfrak{R}_{\mathcal{V}}$ is called the structure sheaf of an affine variety $\mathcal{V}$. The pair $(\mathcal{V}, \mathfrak{R}_{\mathcal{V}})$ is a local-ringed space.

Let us consider a pair $(X, \mathfrak{R})$ of a topological space $X$ and some sheaf $\mathfrak{R}$ of germs of $\mathcal{K}$-valued functions on $X$. This pair is called a prealgebraic variety if $X$ admits a finite open cover $\{U_i\}$ such that each $U_i$ is homeomorphic to some affine variety $\mathcal{V}_i$ and $\mathfrak{R}|_{U_i}$ is isomorphic to the structure sheaf $\mathfrak{R}_{\mathcal{V}_i}$ of $\mathcal{V}_i$. Let us note that the Cartesian product $\mathcal{V} \times \mathcal{V}'$ of affine varieties $\mathcal{V} \in \mathcal{K}^n$ and $\mathcal{V}' \in \mathcal{K}^m$ is an affine variety in $\mathcal{K}^{n+m}$ though the Zariski topology on $\mathcal{V} \times \mathcal{V}'$ is finer than the product topology. Since the Cartesian product $X \times X'$ of prealgebraic varieties is locally a product of affine varieties, this product is a prealgebraic variety. A prealgebraic variety $(X, \mathfrak{R})$ is said to be an algebraic variety if the diagonal map $X \to X \times X$ is closed in the Zariski topology of the product variety. This condition corresponds to Hausdorff's separation axiom. If $\mathcal{W}$ is a locally closed subset (i.e., the intersection of open and closed sets) of an algebraic variety, it becomes an algebraic variety in a natural manner since the germs of regular functions at $x \in \mathcal{W}$ are taken to be the germs of functions on

$\mathcal{W}$ induced by functions in the stalk $\mathfrak{R}_x$. The definitions of irreducibility and local rings of subvarieties for algebraic varieties are given in the same manner as before. From now on, by a variety is meant an algebraic variety. Any variety $(X, \mathfrak{R})$, by definition, is a local-ringed space.

**Remark 3.7:** There is the following correspondence between the affine varieties and the affine schemes. A $\mathcal{K}$-ring $\mathcal{A}$ in Theorem 3.2 is a case an algebraic scheme of finite type over a field $\mathcal{K}$ in Example 3.5, and it is associated to an affine variety in accordance with this theorem. Conversely, every algebraic affine variety $\mathcal{V}$ yields the affine scheme Spec$\mathcal{K}_\mathcal{V}$ such that there is one-to-one correspondence between the points of Spec$\mathcal{K}_\mathcal{V}$ and the irreducible subvarieties of $\mathcal{V}$. $\Box$

Given a variety, one says that its point $x$ is simple and that $\mathcal{V}$ is non-singular or smooth at $x$ if the local ring $R_x$ of $x$ is regular. Since a problem is local, one can assume that $\mathcal{V}$ is an affine variety in $\mathcal{K}^n$. Then the simplicity of $x$ implies that $x$ is contained in only one irreducible component of $\mathcal{V}$ and, if this component is $r$-dimensional, there exists $n-r$ polynomials $f_i(x)$ in the characteristic ideal $I(\mathcal{V})$ of $\mathcal{V}$ such that the rank of a matrix $(\partial f_i/\partial x_j)$ at $x$ equals $n-r$. A point of $\mathcal{V}$ which is not simple is called a singular point. The set of singular points, called the singular locus of $\mathcal{V}$, is a proper closed subset of $\mathcal{V}$. A variety possessing no singular point is said to be smooth. A point $x$ of a variety $\mathcal{V}$ is called normal if the local ring $R(x)$ is normal. A simple point is normal. Normal points make up a non-empty open subset of $\mathcal{V}$. An irreducible variety whose points are all normal is called a normal variety.

**Example 3.8:** When $\mathcal{K} = \mathbb{C}$, an algebraic variety $\mathcal{V}$, called a complex algebraic variety, has the structure of a complex analytic manifold so that the stalk $\mathfrak{R}_{\mathcal{V},x} = R_x$ at $x \in \mathcal{V}$ contains in the stalk $\mathbb{C}^h_{\mathcal{V},x}$ of germs of holomorphic functions at $x$ and their completion coincide. If $x$ is a simple point, $\mathbb{C}^h_{\mathcal{V},x}$ is the ring of converged power series and its completion is the ring of formal power series. $\Box$

A regular morphism $(\mathcal{V}, \mathfrak{R}) \to (\mathcal{V}', \mathfrak{R}')$ of varieties over the same field $\mathcal{K}$ is defined as a morphism of local-ringed spaces

$$\varphi : \mathcal{V} \to \mathcal{V}', \qquad \Phi : \varphi^*\mathfrak{R}' \to \mathfrak{R},$$

where $\varphi^*$ is the pull-back onto $\mathcal{V}$ of $\mathcal{K}$-valued functions on $\mathcal{V}'$. An isomorphism of varieties also is called a biregular morphism.

Let $\mathcal{V}$ and $\mathcal{W}$ be irreducible varieties. Let a closed subset $T \subset \mathcal{V} \times \mathcal{W}$ be an irreducible variety such that a closure of the range of the projection $T \to \mathcal{V}$ coincides with $\mathcal{V}$. Then the function field $R_\mathcal{V}$ of $\mathcal{V}$ can be identified with a subfield of $R_T$. If $R_\mathcal{V} = R_T$, then $T$ is called a rational morphism of $\mathcal{V}$ to $\mathcal{W}$. One can show that, if $T : \mathcal{V} \to \mathcal{W}$ is a rational morphism and $x \in \mathcal{V}$ is a normal point of $\mathcal{V}$ such that $T(x)$ contains an isolated point, then $T$ is regular at $x$.

Let $\mathcal{V}$ be an $m$-dimensional irreducible affine variety. One can associate to $\mathcal{V}$ the two algebras $\mathrm{Diff}_*(R_\mathcal{V})$ and $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ of (linear) differential operators on the function field $R_\mathcal{V}$ and the coordinate ring $\mathcal{K}_\mathcal{V}$ of $\mathcal{V}$, respectively [30].

In this case of the function field $R_\mathcal{V}$, one can choose a separating transcendence $\{x^1, \ldots, x^m\}$ basis for $R_\mathcal{V}$ over $\mathcal{K}$. Let us consider the derivation module $\mathfrak{d}R_\mathcal{V}$ of the $\mathcal{K}$-field $R_\mathcal{V}$. It is finitely generated by the derivations $\partial_i$ of $R_\mathcal{V}$ such that $\partial_i(x^j) = \delta_i^j$. Moreover, any differential operator $\Delta \in \mathrm{Diff}_r(R_\mathcal{V})$ is uniquely expressed as a polynomial of $\partial_i$ with coefficients in $R_\mathcal{V}$. Let $\mathfrak{d}R^*_\mathcal{V}$ be the $R_\mathcal{V}$-dual of the derivation module $\mathfrak{d}R_\mathcal{V}$. Given the above mentioned transcendence basis for $R_\mathcal{V}$, it is finitely generated by the elements $dx_i$ which are the duals of $\partial_i$. As a consequence, the Chevalley–Eilenberg calculus over $R_\mathcal{V}$ coincides with the universal differential calculus $\mathcal{O}^*(R_\mathcal{V})$ over $R_\mathcal{V}$ (Remark 9.1).

The case of the coordinate ring $\mathcal{K}_\mathcal{V}$ is more subtle. In this case, $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ is called the ring of differential operators on an affine variety $\mathcal{V}$. In general, there are no global coordinates on $\mathcal{V}$, but if $\mathcal{K}$ is of characteristic zero and $\mathcal{V}$ is smooth, the structure of $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ is still well understood. Namely, $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ is a simple (left and right) Noetherian ring without divisors of zero, and it is generated by finitely many elements of $\mathrm{Diff}_1(\mathcal{K}_\mathcal{V})$.

If $\mathcal{V}$ is singular, the construction of $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ is less clear. One can show that any differential operator on $\mathcal{K}_\mathcal{V} \subset R_\mathcal{V}$ admits a unique extension to a differential operator of the same order on $R_\mathcal{V}$. Thus, one can regard $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ as a subalgebra of $\mathrm{Diff}_*(R_\mathcal{V})$. Furthermore, a differential operator $\Delta$ on $R_\mathcal{V}$ which preserves $\mathcal{K}_\mathcal{V}$ is a differential operator on $\mathcal{K}_\mathcal{V}$. In particular, it follows that $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ has no zero divisors. In contrast with the smooth case, $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ fails to be generated by elements of $\mathrm{Diff}_1(\mathcal{K}_\mathcal{V})$ in general. One has conjectured that this is true if and only if $\mathcal{V}$ is smooth [33]. This conjecture has been proved for algebraic curves [32] and, more generally, for varieties with smooth normalization [49].

Let us note that, if a field $\mathcal{K}$ is of positive characteristic, the ring $\mathrm{Diff}_*(\mathcal{K}_\mathcal{V})$ is not Noetherian, or finitely generated, or without zero divisors [46].

# 4 Differential geometry of $C^\infty(X)$-modules

Let $X$ be a smooth manifold (Remark 4.1). Similarly to a sheaf $C^0_X$ of continuous functions in Example 3.3, a sheaf $C^\infty_X$ of smooth real functions on $X$ (Example 8.1) provides an important example of local-ringed spaces that this Section is devoted to [16, 39].

**Remark 4.1:** Throughout the work, smooth manifolds are finite-dimensional real manifolds. A smooth real manifold is customarily assumed to be Hausdorff and second-countable (i.e., it has a countable base for topology). Consequently, it is a locally compact space which is a union of a countable number of compact subsets, a separable space (i.e., it has a countable dense subset), a paracompact and completely regular space. Being paracompact, a smooth manifold admits the partition of unity by smooth real functions. One also can show that, given two disjoint closed subsets $N$ and $N'$ of a smooth manifold $X$, there exists a smooth function $f$ on $X$ such that $f|_N = 0$ and $f|_{N'} = 1$. Unless otherwise stated, manifolds are assumed to be connected and, consequently, arcwise connected. We follow the notion of a manifold without boundary. □

Similarly to a sheaf $C^0_X$ of continuous functions, a stalk $C^\infty_x$ of a sheaf $C^\infty_X$ at a point $x \in X$ has a unique maximal ideal of germs of smooth functions vanishing at $x$ (Example 3.3). Therefore, $C^\infty_X$ is a local-ringed space (Definition 3.1).

Though a sheaf $C^\infty_X$ is defined on a topological space $X$, it fixes a unique smooth manifold structure on $X$ as follows.

THEOREM 4.1: Let $X$ be a paracompact topological space and $(X, \mathfrak{R})$ a local-ringed space. Let $X$ admit an open cover $\{U_i\}$ such that a sheaf $\mathfrak{R}$ restricted to each $U_i$ is isomorphic to a local-ringed space $(\mathbb{R}^n, C^\infty_{R^n})$. Then $X$ is an $n$-dimensional smooth manifold together with a natural isomorphism of local-ringed spaces $(X, \mathfrak{R})$ and $(X, C^\infty_X)$. □

One can think of this result as being an alternative definition of smooth real manifolds in terms of local-ringed spaces. A smooth manifold $X$ also is algebraically reproduced as a certain subspace of the spectrum of a real ring $C^\infty(X)$ of smooth real functions on $X$ as follows [1, 16].

Let $\mathcal{A}$ be a commutative real ring and $\mathrm{Specm}\,\mathcal{A}$ its maximal spectrum. The real spectrum of $\mathcal{A}$ is a subspace $\mathrm{Spec}_\mathbb{R}\mathcal{A} \subset \mathrm{Specm}\,\mathcal{A}$ of the maximal ideals $\mathcal{I}$ such that the quotients $\mathcal{A}/\mathcal{I}$ are isomorphic to $\mathbb{R}$. It is endowed with the relative Zariski topology. There is the bijection between the set of real algebra morphisms of $\mathcal{A}$ to a field $\mathbb{R}$ and the real spectrum of $\mathcal{A}$, namely,

$$\begin{aligned} &\mathrm{Hom}_\mathbb{R}(\mathcal{A}, \mathbb{R}) \ni \phi \mapsto \mathrm{Ker}\,\phi \in \mathrm{Spec}_\mathbb{R}\mathcal{A}, \\ &\mathrm{Spec}_\mathbb{R}\mathcal{A} \ni x \mapsto \pi_x \in \mathrm{Hom}_\mathbb{R}(\mathcal{A}, \mathbb{R}), \qquad \pi_x : \mathcal{A} \to \mathcal{A}/x \cong \mathbb{R}. \end{aligned}$$

Any element $a \in \mathcal{A}$ induces a real function

$$f_a : \mathrm{Spec}_\mathbb{R}\mathcal{A} \ni x \mapsto \pi_x(a)$$

on the real spectrum $\mathrm{Spec}_\mathbb{R}\mathcal{A}$. This function need not be continuous with respect to the Zariski topology, but one can provide $\mathrm{Spec}_\mathbb{R}\mathcal{A}$ with another topology, called the Gel'fand one, which is the coarsest topology which makes all such functions continuous. If $\mathcal{A} = C^\infty(X)$, the Zariski and Gel'fand topologies coincide.

THEOREM 4.2: Given a ring $C^\infty(X)$ of smooth real functions on a manifold $X$, let $\mu_x$ denote the maximal ideal of functions vanishing at a point $x \in X$. Then there is a homeomorphism

$$\chi_X : X \ni x \mapsto \mu_x \in \mathrm{Spec}_\mathbb{R}C^\infty(X). \tag{4.1}$$

□

Let $X$ and $X'$ be two smooth manifolds. Any smooth map $\gamma : X \to X'$ induces a $\mathbb{R}$-ring morphism

$$\gamma^* : C^\infty(X') \to C^\infty(X)$$

which associates the pull-back function $\gamma^* f = f \circ \gamma$ on $X$ to a function $f$ on $X'$. Conversely, each $\mathbb{R}$-ring morphism

$$\zeta : C^\infty(X') \to C^\infty(X)$$

yields the continuous map $\zeta^\natural$ (3.7) which sends $\mathrm{Spec}_{\mathbb{R}} C^\infty(X) \subset \mathrm{Spec}\, C^\infty(X)$ to $\mathrm{Spec}_{\mathbb{R}} C^\infty(X') \subset \mathrm{Spec}\, C^\infty(X')$ so that the induced map

$$\chi^{-1}_{X'} \zeta^\natural \circ \chi_X : X \to X'$$

is smooth. Thus, there is one-to-one correspondence between smooth manifold morphisms $X \to X'$ and the $\mathbb{R}$-ring morphisms $C^\infty(X') \to C^\infty(X)$.

**Remark 4.2:** Let $X \times X'$ be a manifold product. A ring $C^\infty(X \times X')$ is constructed from rings $C^\infty(X)$ and $C^\infty(X')$ as follows. Whenever referring to a topology on a ring $C^\infty(X)$, we will mean the topology of compact convergence for all derivatives [36]. The $C^\infty(X)$ is a Fréchet ring with respect to this topology, i.e., a complete metrizable locally convex topological vector space. There is an isomorphism of Fréchet rings

$$C^\infty(X) \widehat{\otimes} C^\infty(X') \cong C^\infty(X \times X'), \tag{4.2}$$

where the left-hand side, called the topological tensor product, is the completion of $C^\infty(X) \otimes C^\infty(X')$ with respect to Grothendieck's topology, defined as follows. If $E_1$ and $E_2$ are locally convex topological vector spaces, Grothendieck's topology is the finest locally convex topology on $E_1 \otimes E_2$ such that the canonical mapping of $E_1 \times E_2$ to $E_1 \otimes E_2$ is continuous [36]. It also is called the $\pi$-topology in contrast with the coarser $\varepsilon$-topology on $E_1 \otimes E_2$ [34, 50]. Furthermore, for any two open subsets $U \subset X$ and $U' \subset X'$, let us consider the topological tensor product of rings $C^\infty(U) \widehat{\otimes} C^\infty(U')$. These tensor products define a local-ringed space $(X \times X', C^\infty_X \widehat{\otimes} C^\infty_{X'})$. Due to the isomorphism (4.2) written for all $U \subset X$ and $U' \subset X'$, we obtain a sheaf isomorphism

$$C^\infty_X \widehat{\otimes} C^\infty_{X'} = C^\infty_{X \times X'}.$$

□

Since a smooth manifold admits the partition of unity by smooth functions, it follows from Proposition 8.9 that any sheaf of $C^\infty_X$-modules on $X$ is fine and, consequently, acyclic.

For instance, let $Y \to X$ be a smooth vector bundle. The germs of its sections form a sheaf $Y_X$ of $C^\infty_X$-modules which, thus, is fine.

In particular, all sheaves $\mathcal{O}^k_X$, $k \in \mathbb{N}_+$, of germs of exterior forms on $X$ are fine. These sheaves constitute the de Rham complex

$$0 \to \mathbb{R} \longrightarrow C^\infty_X \xrightarrow{d} \mathcal{O}^1_X \xrightarrow{d} \cdots \mathcal{O}^k_X \xrightarrow{d} \cdots . \tag{4.3}$$

The corresponding complex of structure modules of these sheaves is the de Rham complex

$$0 \to \mathbb{R} \longrightarrow C^\infty(X) \xrightarrow{d} \mathcal{O}^1(X) \xrightarrow{d} \cdots \mathcal{O}^k(X) \xrightarrow{d} \cdots \tag{4.4}$$

of exterior forms on a manifold $X$. Its cohomology is called the de Rham cohomology $H^*(X)$ of $X$. Due to the Poincaré lemma, the complex (4.3) is exact and, thereby, is a fine resolution of the constant sheaf $\mathbb{R}$ on a manifold. Then a corollary of Theorem 8.7 is the classical de Rham theorem.

THEOREM 4.3: There is an isomorphism

$$H^k(X) = H^k(X;\mathbb{R}) \tag{4.5}$$

of the de Rham cohomology $H^*(X)$ of a manifold $X$ to the cohomology of $X$ with coefficients in the constant sheaf $\mathbb{R}$. □

**Remark 4.3:** Let us consider a short exact sequence of constant sheaves

$$0 \to \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow U(1) \to 0, \tag{4.6}$$

where $U(1) = \mathbb{R}/\mathbb{Z}$ is a circle group of complex numbers of unit module. This exact sequence yields a long exact sequence of sheaf cohomology groups

$$\begin{aligned} 0 \to \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow U(1) \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow H^1(X;\mathbb{R}) \longrightarrow \cdots \\ H^p(X;\mathbb{Z}) \longrightarrow H^p(X;\mathbb{R}) \longrightarrow H^p(X;U(1)) \longrightarrow H^{p+1}(X;\mathbb{Z}) \longrightarrow \cdots, \end{aligned}$$

where

$$H^0(X;\mathbb{Z}) = \mathbb{Z}, \qquad H^0(X;\mathbb{R}) = \mathbb{R}$$

and $H^0(X;U(1)) = U(1)$. This exact sequence defines a homomorphism

$$H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{R}) \tag{4.7}$$

of cohomology with coefficients in the constant sheaf $\mathbb{Z}$ to that with coefficients in $\mathbb{R}$. Combining the isomorphism (4.5) and the homomorphism (4.7) leads to a cohomology homomorphism

$$H^*(X;\mathbb{Z}) \to H^*(X). \tag{4.8}$$

Its kernel contains all cyclic elements of cohomology groups $H^k(X;\mathbb{Z})$. □

Given a vector bundle $Y \to X$, the structure module of a sheaf $Y_X$ of germs of its sections coincides with the structure module $Y(X)$ of global sections of $Y \to X$ (Example 8.2). The forthcoming Serre–Swan theorem (Theorem 4.4), shows that these modules exhaust all projective $C^\infty(X)$-modules of finite rank. This theorem originally has been proved in the case of a compact manifold $X$, but it is generalized to an arbitrary smooth manifold [16, 35].

THEOREM 4.4: Let $X$ be a smooth manifold. A $C^\infty(X)$-module $P$ is isomorphic to the structure module of a smooth vector bundle over $X$ iff it is a projective module of finite rank. □

This theorem states the categorial equivalence between the vector bundles over a smooth manifold $X$ and projective modules of finite rank over the ring

$C^\infty(X)$ of smooth real functions on $X$. The following are *Corollaries* of this equivalence.

(i) The structure module $Y^*(X)$ of the dual $Y^* \to X$ of a vector bundle $Y \to X$ is the $C^\infty(X)$-dual $Y(X)^*$ of the structure module $Y(X)$ of $Y \to X$.

(ii) Any exact sequence of vector bundles

$$0 \to Y \longrightarrow Y' \longrightarrow Y'' \to 0 \tag{4.9}$$

over the same base $X$ yields an exact sequence

$$0 \to Y(X) \longrightarrow Y'(X) \longrightarrow Y''(X) \to 0 \tag{4.10}$$

of their structure modules, and *vice versa*. In accordance with the well-known theorem [28], the exact sequence (4.9) is always split. Every its splitting defines that of the exact sequence (4.10), and *vice versa* (Theorem 2.4).

(iii) In particular, the derivation module of a real ring $C^\infty(X)$ coincides with a $C^\infty(X)$-module $\mathcal{T}_1(X)$ of vector fields on $X$, i.e., with the structure module of sections of the tangent bundle $TX$ of $X$. Hence, it is a projective $C^\infty(X)$-module of finite rank. It is the $C^\infty(X)$-dual $\mathcal{T}_1(X) = \mathcal{O}^1(X)^*$ of the structure module $\mathcal{O}^1(X)$ of the cotangent bundle $T^*X$ of $X$ which is a module of one-forms on $X$ and, conversely, $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$. It follows that the Chevalley–Eilenberg differential calculus over a real ring $C^\infty(X)$ is exactly the differential graded algebra $(\mathcal{O}^*(X), d)$ of exterior forms on $X$, where the Chevalley–Eilenberg coboundary operator $d$ (2.47) coincides with the exterior differential. Accordingly, the de Rham complex (2.59) of a real ring $C^\infty(X)$ is the de Rham complex (4.4) of exterior forms on $X$. Moreover, one can show that $(\mathcal{O}^*(X), d)$ is the minimal differential calculus, i.e., a $C^\infty(X)$-module $\mathcal{O}^1(X)$ is generated by elements $df$, $f \in C^\infty(X)$.

(iv) Let $Y \to X$ be a vector bundle and $Y(X)$ its structure module. An $r$-order jet manifold $J^rY$ of $Y \to X$ consists of equivalence classes $j^r_xs$, $x \in X$, of sections $s$ of $Y \to X$ which are identified by the $r+1$ terms of their Taylor series at points $x \in X$. Since $Y \to X$ is a vector bundle, so is a jet bundle $J^rY \to X$. Its structure module $J^rY(X)$ is exactly the $r$-order jet module $\mathcal{J}^r(Y(X))$ of a $C^\infty(X)$-module $Y(X)$ in Section 2.3 [15, 22]. As a consequence, the notion of a connection on the structure module $Y(X)$ is equivalent to the standard geometric notion of a connection on a vector bundle $Y \to X$ [28]. Indeed, a connection on a fibre bundle $Y \to X$ is defined as a global section $\Gamma$ of an affine jet bundle $J^1Y \to Y$. If $Y \to X$ is a vector bundle, there exists an exact sequence

$$0 \to T^*X \underset{X}{\otimes} Y \longrightarrow J^1Y \longrightarrow Y \to 0 \tag{4.11}$$

over $X$ which is split by $\Gamma$. Conversely, any slitting of this exact sequence yields a connection $Y \to X$. The exact sequence of vector bundles (4.11) induces the exact sequence of their structure modules

$$0 \to \mathcal{O}^1(X) \otimes Y(X) \longrightarrow J^1Y(X) \longrightarrow Y(X) \to 0. \tag{4.12}$$

Then any connection $\Gamma$ on a vector bundle $Y \to X$ defines a splitting of the exact sequence (4.12) which, by Definition 2.7, is a connection on a $C^\infty(X)$-module $Y(X)$, and *vice versa*.

Let now $P$ be an arbitrary $C^\infty(X)$-module. One can reformulate Definitions 2.8 and 2.9 of a connection on $P$ as follows.

DEFINITION 4.1: A connection on a $C^\infty(X)$-module $P$ is a $C^\infty(X)$-module morphism

$$\nabla : P \to \mathcal{O}^1(X) \otimes P,$$

which satisfies the Leibniz rule

$$\nabla(fp) = df \otimes p + f\nabla(p), \qquad f \in C^\infty(X), \qquad p \in P.$$

□

DEFINITION 4.2: A connection on a $C^\infty(X)$-module $P$ associates to any vector field $\tau \in \mathcal{T}_1(X)$ on $X$ a first-order differential operator $\nabla_\tau$ on $P$ which obeys the Leibniz rule

$$\nabla_\tau(fp) = (\tau \rfloor df)p + f\nabla_\tau p.$$

□

Since $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$, Definitions 4.1 and 4.2 are equivalent.

Let us note that a connection on an arbitrary $C^\infty(X)$-module need not exist, unless it is a projective or locally free module (Proposition 4.5).

A curvature of a connection $\nabla$ in Definitions 4.1 and 4.2 is defined as a zero-order differential operator

$$R(\tau,\tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau,\tau']} \tag{4.13}$$

on a module $P$ for all vector fields $\tau, \tau' \in \mathcal{T}_1(X)$ on $X$.

In accordance with Proposition 3.1, we come to the following relation between connections on $C^\infty(X)$-modules and sheaves of $C^\infty_X$-modules (Proposition 4.5).

Let $X$ be a manifold and $C^\infty_X$ a sheaf of smooth real functions on $X$. A sheaf $\mathfrak{d}C^\infty_X$ of its derivations is isomorphic to a sheaf of vector fields on a manifold $X$ (corollary (iii) of Theorem 4.4). It follows that:

- there is the restriction morphism $\mathfrak{d}(C^\infty(U)) \to \mathfrak{d}(C^\infty(V))$ for any open sets $V \subset U$,
- $\mathfrak{d}C^\infty_X$ is a locally free sheaf of $C^\infty_X$-modules of finite rank (Theorem 2.2),
- the sheaves $\mathfrak{d}C^\infty_X$ and $\mathcal{O}^1_X$ are mutually dual (Theorem 2.1).

Let $\mathfrak{P}$ be a locally free sheaf of $C^\infty_X$-modules. In this case, $\mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1_X \otimes \mathfrak{P})$ is a locally free sheaf of $C^\infty_X$-modules. It is fine and acyclic. Its cohomology group

$$H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1_X \otimes \mathfrak{P}))$$

vanishes, and the exact sequence

$$0 \to \mathcal{O}^1_X \otimes \mathfrak{P} \to (C^\infty_X \oplus \mathcal{O}^1_X) \otimes \mathfrak{P} \to \mathfrak{P} \to 0 \tag{4.14}$$

admits a splitting. This proves the following.

PROPOSITION 4.5: Any locally free sheaf of $C^\infty_X$-modules on a manifold $X$ admits a connection and, in accordance with Proposition 3.1, any locally free $C^\infty(X)$-module does well. □

**Example 4.4:** Let $Y \to X$ be a vector bundle and $Y_X$ a sheaf of germs of sections of $Y \to X$ (Example 8.2). Every linear connection $\Gamma$ on $Y \to X$ defines a connection on the structure module $Y(X)$ of sections of $Y \to X$ such that the restriction $\Gamma|_U$ is a connection on a module $Y(U)$ for any open subset $U \subset X$ (corollary (iv) of Theorem 4.4). Then we have a connection on the structure sheaf $Y_X$ of germs of sections of $Y \to X$. Conversely, a connection on the structure sheaf $Y_X$ defines a connection on a module $Y(X)$ and, consequently, a connection on a vector bundle $Y \to X$. □

In conclusion, let us consider a sheaf $S$ of commutative $C^\infty_X$-rings on a manifold $X$. Basing on Definition 2.10, we come to the following notion of a connection on a sheaf $S$ of commutative $C^\infty_X$-rings.

DEFINITION 4.3: Any morphism

$$\mathfrak{d}C^\infty_X \ni \tau \to \nabla_\tau \in \mathfrak{d}S,$$

which is a connection on $S$ as a sheaf of $C^\infty_X$-modules, is called a connection on the sheaf $S$ of rings. □

Its curvature is given by the expression

$$R(\tau,\tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau,\tau']},$$

similar to the expression (2.71) for a curvature of a connection on modules.

# 5 Differential calculus over $\mathbb{Z}_2$-graded commutative rings

This section addresses the differential calculus over Grassmann-graded rings (Definition 1.5) as particular $\mathbb{N}$-graded commutative rings. This also is a special case of $\mathbb{Z}_2$-graded commutative rings which are Grassmann algebras (Definition 5.4).

## 5.1 $\mathbb{Z}_2$-Graded algebraic calculus

Let us summarize the relevant notions of the $\mathbb{Z}_2$-graded algebraic calculus [2, 16, 38, 39].

Recall that the symbol [.] stands for the $\mathbb{Z}_2$-degree.

DEFINITION 5.1: Let $\mathcal{K}$ be a commutative ring without a divisor of zero. A $\mathcal{K}$-module $Q$ is called $\mathbb{Z}_2$-graded if it is endowed with a grading automorphism $\gamma$, $\gamma^2 = \mathrm{Id}$. A $\mathbb{Z}_2$-graded module falls into a direct sum of modules $Q = Q_0 \oplus Q_1$ such that

$$\gamma(q) = (-1)^{[q]}q, \qquad q \in Q_{[q]}.$$

One calls $Q_0$ and $Q_1$ the even and odd parts of $Q$, respectively. $\Box$

**Example 5.1:** Any $\mathbb{N}$-graded $\mathcal{K}$-module $P$ in Definition 1.1) is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module where

$$P_0 = \underset{i\in\mathbb{N}}{\oplus} P^{2i}, \qquad P_1 = \underset{i\in\mathbb{N}}{\oplus} P^{2i+1}.$$

$\Box$

A $\mathbb{Z}_2$-graded $\mathcal{K}$-module is said to be free if it has a basis composed by graded-homogeneous elements.

In particular, by a real $\mathbb{Z}_2$-graded vector space $B = B_0 \oplus B_1$ is meant a graded $\mathbb{R}$-module. A real $\mathbb{Z}_2$-graded vector space is said to be $(n, m)$-dimensional if $B_0 = \mathbb{R}^n$ and $B_1 = \mathbb{R}^m$.

DEFINITION 5.2: A $\mathcal{K}$-ring $\mathcal{A}$ is called $\mathbb{Z}_2$-graded if it is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module such that

$$[aa'] = ([a] + [a'])\mathrm{mod}\, 2,$$

where $a$ and $a'$ are graded-homogeneous elements of $\mathcal{A}$. In particular, $[\mathbf{1}] = 0$. $\Box$

Its even part $\mathcal{A}_0$ is a $\mathcal{K}$-ring and the odd one $\mathcal{A}_1$ is an $\mathcal{A}_0$-module.

**Example 5.2:** Any $\mathbb{N}$-graded ring in Definition 1.2, regarded as $\mathbb{Z}_2$-graded module (Example 5.1) is a $\mathbb{Z}_2$-graded $\mathcal{K}$-ring. The converse need not be true, unless a $\mathbb{Z}_2$-graded $\mathcal{K}$-ring is a Grassmann algebra (Definition 5.4). $\Box$

DEFINITION 5.3: A $\mathbb{Z}_2$-graded ring $\mathcal{A}$ is called graded commutative if

$$aa' = (-1)^{[a][a']}a'a, \qquad a, a' \in \mathcal{A}.$$

$\Box$

In particular, a commutative ring is the even $\mathbb{Z}_2$-graded commutative one $\mathcal{A} = \mathcal{A}_0$.

Every $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\Omega^*$ (Definition 1.3) possesses the associated $\mathbb{Z}_2$-graded commutative structure $\Omega = \Omega_0 \oplus \Omega_1$ (1.4).

**Example 5.3:** Clifford algebras exemplify $\mathbb{Z}_2$-graded rings which are not $\mathbb{Z}_2$-graded commutative. A Clifford $\mathcal{K}$-algebra is defined to be a finitely generated $\mathbb{Z}_2$-graded $\mathcal{K}$-ring which admits a generating basis $\{e^A\}$ of odd elements so that

$$e^Ae^B + e^Be^A = \delta^{AB}\mathbf{1}. \tag{5.1}$$

It also is an $\mathbb{N}$-graded $\mathcal{K}$-module, but not an $\mathbb{N}$-graded ring. $\Box$

Given a $\mathbb{Z}_2$-graded ring $\mathcal{A}$, a left $\mathbb{Z}_2$-graded $\mathcal{A}$-module $Q$ is defined as a left $\mathcal{A}$-module which is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module such that

$$[aq] = ([a] + [q]) \bmod 2.$$

Similarly, right graded $\mathcal{A}$-modules and graded $(\mathcal{A} - \mathcal{A})$-bimodules are defined. If $\mathcal{A}$ is a $\mathbb{Z}_2$-graded commutative ring, a $\mathbb{Z}_2$-graded left or right $\mathcal{A}$-module $Q$ can be provided with a $\mathbb{Z}_2$-graded commutative $\mathcal{A}$-bimodule structure by letting

$$qa = (-1)^{[a][q]}aq, \qquad a \in \mathcal{A}, \qquad q \in Q.$$

Therefore, unless otherwise stated (Section 5.2), any $\mathbb{Z}_2$-graded $\mathcal{A}$-module over a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$ is a $\mathbb{Z}_2$-graded commutative $\mathcal{A}$-bimodule which is called the $\mathcal{A}$-module if there is no danger of confusion.

Given a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$, the following are standard constructions of new $\mathbb{Z}_2$-graded modules from the old ones.

• A direct sum of $\mathbb{Z}_2$-graded modules and a $\mathbb{Z}_2$-graded factor module are defined just as those of modules over a commutative ring.

• A tensor product $P \otimes Q$ of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules $P$ and $Q$ is their tensor product as $\mathcal{A}$-modules such that

$$\begin{aligned}
&[p \otimes q] = ([p] + [q]) \mod 2, \qquad p \in P, \qquad q \in Q, \\
&ap \otimes q = (-1)^{[p][a]}pa \otimes q = (-1)^{[p][a]}p \otimes aq, \qquad a \in \mathcal{A}.
\end{aligned}$$

In particular, the tensor algebra $\otimes P$ of a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $P$ is defined just as that (2.2) of a module over a commutative ring. Its quotient $\wedge P$ with respect to the ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']}p' \otimes p, \qquad p, p' \in P,$$

is the bigraded exterior algebra of a $\mathbb{Z}_2$-graded module $P$ with respect to the graded exterior product

$$p \wedge p' = -(-1)^{[p][p']}p' \wedge p. \tag{5.2}$$

• A morphism $\Phi : P \to Q$ of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules seen as $\mathcal{K}$-modules is said to be even morphism (resp. odd morphism) if $\Phi$ preserves (resp. change) the $\mathbb{Z}_2$-parity of all homogeneous elements of $P$ and the relations

$$\Phi(ap) = (-1)^{[\Phi][a]}a\Phi(p), \qquad p \in P, \qquad a \in \mathcal{A}, \tag{5.3}$$

hold. A morphism $\Phi : P \to Q$ of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules as the $\mathcal{K}$-ones is called a graded $\mathcal{A}$-module morphism if it is represented by a sum of even and odd morphisms. Therefore, a set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of graded morphisms of a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $P$ to a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $Q$ is a $\mathbb{Z}_2$-graded $\mathcal{A}$-module. A $\mathbb{Z}_2$-graded $\mathcal{A}$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $P$.

**Remark 5.4:** Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded commutative ring. A $\mathbb{Z}_2$-graded $\mathcal{A}$-module $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is called the Lie $\mathcal{A}$-superalgebra if it is an $\mathcal{A}$-algebra whose product

$[.,.]$, called the Lie superbracket, obeys the rules

$$[\varepsilon,\varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon',\varepsilon],$$
$$(-1)^{[\varepsilon][\varepsilon'']}[\varepsilon,[\varepsilon',\varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon',[\varepsilon'',\varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'',[\varepsilon,\varepsilon']] = 0.$$

Even and odd parts of a Lie superalgebra $\mathcal{G}$ satisfy supercommutation relations

$$[\mathcal{G}_0,\mathcal{G}_0]\subset\mathcal{G}_0, \qquad [\mathcal{G}_0,\mathcal{G}_1]\subset\mathcal{G}_1, \qquad [\mathcal{G}_1,\mathcal{G}_1]\subset\mathcal{G}_1. \tag{5.4}$$

In particular, an even part $\mathcal{G}_0$ of a Lie $\mathcal{A}$-superalgebra $\mathcal{G}$ is a Lie $\mathcal{A}_0$-algebra. Given an $\mathcal{A}$-superalgebra, a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $P$ is called a $\mathcal{G}$-module if it is provided with an $\mathcal{A}$-bilinear map

$$\mathcal{G}\times P \ni (\varepsilon,p)\to \varepsilon p\in P, \qquad [\varepsilon p] = ([\varepsilon]+[p])\mathrm{mod}\,2,$$
$$[\varepsilon,\varepsilon']p = (\varepsilon\circ\varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon'\circ\varepsilon)p.$$

□

We mainly restrict our consideration to the following particular class of $\mathbb{Z}_2$-graded commutative rings.

DEFINITION 5.4: A $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$ is said to be the Grassmann algebra if it is a free $\mathcal{K}$-module of finite rank so that

$$\mathcal{A}_0 = \mathcal{K}\oplus(\mathcal{A}_1)^2. \tag{5.5}$$

□

It follows from the expression (5.5) that a $\mathcal{K}$-module $\mathcal{A}$ admits a decomposition

$$\mathcal{A} = \mathcal{K}\oplus R, \qquad R = \mathcal{A}_1\oplus(\mathcal{A}_1)^2, \tag{5.6}$$

where $R$ is the ideal of nilpotents of a ring $\mathcal{A}$. The corresponding surjections

$$\sigma:\mathcal{A}\to\mathcal{K}, \qquad s:\mathcal{A}\to R \tag{5.7}$$

are called the body and soul maps, respectively. Automorphisms of a Grassmann algebra preserve its ideal $R$ of nilpotents and the splittings (5.5) and (5.6), but need not the odd sector $\mathcal{A}_1$.

A Grassmann-graded ring $\Omega^*$ in Definition 1.5, seen as a $\mathbb{Z}_2$-graded commutative ring, exemplifies a Grassmann algebra. Conversely, any Grassmann algebra $\mathcal{A}$ admits an associative $\mathbb{N}$-graded structure of a Grassmann-graded ring $\mathcal{A}^*$ by a choice of its minimal generating $\mathcal{K}$-module $\mathcal{A}^1$.

THEOREM 5.1: If $\mathcal{K}$ is a field, all $\mathbb{N}$-graded structures $\mathcal{A}^*$ of a Grassmann algebra $\mathcal{A}^*$ are isomorphic by means of automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ in accordance with Theorem 6.1. □

In particular, we come to the following.

THEOREM 5.2: Given a Grassmann algebra $\mathcal{A}$ over a field $\mathcal{K}$ and an associated Grassmann-graded ring $\mathcal{A}^*$, there exists a finite-dimensional vector space $W$

over $\mathcal{K}$ so that $\mathcal{A}$ and $\mathcal{A}^*$ are isomorphic to the exterior algebra $\wedge W$ of $W$ (Example 2.3) seen as a Grassmann-graded ring $\mathcal{A}^*$ generated by $\mathcal{A}^1 = W$. $\square$

Let us note that, by automorphisms of a Grassmann-graded ring $\mathcal{A}^*$ are meant automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which preserve its $\mathbb{N}$-gradation $\mathcal{A}^*$ (Section 6). Accordingly, automorphisms of a Grassmann algebra $\mathcal{A}$ are automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which preserve its $\mathbb{Z}_2$-gradation $\mathcal{A}^*$. However, there exist automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which do not satisfy this condition as follows.

Given a generating basis $\{c^i\}$ for a $\mathcal{K}$-module $\mathcal{A}^1$, elements of a Grassmann-graded ring $\mathcal{A}^*$ take a form

$$a = \sum_{k=0,1,\ldots} \sum_{(i_1 \cdots i_k)} a_{i_1 \cdots i_k} c^{i_1} \cdots c^{i_k}, \tag{5.8}$$

where the second sum runs through all the tuples $(i_1 \cdots i_k)$ such that no two of them are permutations of each other. We agree to call $\{c^i\}$ the generating basis for a Grassmann algebra $\mathcal{A}$ which brings it into a Grassmann-graded ring $\mathcal{A}^*$.

Given a generating basis $\{c^i\}$ for a Grassmann algebra $\mathcal{A}$, any its $\mathcal{K}$-ring automorphisms are compositions of automorphisms

$$c^i \to c'^i = \rho^i_j c^j + b^i, \tag{5.9}$$

where $\rho$ is an automorphism of $\mathcal{K}$-module $\mathcal{A}^1$ and $b^i$ are odd elements of $\mathcal{A}^{>2}$, and automorphisms

$$c^i \to c'^i = c^i(\mathbf{1} + \kappa), \qquad \kappa \in \mathcal{A}_1. \tag{5.10}$$

Automorphisms (5.9) where $b^i = 0$ are automorphisms of a Grassmann-graded ring $A^*$. If $b^i \neq 0$, the automorphism (5.9) preserve a $\mathbb{Z}_2$-graded structure of $\mathcal{A}$, does not keep its $\mathbb{N}$-graded structure $\mathcal{A}^*$ of, and yields a different $\mathbb{N}$-graded structure $\mathcal{A}'^*$ where $\{c'^i\}$ (5.9) is a basis for $\mathcal{A}'^1$ and the generating basis for $\mathcal{A}'^*$. Automorphisms (5.10) preserve an even sector $\mathcal{A}_0$ of $A$, but not the odd one $\mathcal{A}_1$. However, it follows from Theorem 5.1 that, since $\mathbb{N}$-graded and $\mathbb{Z}_2$-graded structures of a Grassmann algebra are associated, their $\mathbb{Z}_2$-graded structures are isomorphic if $\mathcal{K}$ is a field.

## 5.2 $\mathbb{Z}_2$-Graded differential calculus

The differential calculus on $\mathbb{Z}_2$-graded modules over $\mathbb{Z}_2$-graded commutative rings is defined similarly to that over commutative rings [16, 38, 39], but different from the differential calculus over non-commutative rings in Section 9 (Remark 5.5).

Let $\mathcal{K}$ be a commutative ring without a divisor of zero and $\mathcal{A}$ a $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring. Let $P$ and $Q$ be $\mathbb{Z}_2$-graded $\mathcal{A}$-modules. A $\mathbb{Z}_2$-graded $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathbb{Z}_2$-graded $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with the two $\mathbb{Z}_2$-graded $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P, \tag{5.11}$$

called $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures, respectively. Let us put

$$\delta_a\Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \bullet a, \qquad a\in\mathcal{A}. \tag{5.12}$$

DEFINITION 5.5: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is said to be the $Q$-valued $\mathbb{Z}_2$-graded differential operator of order $s$ on $P$ if

$$\delta_{a_0}\circ\cdots\circ\delta_{a_s}\Delta = 0 \tag{5.13}$$

for any tuple of $s+1$ elements $a_0,\ldots,a_s$ of $\mathcal{A}$. A set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the $\mathbb{Z}_2$-graded $\mathcal{A}$-module structures (5.11). □

**Remark 5.5:** It should be emphasized that, though a $\mathbb{Z}_2$-graded commutative ring is a non-commutative ring, the map (5.12) differs from the map (9.8), and therefore the differential calculus over a $\mathbb{Z}_2$-graded commutative ring is not the particular non-commutative differential calculus in Section 9. Let us note that, though $\mathbb{Z}_2$-graded Clifford algebras (Example 5.3) fail to be graded commutative, the definition of differential operators over them also is based on the formula (5.12), but not (9.8). However, graded derivations of a Clifford algebra do not form a free module over it. □

For instance, zero-order $\mathbb{Z}_2$-graded differential operators obey a condition

$$\delta_a\Delta(p) = a\Delta(p) - (-1)^{[a][\Delta]}\Delta(ap) = 0, \qquad a\in\mathcal{A}, \qquad p\in P,$$

i.e., they coincide with graded $\mathcal{A}$-module morphisms $P\to Q$ (cf. Remark 9.2).

A first-order $\mathbb{Z}_2$-graded differential operator $\Delta$ satisfies a condition

$$\begin{aligned}\delta_a\circ\delta_b\,\Delta(p) = {}& ab\Delta(p) - (-1)^{([b]+[\Delta])[a]}b\Delta(ap) - (-1)^{[b][\Delta]}a\Delta(bp) + \\ & (-1)^{[b][\Delta]+([\Delta]+[b])[a]} = 0, \qquad a,b\in\mathcal{A}, \quad p\in P.\end{aligned}$$

For instance, let $P=\mathcal{A}$. Any zero-order $Q$-valued $\mathbb{Z}_2$-graded differential operator $\Delta$ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is a graded $\mathcal{A}$-module isomorphism $\mathrm{Diff}_0(\mathcal{A},Q)=Q$ via the association

$$Q\ni q\to\Delta_q\in\mathrm{Diff}_0(\mathcal{A},Q),$$

where $\Delta_q$ is given by the equality $\Delta_q(\mathbf{1})=q$. A first-order $Q$-valued $\mathbb{Z}_2$-graded differential operator $\Delta$ on $\mathcal{A}$ fulfils a condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b) - (-1)^{([b]+[a])[\Delta]}ab\Delta(\mathbf{1}), \qquad a,b\in\mathcal{A}.$$

It is called the $Q$-valued $\mathbb{Z}_2$-graded derivation of $\mathcal{A}$ if $\Delta(\mathbf{1})=0$, i.e., the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b), \qquad a,b\in\mathcal{A},$$

holds (cf. (2.13)). One obtains at once that any first-order $\mathbb{Z}_2$-graded differential operator on $\mathcal{A}$ falls into a sum

$$\Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]$$

of a zero-order graded differential operator $\Delta(\mathbf{1})a$ and a graded derivation $\Delta(a) - \Delta(\mathbf{1})a$. If $\partial$ is a $\mathbb{Z}_2$-graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, $\mathbb{Z}_2$-graded derivations of $\mathcal{A}$ constitute a $\mathbb{Z}_2$-graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the graded derivation module.

If $Q = \mathcal{A}$, $\mathbb{Z}_2$-graded derivation of $\mathcal{A}$ obeys the graded Leibniz rule

$$\partial(ab) = \partial(a)b + (-1)^{[a][\partial]}a\partial(b), \qquad a, b \in \mathcal{A}, \tag{5.14}$$

(cf. (9.1)). The derivation module $\mathfrak{d}\mathcal{A}$ also is a Lie superalgebra (Definition 5.4) over a commutative ring $\mathcal{K}$ with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']}u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{5.15}$$

We have an $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}.$$

Graded differential operators on a $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$ form a direct system of $\mathbb{Z}_2$-graded $(\mathcal{A} - \mathcal{A}^\bullet)$-modules.

$$\mathrm{Diff}_0(\mathcal{A}) \xrightarrow{\mathrm{in}} \mathrm{Diff}_1(\mathcal{A}) \cdots \xrightarrow{\mathrm{in}} \mathrm{Diff}_r(\mathcal{A}) \longrightarrow \cdots \tag{5.16}$$

Its direct limit $\mathrm{Diff}_\infty(\mathcal{A})$ (Remark 2.5) is a $\mathbb{Z}_2$-graded module of all $\mathbb{Z}_2$-graded differential operators on $\mathcal{A}$.

Since the graded derivation module $\mathfrak{d}\mathcal{A}$ is a Lie $\mathcal{K}$-superalgebra, let us consider the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$ where a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$ is a regarded as a $\mathfrak{d}\mathcal{A}$-module [13, 39]. It is a complex

$$0 \to \mathcal{A} \xrightarrow{d} C^1[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots \tag{5.17}$$

where

$$C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] = \mathrm{Hom}_{\mathcal{K}}(\overset{k}{\wedge} \mathfrak{d}\mathcal{A}, \mathcal{A})$$

are $\mathfrak{d}\mathcal{A}$-modules of $\mathcal{K}$-linear graded morphisms of the graded exterior products $\overset{k}{\wedge} \mathfrak{d}\mathcal{A}$ of a $\mathbb{Z}_2$-graded $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$. Let us bring homogeneous elements of $\overset{k}{\wedge} \mathfrak{d}\mathcal{A}$ into a form

$$\varepsilon_1 \wedge \cdots \varepsilon_r \wedge \epsilon_{r+1} \wedge \cdots \wedge \epsilon_k, \qquad \varepsilon_i \in \mathfrak{d}\mathcal{A}_0, \quad \epsilon_j \in \mathfrak{d}\mathcal{A}_1.$$

Then the Chevalley–Eilenberg coboundary operator $d$ of the complex (5.17) is given by the expression

$$
\begin{aligned}
&dc(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) = \qquad (5.18)\\
&\sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\epsilon_s) +\\
&\sum_{j=1}^{s}(-1)^{r}\varepsilon_i c(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) +\\
&\sum_{1\leq i<j\leq r}(-1)^{i+j}c([\varepsilon_i,\varepsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\widehat{\varepsilon}_j\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) +\\
&\sum_{1\leq i<j\leq s}c([\epsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_i\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) +\\
&\sum_{1\leq i<r,1\leq j\leq s}(-1)^{i+r+1}c([\varepsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s),
\end{aligned}
$$

where the caret $\widehat{}$ denotes omission. This operator is called the graded Chevalley–Eilenberg coboundary operator.

Let us consider the extended Chevalley–Eilenberg complex

$$0\to\mathcal{K}\xrightarrow{\text{in}}C^*[\mathfrak{d}\mathcal{A};\mathcal{A}].$$

It is easily justified that this complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-linear graded morphisms. The $\mathbb{N}$-graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the structure of a bigraded $\mathcal{A}$-algebra with respect to the graded exterior product

$$
\begin{aligned}
&\phi\wedge\phi'(u_1,...,u_{r+s}) = \qquad (5.19)\\
&\qquad\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}),\\
&\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}],\qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}],\qquad u_k\in\mathfrak{d}\mathcal{A},
\end{aligned}
$$

where $u_1,\ldots,u_{r+s}$ are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$u_1\wedge\cdots\wedge u_{r+s}=\mathrm{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}u_{i_1}\wedge\cdots\wedge u_{i_r}\wedge u_{j_1}\wedge\cdots\wedge u_{j_s}.$$

The graded Chevalley–Eilenberg coboundary operator $d$ (5.18) and the graded exterior product $\wedge$ (5.19) bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential bigraded ring whose elements obey relations

$$\phi\wedge\phi'=(-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \qquad (5.20)$$

$$d(\phi\wedge\phi')=d\phi\wedge\phi'+(-1)^{|\phi|}\phi\wedge d\phi'. \qquad (5.21)$$

It is called the graded Chevalley–Eilenberg differential calculus over a $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$. In particular, we have

$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}]=\mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A})=\mathfrak{d}\mathcal{A}^*. \qquad (5.22)$$

One can extend this duality relation to the graded interior product of $u \in \mathfrak{d}\mathcal{A}$ with any element $\phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$\begin{aligned} & u\rfloor(bda) = (-1)^{[u][b]}u(a), \qquad a, b \in \mathcal{A}, \\ & u\rfloor(\phi \wedge \phi') = (u\rfloor\phi) \wedge \phi' + (-1)^{|\phi|+[\phi][u]}\phi \wedge (u\rfloor\phi'). \end{aligned} \tag{5.23}$$

As a consequence, any $\mathbb{Z}_2$-graded derivation $u \in \mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a graded derivation

$$\begin{aligned} & \mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \phi \in \mathcal{O}^*, \qquad u \in \mathfrak{d}\mathcal{A}, \\ & \mathbf{L}_u(\phi \wedge \phi') = \mathbf{L}_u(\phi) \wedge \phi' + (-1)^{[u][\phi]}\phi \wedge \mathbf{L}_u(\phi'), \end{aligned} \tag{5.24}$$

termed the graded Lie derivative of a differential bigraded ring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

The minimal graded Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$ consists of monomials $a_0da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. The corresponding complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots \tag{5.25}$$

is called the bigraded de Rham complex of a $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

Let us note that, if $\mathcal{A}$ is a commutative ring, $\mathbb{Z}_2$-graded differential operators and the graded Chevalley–Eilenberg differential calculus defined above restart the familiar commutative ones in Sections 2.2 and 2.4.

**Example 5.6:** Let $\mathcal{A}$ be a Grassmann algebra provided with an odd generating basis $\{c^i\}$. Its $\mathbb{Z}_2$-graded derivations are defined in full by their action on the generating elements $c^i$. Let us consider odd derivations

$$\partial_i(c^j) = \delta_i^j, \qquad \partial_i \circ \partial_j = -\partial_j \circ \partial_i. \tag{5.26}$$

Then any $\mathbb{Z}_2$-graded derivation of $\mathcal{A}$ takes a form

$$u = u^i\partial_i, \qquad u_i \in \mathcal{A}. \tag{5.27}$$

Graded derivations (5.27) constitute the free $\mathbb{Z}_2$-graded $\mathcal{A}$-module $\mathfrak{d}\mathcal{A}$ of finite rank. It also is a finite dimensional Lie superalgebra over $\mathcal{K}$ with respect to the superbracket (5.15). Any $\mathbb{Z}_2$-graded differential operator on a Grassmann algebra $\mathcal{A}$ is a composition of $\mathbb{Z}_2$-graded derivations. $\Box$

## 5.3 $\mathbb{Z}_2$-Graded manifolds

As was mentioned in Remark 2.1, the notion of a local ring is extended to the $\mathbb{Z}_2$-graded ones (Definition 2.1), and formalism of $\mathbb{Z}_2$-graded commutative local-ringed spaces can be developed. Grassmann-graded rings in Definition 1.5 and Grassmann algebras in Definition 5.4 are local. Their maximal ideals consist of all nilpotent elements. We restrict our consideration to real Grassmann algebras $\Lambda$. In this case Theorem 2.2 remains true, and we follow the notion of local-ringed spaces in Section 3 [2, 16, 38, 40].

A $\mathbb{Z}_2$-graded manifold of dimension $(n,m)$ is defined as a local-ringed space $(Z,\mathfrak{A})$ (Definition 3.1) where $Z$ is an $n$-dimensional smooth manifold, and $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$ is a sheaf of real Grassmann algebras such that:

• there is the exact sequence of sheaves

$$0\to\mathcal{R}\to\mathfrak{A}\xrightarrow{\sigma}C^\infty_Z\to 0,\qquad \mathcal{R}=\mathfrak{A}_1+(\mathfrak{A}_1)^2, \tag{5.28}$$

where $C^\infty_Z$ is the sheaf of smooth real functions on $Z$;

• $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of $C^\infty_Z$-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the exterior product $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

A sheaf $\mathfrak{A}$ is called the structure sheaf of a $\mathbb{Z}_2$-graded manifold $(Z,\mathfrak{A})$, and a manifold $Z$ is said to be the body of $(Z,\mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are termed graded functions on a $\mathbb{Z}_2$-graded manifold $(Z,\mathfrak{A})$. They make up a $\mathbb{Z}_2$-graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$ called the structure ring of $(Z,\mathfrak{A})$.

By virtue of the well-known Batchelor theorem [2, 39], $\mathbb{Z}_2$-graded manifolds possess the following structure.

THEOREM 5.3: Let $(Z,\mathfrak{A})$ be a $\mathbb{Z}_2$-graded manifold. There exists a vector bundle $E\to Z$ with an $m$-dimensional typical fibre $V$ such that the structure sheaf $\mathfrak{A}$ of $(Z,\mathfrak{A})$ as a sheaf in real rings is isomorphic to the structure sheaf $\mathfrak{A}_E=\wedge E^*_Z$ of germs of sections of the exterior bundle

$$\wedge E^*=(Z\times\mathbb{R})\underset{Z}{\oplus}E^*\underset{Z}{\oplus}\wedge E^*\underset{Z}{\oplus}\overset{2}{\wedge}E^*\cdots, \tag{5.29}$$

whose typical fibre is the Grassmann algebra $\Lambda=\wedge V^*$ in Theorem 5.2. $\Box$

It should be emphasized that Batchelor's isomorphism in Theorem 5.3 fails to be canonical. In applications, it however is fixed from the beginning. Therefore, we restrict our consideration to $\mathbb{Z}_2$-graded manifolds $(Z,\mathfrak{A}_E)$ whose structure sheaf is the sheaf of germs of sections of some exterior bundle $\wedge E^*$.

DEFINITION 5.6: We agree to call $(Z,\mathfrak{A}_E)$ the simple graded manifold modelled over a vector bundle $E\to Z$, called its characteristic vector bundle. $\Box$

Accordingly, the structure ring $\mathcal{A}_E$ of a simple graded manifold $(Z,\mathfrak{A}_E)$ is the structure module

$$\mathcal{A}_E=\mathfrak{A}_E(Z)=\wedge E^*(Z) \tag{5.30}$$

of sections of the exterior bundle $\wedge E^*$. Automorphisms of a simple graded manifold $(Z,\mathfrak{A}_E)$ are restricted to those induced by automorphisms of its characteristic vector bundles $E\to Z$.

**Remark 5.7:** In fact, the structure sheaf $\mathfrak{A}_E$ of a simple $\mathbb{Z}_2$-graded manifold in Definition 5.6 is a sheaf in Grassmann-graded rings $\Lambda^*$ whose $\mathbb{N}$-graded structure is fixed. Therefore, it is an $\mathbb{N}$-graded manifold (Definition 6.2). $\Box$

Combining Batchelor Theorem 5.3 and classical Serre–Swan Theorem 4.4, we come to the following Serre–Swan theorem for $\mathbb{Z}_2$-graded manifolds [3, 16].

THEOREM 5.4: Let $Z$ be a smooth manifold. A $\mathbb{Z}_2$-graded commutative $C^\infty(Z)$-ring $\mathcal{A}$ is isomorphic to the structure ring of a $\mathbb{Z}_2$-graded manifold with a body

$Z$ iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank. $\square$

*Proof:* By virtue of the Batchelor theorem, any $\mathbb{Z}_2$-graded manifold is isomorphic to a simple graded manifold $(Z, \mathfrak{A}_E)$ modelled over some vector bundle $E \to Z$. Its structure ring $\mathcal{A}_E$ (5.30) of graded functions consists of sections of the exterior bundle $\wedge E^*$. The classical Serre–Swan theorem states that a $C^\infty(Z)$-module is isomorphic to the module of sections of a smooth vector bundle over $Z$ iff it is a projective module of finite rank. $\square$

**Remark 5.8:** One can treat a local-ringed space $(Z, \mathfrak{A}_0 = C^\infty_Z)$ as a trivial even $\mathbb{Z}_2$-graded manifold. It is a simple graded manifold whose characteristic bundle is $E = Z \times \{0\}$. Its structure module is a ring $C^\infty(Z)$ of smooth real functions on $Z$. $\square$

Given a simple graded manifold $(Z, \mathfrak{A}_E)$, every trivialization chart $(U; z^A, y^a)$ of a vector bundle $E \to Z$ yields a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$. Graded functions on such a chart are $\Lambda$-valued functions

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \ldots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{5.31}$$

where $f_{a_1 \cdots a_k}(z)$ are smooth functions on $U$ and $\{c^a\}$ is the fibre basis for $E^*$. In particular, the sheaf epimorphism $\sigma$ in (5.28) is induced by the body map of $\Lambda$. One calls $\{z^A, c^a\}$ the local basis for a graded manifold $(Z, \mathfrak{A}_E)$ [2]. Transition functions $y'^a = \rho^a_b(z^A) y^b$ of bundle coordinates on $E \to Z$ induce the corresponding transformation

$$c'^a = \rho^a_b(z^A) c^b \tag{5.32}$$

(cf. (6.1)) of the associated local basis for a simple graded manifold $(Z, \mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (5.31).

**Remark 5.9:** Strictly speaking, elements $c^a$ of the local basis for a simple graded manifold are locally constant sections $c^a$ of $E^* \to X$ such that $y_b \circ c^a = \delta^a_b$. Therefore, graded functions are locally represented by $\Lambda$-valued functions (5.31), but they are not $\Lambda$-valued functions on a manifold $Z$ because of the transformation law (5.32). $\square$

Given a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$, by the sheaf $\mathfrak{d}\mathfrak{A}$ of graded derivations of $\mathfrak{A}$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}$ such that any section $u \in \mathfrak{d}\mathfrak{A}(U)$ of $\mathfrak{d}\mathfrak{A}$ over an open subset $U \subset Z$ is a graded derivation of the real $\mathbb{Z}_2$-graded commutative algebra $\mathfrak{A}(U)$, i.e., $u \in \mathfrak{d}(\mathfrak{A}(U))$. Conversely, one can show that, given open sets $U' \subset U$, there is a surjection of the graded derivation modules $\mathfrak{d}(\mathfrak{A}(U)) \to \mathfrak{d}(\mathfrak{A}(U'))$ [2]. It follows that any graded derivation of a local $\mathbb{Z}_2$-graded algebra $\mathfrak{A}(U)$ also is a local section over $U$ of a sheaf $\mathfrak{d}\mathfrak{A}$. Global sections of $\mathfrak{d}\mathfrak{A}$ are called graded vector fields on a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$. They make up a graded derivation module $\mathfrak{d}\mathfrak{A}(Z)$ of a real $\mathbb{Z}_2$-graded commutative ring $\mathfrak{A}(Z)$. This module is a real Lie superalgebra with respect to the superbracket (5.15).

A key point is that graded vector fields $u \in \mathfrak{d}\mathcal{A}_E$ on a simple graded manifold $(Z, \mathfrak{A}_E)$ can be represented by sections of some vector bundle as follows [16, 28, 39].

Due to the canonical splitting $VE = E \times E$, the vertical tangent bundle $VE$ of $E \to Z$ can be provided with fibre bases $\{\partial/\partial c^a\}$, which are the duals of bases $\{c^a\}$. Then graded vector fields on a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$ read

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \tag{5.33}$$

where $u^\lambda, u^a$ are local graded functions on $U$. In particular,

$$\frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} = -\frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \qquad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A.$$

The graded derivations (5.33) act on graded functions $f \in \mathfrak{A}_E(U)$ (5.31) by a rule

$$u(f_{a\ldots b} c^a \cdots c^b) = u^A \partial_A (f_{a\ldots b}) c^a \cdots c^b + u^k f_{a\ldots b} \frac{\partial}{\partial c^k} \rfloor (c^a \cdots c^b). \tag{5.34}$$

This rule implies a corresponding coordinate transformation law

$$u'^A = u^A, \qquad u'^a = \rho^a_j u^j + u^A \partial_A (\rho^a_j) c^j$$

of graded vector fields. It follows that graded vector fields (5.33) can be represented by sections of a vector bundle

$$\mathcal{V}_E = \wedge E^* \underset{E}{\otimes} TE \to Z. \tag{5.35}$$

Thus, the graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module $\mathcal{V}_E(Z)$ of global sections of the vector bundle $\mathcal{V}_E \to Z$ (5.35).

Given the structure ring $\mathcal{A}_E$ of graded functions on a simple graded manifold $(Z, \mathfrak{A}_E)$ and the real Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, let us consider the graded Chevalley–Eilenberg differential calculus

$$\mathcal{S}^*[E; Z] = \mathcal{O}^*[\mathfrak{d}\mathcal{A}_E] \tag{5.36}$$

over $\mathcal{A}_E$ where $\mathcal{S}^0[E; Z] = \mathcal{A}_E$.

THEOREM 5.5: Since a graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module of sections of a vector bundle $\mathcal{V}_E \to Z$, elements of $\mathcal{S}^*[E; Z]$ are represented by sections of the exterior bundle $\wedge \overline{\mathcal{V}}_E$ of the $\mathcal{A}_E$-dual

$$\overline{\mathcal{V}}_E = \wedge E^* \underset{E}{\otimes} T^* E \to Z \tag{5.37}$$

of $\mathcal{V}_E$. $\square$

With respect to the dual fibre bases $\{dz^A\}$ for $T^*Z$ and $\{dc^b\}$ for $E^*$, sections of $\overline{\mathcal{V}}_E$ (5.37) take a coordinate form

$$\phi = \phi_A dz^A + \phi_a dc^a,$$

together with transition functions

$$\phi'_a = \rho^{-1}{}_a^b \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1}{}_a^b \partial_A(\rho^a_j)\phi_b c^j.$$

The duality isomorphism $\mathcal{S}^1[E;Z] = \mathfrak{d}\mathcal{A}_E^*$ (5.22) is given by the graded interior product

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]}u^a\phi_a.$$

Elements of $\mathcal{S}^*[E;Z]$ are called graded exterior forms on a graded manifold $(Z,\mathfrak{A}_E)$.

Seen as an $\mathcal{A}_E$-algebra, the differential bigraded ring $\mathcal{S}^*[E;Z]$ (5.36) on a splitting domain $(z^A, c^a)$ is locally generated by the graded one-forms $dz^A$, $dc^i$ such that

$$dz^A \wedge dc^i = -dc^i \wedge dz^A, \qquad dc^i \wedge dc^j = dc^j \wedge dc^i.$$

Accordingly, the graded Chevalley–Eilenberg coboundary operator $d$ (5.18), termed the graded exterior differential, reads

$$d\phi = dz^A \wedge \partial_A\phi + dc^a \wedge \frac{\partial}{\partial c^a}\phi,$$

where derivatives $\partial_\lambda$, $\partial/\partial c^a$ act on coefficients of graded exterior forms by the formula (5.34), and they are graded commutative with the graded exterior forms $dz^A$ and $dc^a$. The formulas (5.20) – (5.24) hold.

THEOREM 5.6: The differential bigraded ring $\mathcal{S}^*[E;Z]$ (5.36) is a minimal differential calculus over $\mathcal{A}_E$, i.e., it is generated by elements $df$, $f \in \mathcal{A}_E$. □

*Proof:* Since $\mathfrak{d}\mathcal{A}_E = \mathcal{V}_E(Z)$, it is a projective $C^\infty(Z)$- and $\mathcal{A}_E$-module of finite rank, and so is its $\mathcal{A}_E$-dual $\mathcal{S}^1[E;Z]$ (Theorem 2.1). Hence, $\mathfrak{d}\mathcal{A}_E$ is the $\mathcal{A}_E$-dual of $\mathcal{S}^1[E;Z]$ and, consequently, $\mathcal{S}^1[E;Z]$ is generated by elements $df$, $f \in \mathcal{A}_E$. □

The bigraded de Rham complex (5.25) of a minimal graded Chevalley–Eilenberg differential calculus $\mathcal{S}^*[E;Z]$ reads

$$0 \to \mathbb{R} \to \mathcal{A}_E \xrightarrow{d} \mathcal{S}^1[E;Z] \xrightarrow{d} \cdots \mathcal{S}^k[E;Z] \xrightarrow{d} \cdots . \tag{5.38}$$

Its cohomology $H^*(\mathcal{A}_E)$ is called the de Rham cohomology of a simple graded manifold $(Z,\mathfrak{A}_E)$.

In particular, given a differential graded ring $\mathcal{O}^*(Z)$ of exterior forms on $Z$, there exist a canonical monomorphism

$$\mathcal{O}^*(Z) \to \mathcal{S}^*[E;Z] \tag{5.39}$$

and the body epimorphism $\mathcal{S}^*[E;Z] \to \mathcal{O}^*(Z)$ which are cochain morphisms of the de Rham complexes (5.38) and (4.4).

THEOREM 5.7: The de Rham cohomology of a simple graded manifold $(Z, \mathfrak{A}_E)$ equals the de Rham cohomology of its body $Z$. $\Box$

*Proof:* Let $\mathfrak{A}_E^k$ denote the sheaf of germs of graded $k$-forms on $(Z, \mathfrak{A}_E)$. Its structure module is $\mathcal{S}^k[E;Z]$. These sheaves constitute a complex

$$0 \to \mathbb{R} \longrightarrow \mathfrak{A}_E \xrightarrow{d} \mathfrak{A}_E^1 \xrightarrow{d} \cdots \mathfrak{A}_E^k \xrightarrow{d} \cdots . \tag{5.40}$$

Its members $\mathfrak{A}_E^k$ are sheaves of $C_Z^\infty$-modules on $Z$ and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded exterior forms holds [2]. It follows that the complex (5.40) is a fine resolution of the constant sheaf $\mathbb{R}$ on a manifold $Z$. Then, by virtue of Theorem 8.7, there is an isomorphism

$$H^*(\mathcal{A}_E) = H^*(Z;\mathbb{R}) = H^*_{\rm DR}(Z) \tag{5.41}$$

of the cohomology $H^*(\mathcal{A}_E)$ to the de Rham cohomology $H^*_{\rm DR}(Z)$ of a smooth manifold $Z$. $\Box$

THEOREM 5.8: The cohomology isomorphism (5.41) is accompanied by the cochain monomorphism (5.39). Hence, any closed graded exterior form is decomposed into a sum $\phi = \sigma + d\xi$ where $\sigma$ is a closed exterior form on $Z$. $\Box$

# 6 $\mathbb{N}$-graded manifolds

Let $\mathcal{K}$ be a commutative ring without a divisor of zero, and let $\Omega = \Omega^*$ (1.2) be an $\mathbb{N}$-graded $\mathcal{K}$-ring (Definition 1.2).

As it was observed above, an $\mathbb{N}$-graded ring seen as a $\mathcal{K}$-ring can admit different $\mathbb{N}$-graded structures. In the following case, all these structures are isomorphic [5].

DEFINITION 6.1: An $\mathbb{N}$-graded $\mathcal{K}$-ring $\mathcal{A}^*$ is said to be finitely generated in degree 1 if the following hold:
• $\mathcal{A}^0 = \mathcal{K}$,
• $\mathcal{A}^1$ is a free $\mathcal{K}$-module of finite rank,
• $\mathcal{A}^*$ is generated by $\Omega^1$, namely, if $R$ is an ideal generated by $\mathcal{A}^1$, then there are $\mathcal{K}$-module isomorphism $\Omega/R = \mathcal{K}$, $R/R^2 = \mathcal{A}^1$. $\Box$

THEOREM 6.1: Let $\mathcal{K}$ be a field, and let $\mathcal{A}^*$ and $\Lambda^*$ be $\mathbb{N}$-graded $\mathcal{K}$-rings finitely generated in degree 1. If they are isomorphic as $\mathcal{K}$-rings, there exists their graded isomorphism $\Phi : \mathcal{A}^* \to \Lambda^*$ so that $\Phi(\mathcal{A}^i) = \Lambda^i$ for all $i \in \mathbb{N}$. $\Box$

Let us mention the following particular types of $\mathbb{N}$-graded rings.
• A commutative ring $\mathcal{A}$ is the $\mathbb{N}$-graded ring $\mathcal{A}^*$ where $\mathcal{A}^0 = \mathcal{A}$ and $\mathcal{A}^{>0} = 0$.
• It is a particular commutative $\mathbb{N}$-graded ring $\mathcal{A}^*$ where $\alpha \cdot \beta = \beta \cdot \alpha$ for all elements $\alpha, \beta \in \mathcal{A}^*$ (cf. (1.3)).

• A polynomial $\mathcal{K}$-ring $\mathcal{P}[Q]$ of a $\mathcal{K}$-module $Q$ in Example 2.3 exemplifies a commutative $\mathbb{N}$-graded ring. If $\mathcal{K}$ is a field and $Q$ is a free $\mathcal{K}$-module of finite rank, a polynomial $\mathcal{K}$-ring $\mathcal{P}[Q]$ is finitely generated in degree 1 by virtue of Definition 6.1. If $\mathcal{K}$ is a field, all $\mathbb{N}$-graded structures of this ring are mutually isomorphic in accordance with Theorem 6.1.

• We consider $\mathbb{N}$-graded commutative rings (Definition 1.3). Any commutative $\mathbb{N}$-graded ring $\mathcal{A}^*$ can be regarded as an even $\mathbb{N}$-graded commutative ring $\Lambda^*$ such that $\Lambda^{2i} = \mathcal{A}^i$, $\Lambda^{2i+1} = 0$.

• Grassmann-graded rings (Definition 1.5) exemplify $\mathbb{N}$-graded commutative rings. They are finitely generated in degree 1 (Definition 6.1).

• As was mentioned above, any $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ possesses a structure of a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$ (Definition 5.3). In particular, a Grassmann-graded ring is a Grassmann algebra (Definition 5.4).

Hereafter, we restrict our consideration to $N$-graded commutative rings (Definition 1.3).

Given an $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$, an $\mathbb{N}$-graded $\mathcal{A}^*$-module $Q$ is defined as a graded $\mathcal{A}^*$-bimodule which is an $\mathbb{N}$-graded $\mathcal{K}$-module such that

$$qa = (-1)^{[a][q]}aq, \qquad [aq] = [a] + [q], \qquad a \in \mathcal{A}^*, \qquad q \in Q,$$

and it also is $\mathbb{Z}_2$-graded module.

A direct sum, a tensor product of $\mathbb{N}$-graded modules and the exterior algebra $\wedge P$ of an $\mathbb{N}$-graded module are defined similarly to those of $\mathbb{Z}_2$-graded modules (Section 5.1), and they also are a direct sum, a tensor product and an exterior algebra of associative $\mathbb{Z}_2$-graded modules, respectively.

A morphism $\Phi : P \to Q$ of $\mathbb{N}$-graded $\mathcal{A}^*$-modules seen as $\mathcal{K}$-modules is said to be homogeneous of degree $[\Phi]$ if $[\Phi(p)] = [p] + [\Phi]$ for all homogeneous elements $p \in P$ and the relations (5.3) hold. A morphism $\Phi : P \to Q$ of $\mathbb{N}$-graded $\mathcal{A}^*$-modules as the $\mathcal{K}$-ones is called a $\mathbb{N}$-graded $\mathcal{A}^*$-module morphism if it is represented by a homogeneous morphisms. Therefore, a set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of graded morphisms of an $\mathbb{N}$-graded $\mathcal{A}$-module $P$ to an $\mathbb{N}$-graded $\mathcal{A}^*$-module $Q$ is an $\mathbb{N}$-graded $\mathcal{A}^*$-module. An $\mathbb{N}$-graded $\mathcal{A}^*$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of an $\mathbb{N}$-graded $\mathcal{A}$-module $P$. Certainly, an $\mathbb{N}$-graded $\mathcal{A}^*$-module morphism of $\mathbb{N}$-graded $\mathcal{A}^*$-modules is their $\mathbb{Z}_2$-graded $\mathcal{A}$-module morphism as associative $\mathbb{Z}_2$-graded modules, however the converse need not be true.

By automorphisms of an $\mathbb{N}$-graded ring $\mathcal{A}^*$ are meant automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which preserve its $\mathbb{N}$-gradation $\mathcal{A}^*$. They also preserve the associated $\mathbb{Z}_2$-structure of $\mathcal{A}$. However, there exist automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which do not possess these properties.

For example, let $\mathcal{A}^*$ be the Grassmann-graded ring (5.8). As was mentioned above its automorphisms (5.9) where $b^i \neq 0$ and (5.10) do not preserve an $\mathbb{N}$-graded structure of $\mathcal{A}$, and automorphisms (5.10) also do not keep its $\mathbb{Z}_2$ structure. However, they are morphisms of $\mathcal{A}^*$ as an $\mathbb{N}$-graded module, because can be represented as a certain sum of homogeneous morphisms.

The differential calculus on $\mathbb{N}$-graded modules over $\mathbb{N}$-graded commutative rings is defined just as that over $\mathbb{Z}_2$-graded commutative rings (Section 5.2), but different from the differential calculus over non-commutative rings in Section 9 (Remark 5.5).

However, it should be emphasized that an $\mathbb{N}$-graded differential operator is an $\mathbb{N}$-graded $\mathcal{K}$-module homomorphism which obeys the conditions (5.13), i.e., it is a sum of homogeneous morphisms of fixed $\mathbb{N}$-degrees, but not the $\mathbb{Z}_2$ ones. Therefore, any $\mathbb{N}$-graded differential operator also is a $\mathbb{Z}_2$-graded differential operator, but the converse might not be true (Remark 6.1).

In particular, $\mathbb{N}$-graded derivations of an $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\mathcal{A}^*$ constitute a Lie superalgebra $\mathfrak{d}\mathcal{A}^*$ (Definition 5.4) over a commutative ring $\mathcal{K}$ with respect to the superbracket (5.15). It is a subalgebra of a Lie superalgebra $\mathfrak{d}\mathcal{A}$ of $\mathbb{Z}_2$-graded derivations of a $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$ in general. The Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}^*;\mathcal{A}^*]$ (5.17) and the Chevalley–Eilenberg differential calculus $\mathcal{O}^*[\mathfrak{d}\mathcal{A}^*]$ over an $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ are constructed similarly to those over a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}^*$ in Section 5.2.

As was mentioned above, the notion of a local ring can be extended to the non-commutative ones (Definition 2.1), and formalism of $\mathbb{N}$-graded commutative local-ringed spaces can be developed just as that of commutative local-ringed spaces in Section 3.1 and $\mathbb{Z}_2$-graded manifolds in Section 5.3.

Hereafter, we restrict our consideration of $\mathbb{N}$-graded commutative rings to Grassmann-graded rings and $\mathbb{N}$-graded commutative rings regarded as the even graded ones.

Let $\mathcal{A}^*$ be a Grassmann-graded $\mathcal{K}$-ring (Definition 1.5) whose associated $\mathbb{Z}_2$-graded commutative ring is a Grassmann algebra $\mathcal{A}$ (Definition 5.4). Seen as a $\mathcal{K}$-ring $\mathcal{A}$, it admits different structures of an $\mathbb{N}$-graded commutative ring and a Grassmann algebra, but they are mutually isomorphic if $\mathcal{K}$ is a field in accordance with Theorem 5.1.

A Grassmann-graded ring is local because of a unique maximal ideal $R$ of its nilpotent elements (Remark 2.1). Given an odd generating basis $\{c^i\}$ for a $\mathcal{K}$-module $\mathcal{A}^1$, elements of a Grassmann-graded ring $\mathcal{A}^*$ take the form (5.8). As was mentioned above, $\mathcal{K}$-ring automorphisms of $\mathcal{A}$ are compositions of automorphisms (5.9) and (5.10), but automorphisms of an $\mathbb{N}$-graded ring $\mathcal{A}^*$ take a form

$$c^i \to c'^i = \rho^i_j c^j \tag{6.1}$$

where $\rho$ is an automorphism of $\mathcal{K}$-module $\mathcal{A}^1$.

The differential calculus over a Grassmann-graded $\mathcal{K}$-ring $\mathcal{A}^*$ is exactly the differential calculus over an associated Grassmann algebra $\mathcal{A}$ (Example 5.6). Namely, the derivations (5.27) of $\mathcal{A}$ also are derivations of a Grassmann-graded ring $\mathcal{A}^*$, and any $\mathbb{N}$-graded differential operator on $\mathcal{A}^*$ is a composition of these derivations.

Since Grassmann-graded rings are local, let us now consider local-ringed spaces whose stalks are such kind $\mathbb{N}$-graded commutative rings. Since Grassmann-graded rings also are Grassmann algebras, we follow formalism of $\mathbb{Z}_2$-graded manifolds in Section 5.3.

Let $Z$ be an $n$-dimensional real smooth manifold. Let $\mathcal{A}$ be real Grassmann-graded ring. By virtue of Theorem 5.2), it is isomorphic to the exterior algebra $\wedge W$ of a real vector space $W = \mathcal{A}^1$. Therefore, we come to the following definition.

DEFINITION 6.2: An $N$-graded manifold is a simple $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A}_E)$ modelled over some vector bundle $E \to Z$ (Definition 5.6). $\Box$

In accordance with Definition 6.2, an $\mathbb{N}$-graded manifold is a local-ringed space.

In view of Definition 6.2, Serre–Swan Theorem 5.4 for $\mathbb{Z}_2$-graded manifolds also can be formulated for the $\mathbb{N}$-graded ones.

THEOREM 6.2: Let $Z$ be a smooth manifold. A $\mathbb{N}$-graded commutative $C^\infty(Z)$-ring $\mathcal{A}$ is isomorphic to the structure ring of a $\mathbb{N}$-graded manifold with a body $Z$ iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank. $\Box$

In Section 5.3 on $\mathbb{Z}_2$-graded manifolds, we have restricted our consideration to simple graded manifolds, this Section, in fact, presents formalism of $\mathbb{N}$-graded manifolds.

**Remark 6.1:** Let emphasize the essential peculiarity of an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ in comparison with the $\mathbb{Z}_2$-graded ones. Derivations of its structure module $\mathcal{A}_E$ are represented by sections of the vector bundle $\mathcal{V}_E$ (5.35). Due to this fact the Chevalley–Eilenberg differential calculus $\mathcal{S}^*[E; Z]$ (5.36) over $\mathcal{A}_E$ is minimal, i.e., it is generated by elements $df$, $f \in \mathcal{A}_E$. $\Box$

# 7 $\mathbb{N}$-Graded bundles

An important example of commutative $N$-graded rings are the polynomial ones (Example 2.13). In particular, let $\mathcal{A}$ be a commutative finitely generated ring over an algebraically closed field $\mathcal{K}$, and let it possess no nilpotent elements. Then it is isomorphic to the quotient of some polynomial $\mathcal{K}$-ring which is the coordinate ring (3.9) of a certain affine variety (Theorem 3.2).

A problem is that a polynomial ring $\mathcal{P}[Q]$ is not local. Any its element $q - \lambda\mathbf{1}$, $\lambda \in \mathcal{K}$, generates a maximal ideal. Therefore, a question is how to construct ringed spaces in polynomial rings. Though there is a certain correspondence between affine varieties and affine schemes (Remark 3.7).

In a different way, one can consider a subsheaf of polynomial functions of a sheaf of smooth functions. Let $\pi : Y \to X$ be a vector bundle and $C^\infty_Y$ a sheaf of smooth functions on $Y$. Let $\mathcal{P}_Y$ be a subsheaf of $C^\infty_Y$ of germs of functions which are polynomial on fibres of $Y$. These functions are well defined due to linear transition functions of a vector bundle $Y \to X$. Let $\pi^*\mathcal{P}_Y$ be the direct image of a sheaf $\mathcal{P}_Y$ onto $X$ (Example 3.1). Its stalk $\pi^*\mathcal{P}_x$ at a point $x \in X$ is a polynomial ring $\mathcal{P}[Y_x]$ of a fibre $Y_x$ of $Y$ over $x$ over a local ring $C^\infty_x$ which is a stalk $C^\infty_x$ of a sheaf of smooth real functions on $X$ at a point $x \in X$.

In a more general setting, let us consider $\mathbb{N}$-graded bundles (Definition 7.1).

An epimorphism of $\mathbb{Z}_2$-graded manifolds $(Z,\mathfrak{A})\to(Z',\mathfrak{A}')$ where $Z\to Z'$ is a fibre bundle is called the graded bundle [19, 47]. In this case, a sheaf monomorphism $\widehat\Phi$ induces a monomorphism of canonical presheaves $\overline{\mathfrak{A}}'\to\overline{\mathfrak{A}}$, which associates to each open subset $U\subset Z$ the ring of sections of $\mathfrak{A}'$ over $\phi(U)$. Accordingly, there is a pull-back monomorphism of the structure rings $\mathfrak{A}'(Z')\to\mathfrak{A}(Z)$ of graded functions on graded manifolds $(Z',\mathfrak{A}')$ and $(Z,\mathfrak{A})$.

In particular, let $(Y,\mathfrak{A})$ be an $\mathbb{N}$-graded manifold whose body $Z=Y$ is a fibre bundle $\pi:Y\to X$. Let us consider a trivial graded manifold $(X,\mathfrak{A}_0=C^\infty_X)$ (Remark 5.8). Then we have a graded bundle

$$(Y,\mathfrak{A})\to(X,C^\infty_X). \tag{7.1}$$

Let us denote it by $(X,Y,\mathfrak{A})$. Given a graded bundle $(X,Y,\mathfrak{A})$, the local basis for a graded manifold $(Y,\mathfrak{A})$ can be brought into a form $(x^\lambda,y^i,c^a)$ where $(x^\lambda,y^i)$ are bundle coordinates of $Y\to X$.

DEFINITION 7.1: We agree to call the graded bundle (7.1) over a trivial graded manifold $(X,C^\infty_X)$ the graded bundle over a smooth manifold [41, 42]. □

If $Y\to X$ is a vector bundle, the graded bundle (7.1) is a particular case of graded fibre bundles in [19, 31] when their base is a trivial graded manifold.

**Remark 7.1:** Let $Y\to X$ be a fibre bundle. Then a trivial graded manifold $(Y,C^\infty_Y)$ together with a real ring monomorphism $C^\infty(X)\to C^\infty(Y)$ is a graded bundle $(X,Y,C^\infty_Y)$ of trivial graded manifolds

$$(Y,C^\infty_Y)\to(X,C^\infty_X).$$

□

**Remark 7.2:** A graded manifold $(X,\mathfrak{A})$ itself can be treated as the graded bundle $(X,X,\mathfrak{A})$ (7.1) associated to the identity smooth bundle $X\to X$. □

Let $E\to Z$ and $E'\to Z'$ be vector bundles and $\Phi:E\to E'$ their bundle morphism over a morphism $\phi:Z\to Z'$. Then every section $s^*$ of the dual bundle $E'^*\to Z'$ defines the pull-back section $\Phi^*s^*$ of the dual bundle $E^*\to Z$ by the law

$$v_z\rfloor\Phi^*s^*(z)=\Phi(v_z)\rfloor s^*(\varphi(z)),\qquad v_z\in E_z.$$

It follows that a bundle morphism $(\Phi,\phi)$ yields a morphism of $\mathbb{N}$-graded manifolds

$$(Z,\mathfrak{A}_E)\to(Z',\mathfrak{A}_{E'}). \tag{7.2}$$

This is a pair $(\phi,\widehat\Phi=\phi_*\circ\Phi^*)$ of a morphism $\phi$ of body manifolds and the composition $\phi_*\circ\Phi^*$ of the pull-back $\mathcal{A}_{E'}\ni f\to\Phi^*f\in\mathcal{A}_E$ of graded functions and the direct image $\phi_*$ of a sheaf $\mathfrak{A}_E$ onto $Z'$. Relative to local bases $(z^A,c^a)$ and $(z'^A,c'^a)$ for $(Z,\mathfrak{A}_E)$ and $(Z',\mathfrak{A}_{E'})$, the morphism (7.2) of $\mathbb{N}$-graded manifolds reads $z'=\phi(z)$, $\widehat\Phi(c'^a)=\Phi^a_b(z)c^b$.

The graded manifold morphism (7.2) is a monomorphism (resp. epimorphism) if $\Phi$ is a bundle injection (resp. surjection).

In particular, the graded manifold morphism (7.2) is an $\mathbb{N}$-graded bundle if $\Phi$ is a fibre bundle. Let $\mathcal{A}_{E'} \to \mathcal{A}_E$ be the corresponding pull-back monomorphism of the structure rings. By virtue of Theorem 5.6 it yields a monomorphism of the differential bigraded rings

$$\mathcal{S}^*[E';Z'] \to \mathcal{S}^*[E;Z]. \tag{7.3}$$

Let $(Y, \mathfrak{A}_F)$ be an $\mathbb{N}$-graded manifold modelled over a vector bundle $F \to Y$. This is an $\mathbb{N}$-graded bundle $(X, Y, \mathfrak{A}_F)$:

$$(Y, \mathfrak{A}_F) \to (X, C_X^\infty) \tag{7.4}$$

modelled over a composite bundle

$$F \to Y \to X. \tag{7.5}$$

The structure ring of graded functions on an $\mathbb{N}$-graded manifold $(Y, \mathfrak{A}_F)$ is the graded commutative $C^\infty(X)$-ring $\mathcal{A}_F = \wedge F^*(Y)$ (5.30). Let the composite bundle (7.5) be provided with adapted bundle coordinates $(x^\lambda, y^i, q^a)$ possessing transition functions

$$x'^\lambda(x^\mu), \qquad y'^i(x^\mu, y^j), \qquad q'^a = \rho^a_b(x^\mu, y^j)q^b.$$

The corresponding local basis for an $\mathbb{N}$-graded manifold $(Y, \mathfrak{A}_F)$ is $(x^\lambda, y^i, c^a)$ together with transition functions

$$x'^\lambda(x^\mu), \qquad y'^i(x^\mu, y^j), \qquad c'^a = \rho^a_b(x^\mu, j^j)c^b.$$

We call it the local basis for an $\mathbb{N}$-graded bundle $(X, Y, \mathfrak{A}_F)$.

With respect to this basis, graded functions on $(X, Y, \mathfrak{A}_F)$ take the form (5.31) where $f_{a_1\ldots a_k}(x^\lambda, y^i)$ are local functions on $Y$. Let $\pi : Y \to X$ be a vector bundle. Then one can consider graded functions $f$ whose coefficients $f_{a_1\ldots a_k}(x^\lambda, y^i)$ are polynomial in fibre coordinates $(y^i)$ of $Y$. Let $\mathcal{P}$ be the sheaf of germs of these functions on $Y$. Its direct image $\pi^*\mathcal{P}$ is a sheaf on $X$ whose stalk at $x \in X$ is an $\mathbb{N}$-graded ring of polynomials both in even variables $y^i$ and the odd ones $c^a$ with coefficients in a stalk $C_x^\infty$ of smooth real functions on $X$.

# 8 Appendix. Cohomology

For the sake of convenience of the reader, the relevant topics on cohomology are compiled in this Appendix.

## 8.1 Cohomology of complexes

We start with cohomology of complexes of modules over a commutative ring [27, 29, 39].

Let $\mathcal{K}$ be a commutative ring. A sequence

$$0 \to B^0 \xrightarrow{\delta^0} B^1 \xrightarrow{\delta^1} \cdots B^p \xrightarrow{\delta^p} \cdots \tag{8.1}$$

of modules $B^p$ and their homomorphisms $\delta^p$ is said to be the cochain complex (henceforth, simply, a complex) if

$$\delta^{p+1} \circ \delta^p = 0, \qquad p \in \mathbb{N},$$

i.e., $\operatorname{Im} \delta^p \subset \operatorname{Ker} \delta^{p+1}$. Homomorphisms $\delta^p$ are called the coboundary operators. Elements of a module $B^p$ are said to be the $p$-cochains, whereas elements of its submodules $\operatorname{Ker} \delta^p \subset B^p$ and $\operatorname{Im} \delta^{p-1} \subset \operatorname{Ker} \delta^p$ are called the $p$-cocycles and $p$-coboundaries, respectively. The $p$-th cohomology group of the complex $B^*$ (8.1) is the factor module

$$H^p(B^*) = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}.$$

It is a $\mathcal{K}$-module. In particular, $H^0(B^*) = \operatorname{Ker} \delta^0$.

The complex (8.1) is said to be exact at a term $B^p$ if $H^p(B^*) = 0$. It is an exact sequence if all cohomology groups are trivial.

A complex $(B^*, \delta^*)$ is called acyclic if its cohomology groups $H^{p>0}(B^*)$ are trivial. A complex $(B^*, \delta^*)$ is said to be the resolution of a module $B$ if it is acyclic and $H^0(B^*) = B$.

A cochain morphism of complexes

$$\gamma : B_1^* \to B_2^* \tag{8.2}$$

is defined as a family of degree-preserving homomorphisms

$$\gamma^p : B_1^p \to B_2^p, \qquad p \in \mathbb{N},$$

which commute with the coboundary operators, i.e.,

$$\delta_2^p \circ \gamma^p = \gamma^{p+1} \circ \delta_1^p, \qquad p \in \mathbb{N}.$$

It follows that if $b^p \in B_1^p$ is a cocycle or a coboundary, then $\gamma^p(b^p) \in B_2^p$ is so. Therefore, the cochain morphism of complexes (8.2) yields an induced homomorphism of their cohomology groups

$$[\gamma]^* : H^*(B_1^*) \to H^*(B_2^*).$$

Let us consider a short sequence of complexes

$$0 \to C^* \xrightarrow{\gamma} B^* \xrightarrow{\zeta} F^* \to 0, \tag{8.3}$$

represented by the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^p \arrow[r, "\delta_C^p"] \arrow[d, "\gamma_p"'] & C^{p+1} \arrow[r] \arrow[d, "\gamma_{p+1}"'] & \cdots \\
\cdots \arrow[r] & B^p \arrow[r, "\delta_B^p"] \arrow[d, "\zeta_p"'] & B^{p+1} \arrow[r] \arrow[d, "\zeta_{p+1}"'] & \cdots \\
\cdots \arrow[r] & F^p \arrow[r, "\delta_F^p"] \arrow[d] & F^{p+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

It is said to be exact if all columns of this diagram are exact, i.e., $\gamma$ is a cochain monomorphism and $\zeta$ is a cochain epimorphism onto the quotient $F^* = B^*/C^*$.

THEOREM 8.1: The short exact sequence of complexes (8.3) yields a long exact sequence of their cohomology groups

$$\begin{aligned}
0 \to H^0(C^*) \xrightarrow{[\gamma]^0} H^0(B^*) \xrightarrow{[\zeta]^0} H^0(F^*) \xrightarrow{\tau^0} H^1(C^*) \longrightarrow \cdots \\
\longrightarrow H^p(C^*) \xrightarrow{[\gamma]^p} H^p(B^*) \xrightarrow{[\zeta]^p} H^p(F^*) \xrightarrow{\tau^p} H^{p+1}(C^*) \longrightarrow \cdots .
\end{aligned}$$

□

THEOREM 8.2: A direct sequence of complexes

$$B_0^* \longrightarrow B_1^* \longrightarrow \cdots B_k^* \xrightarrow{\gamma_{k+1}^k} B_{k+1}^* \longrightarrow \cdots$$

admits a direct limit $B_\infty^*$ which is a complex whose cohomology $H^*(B_\infty^*)$ is a direct limit of the direct sequence of cohomology groups

$$H^*(B_0^*) \longrightarrow H^*(B_1^*) \longrightarrow \cdots H^*(B_k^*) \xrightarrow{[\gamma_{k+1}^k]} H^*(B_{k+1}^*) \longrightarrow \cdots .$$

This statement also is true for a direct system of complexes indexed by an arbitrary directed set (Remark 2.4). □

## 8.2 Cohomology of Lie algebras

One can associate the Chevalley–Eilenberg cochain complex (8.4) to an arbitrary Lie algebra [13, 39]. In this Section, $\mathcal{G}$ denotes a Lie algebra (not necessarily finite-dimensional) over a commutative ring $\mathcal{K}$.

Let $P$ be a $\mathcal{K}$-module, and let $\mathcal{G}$ act on $P$ on the left by endomorphisms

$$\begin{aligned}
&\mathcal{G} \times P \ni (\varepsilon, p) \to \varepsilon p \in P, \\
&[\varepsilon, \varepsilon']p = (\varepsilon \circ \varepsilon' - \varepsilon' \circ \varepsilon)p, \qquad \varepsilon, \varepsilon' \in \mathcal{G}.
\end{aligned}$$

One says that $P$ is a $\mathcal{G}$-module. A $\mathcal{K}$-multilinear skew-symmetric map

$$c^k : \overset{k}{\times}\mathcal{G} \to P$$

is called the $P$-valued $k$-cochain on a Lie algebra $\mathcal{G}$. These cochains form a $\mathcal{G}$-module $C^k[\mathcal{G};P]$. Let us put $C^0[\mathcal{G};P] = P$. We obtain the cochain complex

$$0 \to P \xrightarrow{\delta^0} C^1[\mathcal{G};P] \xrightarrow{\delta^1} \cdots C^k[\mathcal{G};P] \xrightarrow{\delta^k} \cdots, \tag{8.4}$$

with respect to the Chevalley–Eilenberg coboundary operators

$$\delta^k c^k(\varepsilon_0,\ldots,\varepsilon_k) = \sum_{i=0}^{k}(-1)^i \varepsilon_i c^k(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_k) + \tag{8.5}$$
$$\sum_{1\leq i<j\leq k}(-1)^{i+j}c^k([\varepsilon_i,\varepsilon_j],\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_k),$$

where the caret $\widehat{}$ denotes omission. The complex (8.4) is called the Chevalley–Eilenberg complex with coefficients in a module $P$. Cohomology $H^*(\mathcal{G};P)$ of the complex $C^*[\mathcal{G};P]$ is called the Chevalley–Eilenberg cohomology of a Lie algebra $\mathcal{G}$ with coefficients in a module $P$.

## 8.3 Sheaf cohomology

In this Section, we follow the terminology of [7, 20]. All presheaves and sheaves are considered on the same topological space $X$.

A sheaf on a topological space $X$ is a topological fibre bundle $\pi : S \to X$ in modules over a commutative ring $\mathcal{K}$, where a surjection $\pi$ is a local homeomorphism and fibres $S_x$, $x \in X$, called the stalks, are provided with the discrete topology. Global sections of a sheaf $S$ constitute a $\mathcal{K}$-module $S(X)$, called the structure module of $S$.

Any sheaf is generated by a presheaf. A presheaf $S_{\{U\}}$ on a topological space $X$ is defined if a module $S_U$ over a commutative ring $\mathcal{K}$ is assigned to every open subset $U \subset X$ ($S_\emptyset = 0$) and if, for any pair of open subsets $V \subset U$, there exists a restriction morphism $r^U_V : S_U \to S_V$ such that

$$r^U_U = \mathrm{Id}\, S_U, \qquad r^U_W = r^V_W r^U_V, \qquad W \subset V \subset U.$$

Every presheaf $S_{\{U\}}$ on a topological space $X$ yields a sheaf on $X$ whose stalk $S_x$ at a point $x \in X$ is the direct limit of modules $S_U$, $x \in U$, with respect to restriction morphisms $r^U_V$. It means that, for each open neighborhood $U$ of a point $x$, every element $s \in S_U$ determines an element $s_x \in S_x$, called the germ of $s$ at $x$. Two elements $s \in S_U$ and $s' \in S_V$ belong to the same germ at $x$ iff there exists an open neighborhood $W \subset U \cap V$ of $x$ such that $r^U_W s = r^V_W s'$.

**Example 8.1:** Let $C^0_{\{U\}}$ be a presheaf of continuous real functions on a topological space $X$. Two such functions $s$ and $s'$ define the same germ $s_x$ if they coincide on an open neighborhood of $x$. Hence, we obtain a sheaf $C^0_X$ of continuous functions on $X$. Similarly, a sheaf $C^\infty_X$ of smooth functions on a smooth manifold $X$ is defined. Let us also mention a presheaf of real functions which are constant on connected open subsets of $X$. It generates the constant sheaf on $X$ denoted by $\mathbb{R}$. □

**Example 8.2:** Let $Y \to X$ be a smooth vector bundle. A sheaf of germs of its sections is denoted $Y_X$. Its structure module is a $C^\infty(X)$-ring $Y(X)$ of global sections of $Y \to X$. □

Two different presheaves may generate the same sheaf. Conversely, every sheaf $S$ defines a presheaf $S(\{U\})$ of modules $S(U)$ of its local sections. It is called the canonical presheaf of a sheaf $S$. Global sections of $S$ constitute the structure module $S(X)$ of $S$. If a sheaf $S$ is constructed from a presheaf $S_{\{U\}}$, there are natural module morphisms

$$S_U \ni s \to s(U) \in S(U), \qquad s(x) = s_x, \quad x \in U,$$

which are neither monomorphisms nor epimorphisms in general. For instance, it may happen that a non-zero presheaf defines a zero sheaf. The sheaf generated by the canonical presheaf of a sheaf $S$ coincides with $S$.

A direct sum and a tensor product of presheaves (as families of modules) and sheaves (as fibre bundles in modules) are naturally defined. By virtue of Theorem 2.6, a direct sum (resp. a tensor product) of presheaves generates a direct sum (resp. a tensor product) of the corresponding sheaves.

**Remark 8.3:** In the terminology of [48], a sheaf is introduced as a presheaf which satisfies the following additional axioms.

(S1) Suppose that $U \subset X$ is an open subset and $\{U_\alpha\}$ is its open cover. If $s, s' \in S_U$ obey a condition $r^U_{U_\alpha}(s) = r^U_{U_\alpha}(s')$ for all $U_\alpha$, then $s = s'$.

(S2) Let $U$ and $\{U_\alpha\}$ be as in previous item. Suppose that we are given a family of presheaf elements $\{s_\alpha \in S_{U_\alpha}\}$ such that

$$r^{U_\alpha}_{U_\alpha\cap U_\lambda}(s_\alpha) = r^{U_\lambda}_{U_\alpha\cap U_\lambda}(s_\lambda)$$

for all $U_\alpha$, $U_\lambda$. Then there exists an element $s \in S_U$ such that $s_\alpha = r^U_{U_\alpha}(s)$.

Canonical presheaves are in one-to-one correspondence with presheaves obeying these axioms. For instance, the presheaves of continuous, smooth and locally constant functions in Example 8.1 satisfy the axioms (S1) – (S2). □

A morphism of a presheaf $S_{\{U\}}$ to a presheaf $S'_{\{U\}}$ on a topological space $X$ is defined as a set of module morphisms $\gamma_U : S_U \to S'_U$ which commute with restriction morphisms. A morphism of presheaves yields a morphism of sheaves generated by these presheaves. This is a bundle morphism over $X$ such that $\gamma_x : S_x \to S'_x$ is the direct limit of morphisms $\gamma_U$, $x \in U$. Conversely, any morphism of sheaves $S \to S'$ on a topological space $X$ yields a morphism of canonical presheaves of local sections of these sheaves. Let $\mathrm{Hom}\,(S|_U, S'|_U)$ be a commutative group of sheaf morphisms $S|_U \to S'|_U$ for any open subset $U \subset X$.

These groups are assembled into a presheaf, and define the sheaf Hom $(S, S')$ on $X$. There is a monomorphism

$$\mathrm{Hom}\,(S, S')(U) \to \mathrm{Hom}\,(S(U), S'(U)), \tag{8.6}$$

which need not be an isomorphism.

By virtue of Theorem 2.7, if a presheaf morphism is a monomorphism or an epimorphism, so is the corresponding sheaf morphism. Furthermore, the following holds.

THEOREM 8.3: A short exact sequence

$$0 \to S'_{\{U\}} \to S_{\{U\}} \to S''_{\{U\}} \to 0 \tag{8.7}$$

of presheaves on a topological space $X$ yields a short exact sequence of sheaves generated by these presheaves

$$0 \to S' \to S \to S'' \to 0, \tag{8.8}$$

where the factor sheaf $S'' = S/S'$ is isomorphic to that generated by the factor presheaf $S''_{\{U\}} = S_{\{U\}}/S'_{\{U\}}$. If the exact sequence of presheaves (8.7) is split, i.e.,

$$S_{\{U\}} \cong S'_{\{U\}} \oplus S''_{\{U\}},$$

the corresponding splitting $S \cong S' \oplus S''$ of the exact sequence of sheaves (8.8) holds. □

The converse is more intricate. A sheaf morphism induces a morphism of the corresponding canonical presheaves. If $S \to S'$ is a monomorphism, $S(\{U\}) \to S'(\{U\})$ also is a monomorphism. However, if $S \to S'$ is an epimorphism, $S(\{U\}) \to S'(\{U\})$ need not be so. Therefore, the short exact sequence (8.8) of sheaves yields the exact sequence of the canonical presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''(\{U\}), \tag{8.9}$$

where $S(\{U\}) \to S''(\{U\})$ is not necessarily an epimorphism. At the same time, there is the short exact sequence of presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''_{\{U\}} \to 0, \tag{8.10}$$

where the factor presheaf

$$S''_{\{U\}} = S(\{U\})/S'(\{U\})$$

generates the factor sheaf $S'' = S/S'$, but need not be its canonical presheaf.

THEOREM 8.4: Let the exact sequence of sheaves (8.8) be split. Then

$$S(\{U\}) \cong S'(\{U\}) \oplus S''(\{U\}),$$

and the canonical presheaves make up the short exact sequence

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''(\{U\}) \to 0.$$

□

Let us turn now to sheaf cohomology. We follow its definition in [20].

Let $S_{\{U\}}$ be a presheaf of modules on a topological space $X$, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of $X$. One constructs a cochain complex where a $p$-cochain is defined as a function $s^p$ which associates an element

$$s^p(i_0, \ldots, i_p) \in S_{U_{i_0}\cap\cdots\cap U_{i_p}}$$

to each $(p+1)$-tuple $(i_0, \ldots, i_p)$ of indices in $I$. These $p$-cochains are assembled into a module $C^p(\mathfrak{U}, S_{\{U\}})$. Let us introduce a coboundary operator

$$\begin{aligned}
&\delta^p : C^p(\mathfrak{U}, S_{\{U\}}) \to C^{p+1}(\mathfrak{U}, S_{\{U\}}), \\
&\delta^p s^p(i_0, \ldots, i_{p+1}) = \sum_{k=0}^{p+1} (-1)^k r_W^{W_k} s^p(i_0, \ldots, \widehat{i}_k, \ldots, i_{p+1}), \qquad (8.11) \\
&W = U_{i_0} \cap \ldots \cap U_{i_{p+1}}, \qquad W_k = U_{i_0} \cap \cdots \cap \widehat{U}_{i_k} \cap \cdots \cap U_{i_{p+1}}.
\end{aligned}$$

It is easily justified that $\delta^{p+1} \circ \delta^p = 0$. Thus, we obtain a cochain complex of modules

$$0 \to C^0(\mathfrak{U}, S_{\{U\}}) \xrightarrow{\delta^0} \cdots C^p(\mathfrak{U}, S_{\{U\}}) \xrightarrow{\delta^p} C^{p+1}(\mathfrak{U}, S_{\{U\}}) \longrightarrow \cdots . \qquad (8.12)$$

Its cohomology groups

$$H^p(\mathfrak{U}; S_{\{U\}}) = \mathrm{Ker}\, \delta^p / \mathrm{Im}\, \delta^{p-1}$$

are modules. Certainly, they depend on a cover $\mathfrak{U}$ of a topological space $X$.

**Remark 8.4:** Only proper covers throughout are considered, i.e., $U_i \neq U_j$ if $i \neq j$. A cover $\mathfrak{U}'$ is said to be the refinement of a cover $\mathfrak{U}$ if, for each $U' \in \mathfrak{U}'$, there exists $U \in \mathfrak{U}$ such that $U' \subset U$. □

If $\mathfrak{U}'$ is a refinement of a cover $\mathfrak{U}$, there is a morphism of cohomology groups

$$H^*(\mathfrak{U}; S_{\{U\}}) \to H^*(\mathfrak{U}'; S_{\{U\}}).$$

Let us take the direct limit of cohomology groups $H^*(\mathfrak{U}; S_{\{U\}})$ with respect to these morphisms, where $\mathfrak{U}$ runs through all open covers of $X$. This limit $H^*(X; S_{\{U\}})$ is called the cohomology of $X$ with coefficients in a presheaf $S_{\{U\}}$.

Let $S$ be a sheaf on a topological space $X$. Cohomology of $X$ with coefficients in $S$ or, simply, sheaf cohomology is defined as cohomology

$$H^*(X; S) = H^*(X; S(\{U\}))$$

with coefficients in the canonical presheaf $S(\{U\})$ of a sheaf $S$.

In this case, a $p$-cochain $s^p \in C^p(\mathfrak{U}, S(\{U\}))$ is a family $s^p = \{s^p(i_0,\ldots,i_p)\}$ of local sections $s^p(i_0,\ldots,i_p)$ of a sheaf $S$ over $U_{i_0}\cap\cdots\cap U_{i_p}$ for each $(p+1)$-tuple $(U_{i_0},\ldots,U_{i_p})$ of elements of a cover $\mathfrak{U}$. The coboundary operator (8.11) reads

$$\delta^p s^p(i_0,\ldots,i_{p+1}) = \sum_{k=0}^{p+1}(-1)^k s^p(i_0,\ldots,\widehat{i}_k,\ldots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

For instance,

$$\delta^0 s^0(i,j) = [s^0(j) - s^0(i)]|_{U_i\cap U_j}, \tag{8.13}$$

$$\delta^1 s^1(i,j,k) = [s^1(j,k) - s^1(i,k) + s^1(i,j)]|_{U_i\cap U_j\cap U_k}. \tag{8.14}$$

A glance at the expression (8.13) shows that a zero-cocycle is a collection $s = \{s(i)\}_I$ of local sections of a sheaf $S$ over $U_i \in \mathfrak{U}$ such that $s(i) = s(j)$ on $U_i\cap U_j$. It follows from the axiom (S2) in Remark 8.3 that $s$ is a global section of a sheaf $S$, while each $s(i)$ is its restriction $s|_{U_i}$ to $U_i$. Consequently, the cohomology group $H^0(\mathfrak{U}, S(\{U\}))$ is isomorphic to the structure module $S(X)$ of global sections of a sheaf $S$. A one-cocycle is a collection $\{s(i,j)\}$ of local sections of a sheaf $S$ over overlaps $U_i\cap U_j$ which satisfy the cocycle condition

$$[s(j,k) - s(i,k) + s(i,j)]|_{U_i\cap U_j\cap U_k} = 0.$$

If $X$ is a paracompact space, the study of its sheaf cohomology is essentially simplified due to the following fact [20].

THEOREM 8.5: Cohomology of a paracompact space $X$ with coefficients in a sheaf $S$ coincides with cohomology of $X$ with coefficients in any presheaf generating a sheaf $S$. □

**Remark 8.5:** We follow the definition of a paracompact topological space in [20] as a Hausdorff space such that any its open cover admits a locally finite open refinement, i.e., any point has an open neighborhood which intersects only a finite number of elements of this refinement. A topological space $X$ is paracompact iff any cover $\{U_\xi\}$ of $X$ admits the subordinate partition of unity $\{f_\xi\}$, i.e.:

(i) $f_\xi$ are real positive continuous functions on $X$;

(ii) $\operatorname{supp} f_\xi \subset U_\xi$;

(iii) each point $x \in X$ has an open neighborhood which intersects only a finite number of the sets $\operatorname{supp} f_\xi$;

(iv) $\sum\limits_\xi f_\xi(x) = 1$ for all $x \in X$. □

A key point of the analysis of sheaf cohomology is that short exact sequences of presheaves and sheaves yield long exact sequences of sheaf cohomology groups.

Let $S_{\{U\}}$ and $S'_{\{U\}}$ be presheaves on a topological space $X$. It is readily observed that, given an open cover $\mathfrak{U}$ of $X$, any morphism $S_{\{U\}} \to S'_{\{U\}}$ yields a cochain morphism of complexes

$$C^*(\mathfrak{U}, S_{\{U\}}) \to C^*(\mathfrak{U}, S'_{\{U\}})$$

and the corresponding morphism

$$H^*(\mathfrak{U}, S_{\{U\}}) \to H^*(\mathfrak{U}, S'_{\{U\}})$$

of cohomology groups of these complexes. Passing to the direct limit through all refinements of $\mathfrak{U}$, we come to a morphism of the cohomology groups

$$H^*(X, S_{\{U\}}) \to H^*(X, S'_{\{U\}})$$

of $X$ with coefficients in the presheaves $S_{\{U\}}$ and $S'_{\{U\}}$. In particular, any sheaf morphism $S \to S'$ yields a morphism of canonical presheaves $S(\{U\}) \to S'(\{U\})$ and the corresponding cohomology morphism

$$H^*(X, S) \to H^*(X, S').$$

By virtue of Theorems 8.1 and 8.2, every short exact sequence

$$0 \to S'_{\{U\}} \longrightarrow S_{\{U\}} \longrightarrow S''_{\{U\}} \to 0$$

of presheaves on a topological space $X$ and the corresponding exact sequence of complexes (8.12) yield the long exact sequence

$$\begin{aligned} 0 \to H^0(X; S'_{\{U\}}) &\longrightarrow H^0(X; S_{\{U\}}) \longrightarrow H^0(X; S''_{\{U\}}) \longrightarrow \\ H^1(X; S'_{\{U\}}) &\longrightarrow \cdots H^p(X; S'_{\{U\}}) \longrightarrow H^p(X; S_{\{U\}}) \longrightarrow \\ H^p(X; S''_{\{U\}}) &\longrightarrow H^{p+1}(X; S'_{\{U\}}) \longrightarrow \cdots \end{aligned}$$

of the cohomology groups of $X$ with coefficients in these presheaves. This result however is not extended to an exact sequence of sheaves, unless $X$ is a paracompact space. Let

$$0 \to S' \longrightarrow S \longrightarrow S'' \to 0$$

be a short exact sequence of sheaves on $X$. It yields the short exact sequence of presheaves (8.10) where the presheaf $S''_{\{U\}}$ generates the sheaf $S''$. If $X$ is paracompact,

$$H^*(X; S''_{\{U\}}) = H^*(X; S'')$$

in accordance with Theorem 8.5, and we have the exact sequence of sheaf cohomology

$$\begin{aligned} 0 \to H^0(X; S') &\longrightarrow H^0(X; S) \longrightarrow H^0(X; S'') \longrightarrow \\ H^1(X; S') &\longrightarrow \cdots H^p(X; S') \longrightarrow H^p(X; S) \longrightarrow \\ H^p(X; S'') &\longrightarrow H^{p+1}(X; S') \longrightarrow \cdots . \end{aligned} \tag{8.15}$$

Let us point out the following isomorphism between the sheaf cohomology and the singular (Čhech and Alexandery) cohomology of a paracompact space [7, 45].

THEOREM 8.6: The sheaf cohomology $H^*(X; \mathbb{Z})$ (resp. $H^*(X; \mathbb{R})$) of a paracompact topological space $X$ with coefficients in the constant sheaf $\mathbb{Z}$ (resp. $\mathbb{R}$)

is isomorphic to the singular cohomology of $X$ with coefficients in a ring $\mathbb{Z}$ (resp. $\mathbb{R}$). □

Since singular cohomology is a topological invariant (i.e., homotopic topological spaces have the same singular cohomology) [45], the sheaf cohomology groups $H^*(X;\mathbb{Z})$ and $H^*(X;\mathbb{R})$ of a paracompact space also are topological invariants.

Let us turn now to the abstract de Rham theorem which provides a powerful tool of studying algebraic systems on paracompact spaces.

Let us consider an exact sequence of sheaves

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots S_p \xrightarrow{h^p} \cdots . \tag{8.16}$$

It is said to be the resolution of a sheaf $S$ if each sheaf $S_{p\geq 0}$ is acyclic, i.e., its cohomology groups $H^{k>0}(X;S_p)$ vanish.

Any exact sequence of sheaves (8.16) yields a sequence of their structure modules

$$0 \to S(X) \xrightarrow{h_*} S_0(X) \xrightarrow{h^0_*} S_1(X) \xrightarrow{h^1_*} \cdots S_p(X) \xrightarrow{h^p_*} \cdots \tag{8.17}$$

which always is exact at terms $S(X)$ and $S_0(X)$ (see the exact sequence (8.9)). The sequence (8.17) is a cochain complex because $h^{p+1}_* \circ h^p_* = 0$. If $X$ is a paracompact space and the exact sequence (8.16) is a resolution of $S$, the abstract de Rham theorem establishes an isomorphism of cohomology of the complex (8.17) to cohomology of $X$ with coefficients in a sheaf $S$ as follows [20].

Theorem 8.7: Given a resolution (8.16) of a sheaf $S$ on a paracompact topological space $X$ and the induced complex (8.17), there are isomorphisms

$$H^0(X;S) = \operatorname{Ker} h^0_*, \qquad H^q(X;S) = \operatorname{Ker} h^q_*/\operatorname{Im} h^{q-1}_*, \qquad q > 0. \tag{8.18}$$

□

We will also refer to the following minor modification of Theorem 8.7 [14].

Theorem 8.8: Let

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \qquad p > 1,$$

be an exact sequence of sheaves on a paracompact topological space $X$, where the sheaves $S_q$, $0 \leq q < p$, are acyclic, and let

$$0 \to S(X) \xrightarrow{h_*} S_0(X) \xrightarrow{h^0_*} S_1(X) \xrightarrow{h^1_*} \cdots \xrightarrow{h^{p-1}_*} S_p(X) \xrightarrow{h^p_*} S_{p+1}(X)$$

be the corresponding cochain complex of structure modules of these sheaves. Then the isomorphisms (8.18) hold for $0 \leq q \leq p$. □

Let us mention a fine resolution of sheaves, i.e., a resolution by fine sheaves. A sheaf $S$ on a paracompact space $X$ is called fine if, for each locally finite open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of $X$, there exists a system $\{h_i\}$ of endomorphisms $h_i : S \to S$ such that:

(i) there is a closed subset $V_i \subset U_i$ and $h_i(S_x) = 0$ if $x \notin V_i$,

(ii) $\sum\limits_{i\in I} h_i$ is the identity map of $S$.

A fine sheaf on a paracompact space is acyclic. There are the following important examples of fine sheaves [20].

Proposition 8.9: Let $X$ be a paracompact topological space which admits the partition of unity performed by elements of the structure module $\mathfrak{A}(X)$ of some sheaf $\mathfrak{A}$ of real functions on $X$. Then any sheaf $S$ of $\mathfrak{A}$-modules on $X$, including $\mathfrak{A}$ itself, is fine. $\Box$

In particular, a sheaf $C^0_X$ of continuous functions on a paracompact topological space is fine, and so is any sheaf of $C^0_X$-modules. A smooth manifold $X$ admits the partition of unity performed by smooth real functions. It follows that a sheaf $C^\infty_X$ of smooth real functions on $X$ is fine, and so is any sheaf of $C^\infty_X$-modules, e.g., the sheaves of sections of smooth vector bundles over $X$.

# 9 Appendix. Non-commutative differential calculus

The notion of the graded differential calculus (Definition 2.5) has been formulated for any $\mathcal{K}$-ring. One can generalize the Chevalley–Eilenberg differential calculus over a commutative ring in Section 2.4 to a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ [11, 16, 39]. As was mentioned above, the extension of the notion of a differential operator to modules over a non-commutative ring however meets difficulties [16, 39]. A key point is that a multiplication in a non-commutative ring is not a zero-order differential operator (Remark 9.2). One overcomes this difficulty in a case of graded commutative rings by means of a reformulation (Definition 5.5) of the notion of differential operators (Definition 9.1).

Namely, let $\mathcal{A}$ further be an arbitrary non-commutative $\mathcal{K}$-ring. Its derivation $u \in \mathfrak{d}\mathcal{A}$, by definition, obeys the Leibniz rule

$$u(ab) = u(a)b + au(b), \qquad a, b \in \mathcal{A}. \tag{9.1}$$

However, if $\mathcal{A}$ is a graded commutative ring, its Leibniz rule (5.14) differs from the above one (9.1).

By virtue of the relation (9.1), derivations preserve the center $\mathcal{Z}_\mathcal{A}$ of $\mathcal{A}$. A set of derivations $\mathfrak{d}\mathcal{A}$ is both a $\mathcal{Z}_\mathcal{A}$-bimodule and a Lie $\mathcal{K}$-algebra with respect to the Lie bracket (2.15).

Given a Lie $\mathcal{K}$-algebra $\mathfrak{d}\mathcal{A}$, let us consider its Chevalley–Eilenberg complex (2.44) with coefficients in a ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{Z}_\mathcal{A}$-multilinear skew-symmetric maps (2.46) with respect to the Chevalley–Eilenberg coboundary operator $d$ (2.47). Its terms $\mathcal{O}^k[\mathfrak{d}\mathcal{A}]$ are $\mathcal{A}$-bimodules. The graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the product (2.51) which obeys the relation (2.52) and brings $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential graded ring. Let us note that, if $\mathcal{A}$ is not commutative, there is nothing like the graded commutativity of forms (2.53). Therefore, a differential graded ring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ fails to be $\mathbb{N}$-graded commutative.

The minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$ consists of monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$, whose product $\wedge$ (2.51) obeys a juxtaposition rule

$$(a_0 da_1) \wedge (b_0 db_1) = a_0 d(a_1 b_0) \wedge db_1 - a_0 a_1 db_0 \wedge db_1, \qquad a_i, b_i \in \mathcal{A}.$$

For instance, it follows from the product (2.51) that, if $a, a' \in \mathcal{Z}_\mathcal{A}$, then

$$da \wedge da' = -da' \wedge da, \qquad a da' = (da')a. \tag{9.2}$$

**Remark 9.1:** Let us mention a different graded differential calculus over a non-commutative ring which often is used in non-commutative geometry [9, 23, 39]. Given a $\mathcal{K}$-ring $\mathcal{A}$, let us consider a tensor product $\mathcal{A} \underset{\mathcal{K}}{\otimes} \mathcal{A}$ of $\mathcal{K}$-modules. It is brought into an $\mathcal{A}$-bimodule with respect to the multiplication

$$b(a \otimes a')c = (ba) \otimes (a'c), \qquad a, a', b, c \in \mathcal{A}.$$

Let us consider its submodule $\Omega^1(\mathcal{A})$ generated by the elements $\mathbf{1} \otimes a - a \otimes \mathbf{1}$, $a \in \mathcal{A}$. It is readily observed that

$$d : \mathcal{A} \ni a \to \mathbf{1} \otimes a - a \otimes \mathbf{1} \in \Omega^1(\mathcal{A}) \tag{9.3}$$

is a $\Omega^1(\mathcal{A})$-valued derivation of $\mathcal{A}$. Thus, $\Omega^1(\mathcal{A})$ is an $\mathcal{A}$-bimodule generated by elements $da$, $a \in \mathcal{A}$, such that the relation

$$(da)b = d(ab) - adb, \qquad a, b \in \mathcal{A}, \tag{9.4}$$

holds. Let us consider the tensor algebra $\Omega^*(\mathcal{A}) = \otimes\Omega^1(\mathcal{A})$ (2.2) of an $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A})$ (Example 2.3). It consists of the monomials

$$a_0 da_1 \cdots da_k, \qquad a_i \in \mathcal{A}, \tag{9.5}$$

whose product obeys the juxtaposition rule

$$(a_0 da_1)(b_0 db_1) = a_0 d(a_1 b_0) db_1 - a_0 a_1 db_0 b_1, \qquad a_i, b_i \in \mathcal{A},$$

because of the relation (9.4). The operator $d$ (9.3) is extended to $\Omega^*(\mathcal{A})$ by the law

$$d(a_0 da_1 \cdots da_k) = da_0 da_1 \cdots da_k, \tag{9.6}$$

that makes $\Omega^*(\mathcal{A})$ into a differential graded ring. Of course, $\Omega^*(\mathcal{A})$ is a minimal differential calculus. One calls it the universal differential calculus. It differs from the Chevalley–Eilenberg differential calculus. For instance, the monomials $da$, $a \in \mathcal{Z}_\mathcal{A}$, of the universal differential calculus do not satisfy the relations (9.2). $\Box$

It seems natural to regard the derivations (9.1) of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ and the Chevalley–Eilenberg coboundary operator $d$ (2.47) as particular differential operators over $\mathcal{A}$. Definition 2.2 provides a standard notion of differential operators on modules over a commutative ring. However, there exist its different generalizations to modules over a non-commutative ring [6, 10, 11, 16, 26, 39].

Let $P$ and $Q$ be $\mathcal{A}$-bimodules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-linear homomorphisms $\Phi : P \to Q$ can be provided with the left $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (2.9) and the similar right module structures

$$(\Phi a)(p) = \Phi(p)a, \qquad (a \bullet \Phi)(p) = \Phi(pa), \quad a \in \mathcal{A}, \qquad p \in P. \tag{9.7}$$

For the sake of convenience, we will refer to the module structures (2.9) and (9.7) as the left and right $\mathcal{A} - \mathcal{A}^\bullet$ structures, respectively. Let us put

$$\delta_a \Phi = a\Phi - \Phi \bullet a, \qquad a \in \mathcal{A}, \tag{9.8}$$

$$\overline{\delta}_a \Phi = \Phi a - a \bullet \Phi, \qquad a \in \mathcal{A}, \qquad \Phi \in \mathrm{Hom}_{\mathcal{K}}(P,Q). \tag{9.9}$$

It is readily observed that

$$\delta_a \circ \overline{\delta}_b = \overline{\delta}_b \circ \delta_a, \qquad a, b \in \mathcal{A}.$$

The left $\mathcal{A}$-module homomorphisms $\Delta : P \to Q$ obey conditions $\delta_a \Delta = 0$, for all $a \in \mathcal{A}$ and, consequently, they can be regarded as left zero-order $Q$-valued differential operators on $P$. Similarly, right zero-order differential operators are defined.

**Remark 9.2:** However, a left (resp., right) multiplication fails to be a left (resp., right) zero-order differential operator. □

Utilizing the condition (2.11) as a definition of a first-order differential operator, one meets difficulties too. If $P = \mathcal{A}$ and $\Delta(\mathbf{1}) = 0$, the condition (2.11) does not lead to the Leibniz rule (2.13), i.e., derivations of a $\mathcal{K}$-ring $\mathcal{A}$ are not first-order differential operators. In order to overcome these difficulties, one can replace the condition (2.11) with the following one [10].

Definition 9.1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called the first-order differential operator of a bimodule $P$ over a non-commutative ring $\mathcal{A}$ if it obeys a condition

$$\delta_a \circ \overline{\delta}_b \Delta = \overline{\delta}_b \circ \delta_a \Delta = 0, \qquad a, b \in \mathcal{A},$$
$$a\Delta(p)b - a\Delta(pb) - \Delta(ap)b + \Delta(apb) = 0, \qquad p \in P.$$

□

If $P$ is a commutative bimodule over a commutative ring $\mathcal{A}$, then $\delta_a = \overline{\delta}_a$ and Definition 9.1 comes to Definition 2.2 for first-order differential operators.

First-order $Q$-valued differential operators on a module $P$ constitute a $\mathcal{Z}_{\mathcal{A}}$-module $\mathrm{Diff}_1(P,Q)$.

Derivations of $\mathcal{A}$ are first-order differential operators in accordance with Definition 9.1. The Chevalley–Eilenberg coboundary operator $d$ (2.47) also is well.

If derivations of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ are first-order differential operators on $\mathcal{A}$, it seems natural to think of their compositions as being particular higher order differential operators on $\mathcal{A}$.

By analogy with Definition 2.2, one may try to generalize Definition 9.1 by means of the maps $\delta_a$ (9.8) and $\overline{\delta}_a$ (9.9). A problem lies in the fact that, if $P = Q = \mathcal{A}$, the compositions $\delta_a \circ \delta_b$ and $\overline{\delta}_a \circ \overline{\delta}_b$ do not imply the Leibniz rule and, as a consequence, compositions of derivations of $\mathcal{A}$ fail to be differential operators [16, 39].

This problem can be solved if $P$ and $Q$ are regarded as left $\mathcal{A}$-modules [26]. Let us consider a $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ provided with the left $\mathcal{A} - \mathcal{A}^\bullet$ module structure (2.9). We denote by $\mathcal{Z}_0$ its center, i.e., $\delta_a \Phi = 0$ for all $\Phi \in \mathcal{Z}_0$ and $a \in \mathcal{A}$. Let $\mathcal{I}_0 = \overline{\mathcal{Z}}_0$ be an $\mathcal{A} - \mathcal{A}^\bullet$ submodule of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ generated by $\mathcal{Z}_0$. Let us consider:

(i) the quotient $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_0$,

(ii) its center $\mathcal{Z}_1$,

(iii) an $\mathcal{A} - \mathcal{A}^\bullet$ submodule $\overline{\mathcal{Z}}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_0$ generated by $\mathcal{Z}_1$,

(iv) an $\mathcal{A} - \mathcal{A}^\bullet$ submodule $\mathcal{I}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ given by the relation $\mathcal{I}_1/\mathcal{I}_0 = \overline{\mathcal{Z}}_1$.

Then we define the $\mathcal{A} - \mathcal{A}^\bullet$ submodules $\mathcal{I}_r$, $r = 2, \ldots$, of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ by induction as $\mathcal{I}_r/\mathcal{I}_{r-1} = \overline{\mathcal{Z}}_r$, where $\overline{\mathcal{Z}}_r$ is the $\mathcal{A} - \mathcal{A}^\bullet$ module generated by the center $\mathcal{Z}_r$ of the quotient $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_{r-1}$.

Definition 9.2: 9.2 Elements of the submodule $\mathcal{I}_r$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ are said to be left $r$-order $Q$-valued differential operators of an $\mathcal{A}$-bimodule $P$ [26]. □

If $\mathcal{A}$ is a commutative ring, Definition 9.2 comes to Definition 2.2.

Let $P = Q = \mathcal{A}$. Any zero-order differential operator on $\mathcal{A}$ in accordance with Definition 9.2 takes a form $a \to cac'$ for some $c, c' \in \mathcal{A}$. Any derivation $u \in \mathfrak{d}\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ is a first-order differential operator in accordance with Definition 9.2. Indeed, it is readily observed that

$$(\delta_a u)(b) = au(b) - u(ab) = -u(a)b, \qquad b \in \mathcal{A},$$

is a zero-order differential operator for all $a \in \mathcal{A}$. The compositions $au$, $u \bullet a$ (2.9), $ua$, $a \bullet u$ (9.7) for any $u \in \mathfrak{d}\mathcal{A}$, $a \in \mathcal{A}$ and the compositions of derivations $u_1 \circ \cdots \circ u_r$ also are differential operators on $\mathcal{A}$ in accordance with Definition 9.2.

By analogy with Definition 9.2, one can define differential operators on right $\mathcal{A}$-modules as follows.

Definition 9.3: Let $P$, $Q$ be seen as right $\mathcal{A}$-modules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be the right zero-order $Q$-valued differential operator on $P$ if it is a finite sum $\Delta = \Phi^i b_i$, $b_i \in \mathcal{A}$, where $\overline{\delta}_a \Phi^i = 0$ for all $a \in \mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called the right differential operator of order $r > 0$ on $P$ if it is a finite sum

$$\Delta(p) = \Phi^i(p)b_i + \Delta_{r-1}(p), \qquad b_i \in \mathcal{A},$$

where $\Delta_{r-1}$ and $\overline{\delta}_a\Phi^i$ for all $a \in \mathcal{A}$ are right $(r-1)$-order differential operators. $\Box$

Definition 9.2 and Definition 9.3 of left and right differential operators on $\mathcal{A}$-bimodules are not equivalent, but one can combine them as follows.

DEFINITION 9.4: Let $P$ and $Q$ be bimodules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is a two-sided zero-order $Q$-valued differential operator on $P$ if it is either a left or right zero-order differential operator. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is said to be the two-sided differential operator of order $r > 0$ on $P$ if it is brought both into the form

$$\Delta = b_i\Phi^i + \Delta_{r-1}, \qquad b_i \in \mathcal{A},$$
$$\Delta = \overline{\Phi}^i\overline{b}_i + \overline{\Delta}_{r-1}, \qquad \overline{b}_i \in \mathcal{A},$$

where $\Delta_{r-1}$, $\overline{\Delta}_{r-1}$ and $\delta_a\Phi^i$, $\overline{\delta}_a\overline{\Phi}^i$ for all $a \in \mathcal{A}$ are two-sided $(r-1)$-order differential operators. $\Box$

One can think of this definition as a generalization of Definition 9.1 to higher order differential operators.

It is readily observed that two-sided differential operators described by Definition 9.4 need not be left or right differential operators, and *vice versa*. At the same time, derivations of a $\mathcal{K}$-ring $\mathcal{A}$ and their compositions obey Definition 9.4.

# References


[1] M.Atiyah and I. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley, London, 1969).

[2] C.Bartocci, U.Bruzzo and D.Hernández Ruipérez, *The Geometry of Supermanifolds* (Kluwer Academic Publ., Dordrecht, 1991).

[3] D.Bashkirov, G.Giachetta, L.Mangiarotti and G. Sardanashvily, The antifield Koszul–Tate complex of reducible Noether identities, *J. Math. Phys.* **46** (2005) 103513; *arXiv*: math-ph/ 0506034.

[4] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, The KT-BRST complex of a degenerate Lagrangian system, *Lett. Math. Phys.* **83** (2008); *arXiv*; math-ph/0702097.

[5] J.Bell and J. Zhang, An isomorphism lemma for graded rings, *arXiv*: 1509.08812.

[6] A.Borowiec, Vector fields and differential operators: Noncommutative case, *Cech. J. Phys.* **47** (1997) 1093.

[7] G.Bredon, *Sheaf theory* (McGraw-Hill, N.-Y., 1967).

[8] A.Bruce, K.Grabowska, J.Grabowski, Introduction to graded bundles, *arXiv*: 1605.03296.

[9] A.Connes, *Noncommutative Geometry* (Academic Press, New York, 1994).

[10] M.Dubois-Violette and T.Masson, On the first-order operators on bimodules, *Lett. Math. Phys.* **37** (1996) 467.

[11] M.Dubois-Violette, Lectures on graded differential algebras and noncommutative geometry, In: *Noncommutative Differential Geometry and Its Applications to Physics*, Y. Maeda et al (eds) (Kluwer, 2001) pp. 245-306.

[12] M.Farion, Introduction to graded geometry, *arXiv*; 1512.02810.

[13] D.Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* (Consultants Bureau, New York, 1986).

[14] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Cohomology of the infinite-order jet space and the inverse problem, *J. Math. Phys.* **42** (2001) 4272.

[15] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *New Lagrangian and Hamiltonia Methods in Field Theory* (World Scientific, Singapore, 1997).

[16] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Geometric and Algebraic Topological Methods in Quantum Mechanics* (World Scientific, Singapore, 2005).

[17] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Advanced Classical Field Theory* (World Scientific, Singapore, 2009).

[18] A.Grothendieck, *Eléments de Géométrie Algébrique IV*, Publ. Math. **32** (IHES, Paris, 1967).

[19] D.Hernández Ruipérez, J.Muñoz Masqué, Global variational calculus on graded manifolds. *J. Math. Pures Appl.* **63** (1984) 283.

[20] F.Hirzebruch, *Topological Methods in Algebraic Geometry* (Springer, Berlin, 1966).

[21] J.Koszul, *Lectures on Fibre Bundles and Differential Geometry* (Tata University, Bombay, 1960).

[22] K.Krasil'shchik, V.Lychagin and A.Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, Glasgow, 1985).

[23] G.Landi, *An Introduction to Noncommutative Spaces and their Geometries*, Lect. Notes in Physics, New series m: Monographs, **51** (Springer, Berlin, 1997).

[24] S.Lang, *Algebra* (Addison–Wisley, New York, 1993).

[25] H.Lombardi and C. Quitté, *Commutative Algebra: Constructive Methods. Finite Projective Modules* (Springer, 2015).

[26] V.Lunts and A.Rosenberg, Differential operators on noncommutative rings, *Selecta Mathematica, New Series* **3** (1997) 335.

[27] S.Mac Lane, *Homology* (Springer, Berlin, 1967).

[28] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[29] W.Massey, *Homology and Cohomology Theory* (Marcel Dekker, Inc., New York, 1978).

[30] J.McConnell and J.Robson, *Noncommutative Noetherian Rings*, Graduate Stud. in Math. **30** (AMS, Providence, RI, 2001).

[31] J.Monterde, J.Munos Masque, J.Vallejo, The Poincaré–Cartan form in superfield theory, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 775.

[32] K.Mount and O.Villamayor, On a conjecture of Y.Nakai, *Osaka J. Math.* **10** (1973) 325.

[33] Y.Nakai, High order derivations I, *Osaka J. Math.* **7** (1970) 1.

[34] A.Pietsch, *Nuclear Locally Convex Spaces* (Springer, Berlin, 1972).

[35] A.Rennie, Smoothness and locality for nonunital spectral triples, *K-Theory* **28** (2003) 127.

[36] A.Robertson and W.Robertson, *Topological Vector Spaces* (Cambridge Univ. Press., Cambridge, 1973).

[37] D.Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids, *arXiv*: math/0203110.

[38] G.Sardanashvily, Lectures on supergeometry, *arXiv:* 0910.0092.

[39] G.Sardanashvily, *Lectures on Differential Geometry of Modules and Rings. Application to Quantum Theory* (Lambert Academic Publishing, Saarbrucken, 2012); *arXiv*: 0910.1515.

[40] G.Sardanashvily, Graded Lagrangian formalism, *Int. J. Geom. Methods Mod. Phys.* **10** (2013) 1350016; *arXiv*: 1206.2508.

[41] G.Sardanashvily, W.Wachowski, SUSY gauge theory on graded manifolds, *arXiv*: 1406.6318.

[42] G.Sardanashvily, Higher-stage Noether identities and second Noether theorems, *Adv. Math. Phys.* **2015** (2015) 127481.

[43] G.Sardanashvily, *Noether's Theorems. Applications in Mechanics and Field Theory* (Springer, 2016).

[44] I.Shafarevich, *Basic Algebraic Geometry* (Springer, Berlin, 1974).

[45] E.Spanier, *Algebraic Topology* (McGraw-Hill, N.Y., 1966).

[46] S.Smith, Differential operators on commutative algebras, In: *Ring Theory (Antwerp, 1985)*, Lect. Notes in Math. **1197** (Springer, Berlin, 1986) p. 165.

[47] T.Stavracou, Theory of connections on graded principal bundles, *Rev. Math. Phys.* **10** (1998) 47-79.

[48] B.Tennison, *Sheaf Theory* (Cambridge Univ. Press, Cambridge, 1975).

[49] W.Traves, Nakai's conjecture for varieties smoothed by normalization, *Proc. Amer. Math. Soc.* **127** (1999) 2245.

[50] F.Trevers, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).

[51] K.Ueno, *Algebraic Geometry 1. From Algebraic Varieties to Schemes*, Transl. Math. Monographs **185** (Amer. Math. Soc., Providence, RI, 1999).

[52] L.Vitagliano, Vector bundle valued differential forms on $NQ$-manifolds, *arXiv*: 1404.6256.