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II. MATHEMATICAL MODELING

INVESTIGATION OF A MATHEMATICAL MODEL LINKING GDP GROWTH WITH CHANGES IN NATIONAL DEBT

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The article proposes a mathematical model linking GDP growth with changes in national debt. The model is based on a system of two linear ordinary differential equations. The rate of GDP growth is defined in the model as the difference between aggregate revenues and aggregate expenses, including debt service expenses. Possible types of GDP and national debt dynamics are investigated as a function of parameters. External investments increasing national debt may accelerate economic growth, whereas without new external loans GDP either does not grow or grows slowly. An equation describing the variation of relative external debt has been derived and fully investigated. It is shown that the external debt is never fully repaid, which is consistent with other models. Conditions of stable economic growth are derived, when GDP grows faster than or at the same rate as national debt, whereas the relative debt approaches a constant value. We investigate conditions that preclude a Ponzi game, so that a country cannot use new external debt to build a financial pyramid to repay old debt. The model features are demonstrated in application to statistical data from a number of countries. The model parameters are determined and growth trajectories are calculated for GDP and national debt.

Keywords: mathematical model, economic growth, linear system of ordinary differential equations, GDP, national debt.

Introduction

International investments have begun to play an increasing role as a factor in a country's GDP growth. Yet, they increase the national debt and its service burden. Paradoxically, huge national debt is accumulated by the leading world economies, less so by the weak developing countries [1–9]. In many European countries, the ratio of national debt (ND) to Gross Domestic Product (GDP) has been growing after World War II. In some countries, such as Italy, Greece, Japan, and the USA, the national debt has exceeded GDP. The USA national debt and GDP at the end of 2017 were \$21.676 trillion and \$19.5 trillion, respectively (giving a GDP-to-ND ratio of more than 111%). The national debt of Japan is almost half this amount (\$12.9 trillion), but with GDP of \$5.1 trillion this country has the highest ratio of national debt to GDP (253%). The last of the big three is China, with national debt of \$2.991 trillion and GDP of \$18.088 trillion. The total world debt had reached \$60 trillion in 2017.

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The critical debt level is 60% of GDP according to the IMF and 80%–100% according to the World Bank. When this level is exceeded, the repayment of external debt involves transfer of resources, and instead of producing goods and services for domestic needs the state produces goods for export trade.

It would seem that financial indebtedness to foreign countries is not a positive attribute: this involves inefficient use of credit, loan service, and economic dependence on the creditor country affecting international relations. But economists and finance experts find also positive aspects in external debt:

- any foreign loan improves the economic situation of the borrower country;
- inflow of foreign capital helps to develop certain economic sectors (such as transport, energy, and others);
- the total state budget receives an injection.

But these positive aspects manifest themselves only if the loans are efficiently allocated. Efficient loan use can act as a powerful growth factor, smoothing out economic fluctuations and generating additional financial resources. On the other hand, raising national debt to the critical level may prove a grave negative factor from both economic and political considerations. It is therefore essential to be able be manage and forecast the national debt, a task for mathematical modeling. An adequate dynamic mathematical model can be used to study the degree of influence of various macroeconomic parameters on the growth of national debt and to design an optimal strategy.

We propose a fairly general mathematical model that links GDP growth to changes in national debt. The model is based on a system of two linear ordinary differential equations. Our objective is to study the effect of external investments — the cause of national debt — on GDP growth. The model dynamics is analyzed for various parameter values. Conditions are determined when external loans are a powerful factor of stable economic growth. The critical parameters are found when the state becomes bankrupt and cannot service its debt. Conditions enabling a Ponzi game (a financial pyramid) under increasing government indebtedness are investigated.

1. A Model of Joint GDP and National Debt Growth

GDP growth is driven by investments, their share in GDP, and the excess of total investment volume over the value of capital consumed in production. Leading factors of growth include attraction of additional production resources and also improved resource productivity due to scientific progress, i.e., innovation in production, which also requires investment. Government borrowing is an important factor in this process, and it creates national debt.

Time series of national debt and GDP of various countries are available on the Internet in the websites of known international organizations [1–9]. Analysis of these data shows that economic development and GDP growth are closely linked with international borrowing and thus with the growth of national debt. Even when the loans are not invested directly in the economy but are used instead to address major social, environmental, and other problems, they help to balance the budget and maintain economic growth.

National debt may be external or internal. Internal (domestic) debt is owed by the state to its citizens; it constitutes internal transfers among residents of a single country. External debt is a net transfer of the country, which imposes a heady burden on the economy. Furthermore, contrary to internal debt, external debt cannot be repaid by printing money, as it is denominated in foreign currency.

The country's GDP is the source of debt repayment in the most general sense. The inflows to GDP from international activity and domestic revenues are sources for the state budget. Both external and internal debt are serviced from the initial budget surplus. Before investing, creditor countries and international financial institutions assess the borrowing country's creditworthiness and agree on repayment terms and conditions. Debt management models and solvency models are available to estimate the repayment ability of the borrowing state [10–12]. They allow for many economic parameters, such as exchange rates, inflation level, real GDP growth rate, net trade balance, etc. A weakness of all these models is the long-term forecasting of the parameter values and thus the ability to predict the borrower's financial state and debt repayment capacity in the future.

In the present article, we propose a fairly general mathematical model that describes the joint growth of (external) national debt and GDP. We examine GDP growth due to increasing debt, the possibilities of debt repayment, and other interrelated factors.

We denote by $Q(t)$ the country's GDP and by $B(t)$ the national debt; $Q(t)$ and $B(t)$ are measured in US dollars. The variation of GDP and national debt is described by the following equations:

$$\frac{dQ}{dt} = q(\alpha Q + I) - \mu Q - \beta Q - (r+w)B, \tag{1}$$

$$\frac{dB}{dt} = -wB + I + S, \tag{2}$$

where αQ is the GDP share reinvested in economic development, $0 \le \alpha < 1$;

I is the investment in the economy due to external government borrowing; here $I = \gamma Q$, $0 \le \gamma < 1$;

q is the investment efficiency (return rate);

μQ are outflows from GDP used to service the national debt, $0 < \mu < 1$;

βQ, S respectively are the GDP share and the share of the debt increment (also expressed as a share of GDP) directed to social or military expenses or to disaster relief and other expenses not directly related with economic development, $0 \le \beta < 1$, $S = \delta Q$, $0 \le \delta < 1$;

$(r+w)B$ are the debt-service payments to creditors, which include interest rate expense $r(t)$ and partial repayment of principal $w(t)$, $0 < (r+w) < 1$.

In this model, national debt arises when the budget does not have sufficient funds to cover investments in the economy (αQ) or other expenditures not directly related to GDP growth (βQ). In such a case, the government takes a loan. Loans may be one-time, when the national debt increases in a jump, changing the debt-service rates r and w and possibly affecting the GDP growth rate, or recurring, in the form of periodic tranches received over a certain time. The loan amounts depend on the country's GDP, and we naturally assume they are a fraction of GDP: $I = \gamma Q$ and $S = \delta Q$.

Debt service requires periodic payments of $(r+w)B$ from the budget; wB is the share of the principal repaid each period (this is the only factor that reduces the national debt).

All the parameters in system (1), (2) may be functions of time.

What is the critical debt level and how does it depend on the system parameters and on GDP? Should we make payments to reduce the principal, or is it better to only cover the interest?

We start our analysis at the initial time $t = 0$. The GDP and the national debt at the initial time are respectively

$$Q(0) = Q_0, \qquad B(0) = B_0. \tag{3}$$

We rewrite the linear system (1), (2) as

$$\frac{dQ}{dt} = (\alpha_1 + q\gamma + \delta)\,Q - (r + w)B, \tag{4.1}$$

$$\frac{dB}{dt} = (\gamma + \delta)\,Q - wB, \tag{4.2}$$

where $\alpha_1 = q\alpha - \mu - \beta$ is the GDP growth factor due to internal resources (it may be negative). The coefficients of the linear system (4) have the form

$$a_{11} = \alpha_1 + q\gamma + \delta, \qquad a_{12} = -(r + w) < 0;$$

$$a_{21} = \gamma + \delta \geq 0, \qquad a_{22} = -w < 0.$$

We stipulate that a government cannot use new external borrowing to build a financial pyramid for the repayment of outstanding debt [10]. Without a Ponzi game, the interest expenses on accumulated debt in the long term from t_0 to some time $t_{\max}$ do not exceed new borrowing:

$$Z_B = \int_{t_0}^{t_{\max}} (r + w)\,B\,dt > Z_Q = \int_{t_0}^{t_{\max}} (\gamma + \delta)\,Q\,dt. \tag{5}$$

This implies that, starting with some $t^* < t_{\max}$, we should have

$$(r + w)B > (\gamma + \delta)Q, \tag{6}$$

or the Ponzi factor po should be less than 1:

$$po = \frac{(\gamma + \delta)Q}{(r + w)B} < 1. \tag{7}$$

Assume that the coefficients remain constant over a certain time interval. Then the solution of system (4) with initial values (3) has the form

$$Q(t) = \frac{Q_0}{\lambda_2-\lambda_1}\left((\lambda_2-a_{11})\,e^{\lambda_1 t}-(\lambda_1-a_{11})\,e^{\lambda_2 t}\right)+\frac{B_0 a_{12}}{\lambda_2-\lambda_1}\left(e^{\lambda_2 t}-e^{\lambda_1 t}\right),$$

$$B(t) = \frac{Q_0 a_{21}}{\lambda_2-\lambda_1}\left(e^{\lambda_2 t}-e^{\lambda_1 t}\right)+\frac{B_0}{\lambda_2-\lambda_1}\left((\lambda_2-a_{22})\,e^{\lambda_2 t}-(\lambda_1-a_{22})\,e^{\lambda_1 t}\right), \tag{8.1}$$

where λ_2 and λ_1 are the eigenvalues:

$$\lambda_{1,2} = 0.5\left(Sp \mp \sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}\right), \tag{8.2}$$

$$Sp = \alpha_1 + q\gamma + \delta - w, \qquad \Delta = -(\alpha_1+q\gamma+\delta)w + (\gamma+\delta)(r+w).$$

Let us investigate the solution of this model as a function of the parameter values. Before that, we derive and investigate the equation for the relative debt $b(t)$, which is a major economic indicator:

$$b(t) = \frac{B(t)}{D(t)}. \tag{9}$$

2. The Dynamics of Relative Debt

Differentiating (9) with respect to time and substituting the derivatives $\frac{dQ}{dt}$ and $\frac{dB}{dt}$ from (4.1), (4.2), we obtain an equation for the variation of relative debt:

$$\frac{db(t)}{dt} = (r+w)b^2 - (\alpha_1+q\gamma+\delta+w)b + (\gamma+\delta), \tag{10}$$

$$b_0 = b(0) = \frac{B_0}{D_0}.$$

are the initial conditions.

The right-hand side of Eq. (10) is a quadratic trinomial. Its roots are given by

$$b_1 = \frac{\left(s-\sqrt{s^2-4(\gamma+\delta)(r+w)}\right)}{2(r+w)}, \tag{11.1}$$

$$b_2 = \frac{(\gamma+\delta)}{(r+w)b_1} = po(b_1), \tag{11.2}$$

where $s = a_{11}-a_{22} = \alpha_1+q\gamma+\delta+w$.

The roots b_1 and b_2 exist and are positive if the discriminant is greater than zero:

$$Dic = s^2 - 4(\gamma+\delta)(r+w) > 0,$$

i.e.,

$$(\alpha_1 + q\gamma + \delta + w)^2 > 4(\gamma+\delta)(r+w).$$

The last inequality is satisfied if the following two inequalities are true:

$$\alpha_1 + q\gamma + \delta + w > 2(r+w), \tag{12.1}$$

$$\alpha_1 + q\gamma + \delta + w > 2(\gamma+\delta). \tag{12.2}$$

The left-hand sides of inequalities (12) contain parameters that determine economic growth; the right-hand sides are double the service amounts for outstanding debt or new borrowing. Under normal conditions, these equalities are satisfied. If they are not satisfied, this implies that the economy cannot carry the debt service burden. In this event, the right-hand side of Eq. (10) is greater than zero and the relative national debt grows without bound: the national debt always grows faster than GDP.

Now consider the case when inequalities (12) are satisfied. Then the roots b_1 and b_2 exist. They describe stationary states of Eq. (10). The stationary solution b_1 is stable and b_2 is unstable. If initially $b_0 < b_1$, the relative debt increases up to b_1 and stabilizes thereafter. If $b_1 < b_0 < b_2$, the relative debt decreases, approaching b_1 over time. With $b_0 < b_2$, the country's economy grows due to external investments and successfully manages the service of national debt. GDP and the national debt increase over time at the same rate. If $b_0 > b_2$, then the relative debt increases without bound. The critical value is thus $b_0 = b_2$.

Let us find the conditions that rule out a Ponzi game. To this end, inequality (5) describing the ratio between total borrowing Z_Q and total debt service Z_B is transformed to the form

$$\int_{t_0}^{t_{\max}} (r+w)B\left(1 - \frac{(\gamma+\delta)}{(r+w)\,b(t)}\right)dt > 0. \tag{13}$$

If the relative debt increases without bound, i.e., the economy fails to manage its debt service, inequality (13) is satisfied and no Ponzi game is possible.

Assume that the stationary solutions exist and $b_0 < b_2$. Then the relative debt goes to b_1 $(b(t) \to b_1)$ and the ratio $\dfrac{(\gamma+\delta)}{(r+w)\,b(t)} \to b_2$, i.e., the ratio goes to the value of the second stationary solution equal to the coefficient $po(b_1)$ (11.2). If $po(b_1) > 1$, then the integral (13) is less than zero and a financial pyramid is constructed. If $po(b_1) < 1$, inequality (13) holds and no Ponzi game is possible. Thus, a necessary and sufficient condition for a Ponzi game is that $b_2 < 1$. Since $b_1 < b_2$, then also $b_1 < 1$ and their product is less than 1. Hence we obtain a necessary (but not sufficient) condition to rule out the existence of a Ponzi game:

$$b_1 * b_2 = \frac{(\gamma+\delta)}{(r+w)} < 1. \tag{14}$$

Inequality (14) implies that new borrowing as a share of GDP should be less than the share of debt service or the external debt interest rate should be greater than the rate of GDP growth in foreign currency due to new borrowing.

3. Conditions for GDP Growth Driven by National Debt

Let us analyze the solutions of Eqs. (4), (3), and (9) for various parameter values and initial conditions.

(i) GDP growth with no new borrowing. If there is no new borrowing $(\gamma = 0,\ \delta = 0)$, the previously accumulated national debt decreases exponentially with exponent w:

$$B(t) = B_0 e^{-wt}.$$

The GDP growth in this case is described by the function

$$Q(t) = Q_0 e^{\alpha_1 t}\left(1 - \frac{(r+w)B_0}{(\alpha_1 + w)Q_0}\left(1 - e^{-(w+\alpha_1)t}\right)\right). \tag{15}$$

Let us analyze expression (15). If $\alpha_1 > 0$ (i.e., $q\alpha > \mu + \beta$, revenues exceed expenses), then with no initial debt $B_0 = 0$, the country's GDP increases exponentially with exponent α_1. If there is some initial debt $B_0 > 0$, the amount of debt determines whether GDP grows or not. With a moderate initial debt $B_0 < Q_0 \frac{\alpha_1 + w}{r + w}$, GDP grows; otherwise $\left(B_0 > Q_0 \frac{\alpha_1 + w}{r + w}\right)$, GDP decreases.

(ii) *Conditions when national debt accelerates GDP growth.* Government borrowing, on the one hand, increases the national debt and its service expenditure. On the other hand, it may lead to acceleration of GDP growth. Let us derive the parameter values for these two eventualities.

Let both GDP and the national debt increase $\left(\frac{dQ}{dt} > 0,\ \frac{dB}{dt} > 0\right)$. GDP grows faster than the national debt if

$$(\alpha_1 + (q-1)\gamma)Q - rB > 0.$$

α_1 is the rate of GDP growth due to internal resources (in the absence of debt), $(q-1)\gamma$ is the rate of GDP growth due to debt growth when $q > 1$. If the investment return is low $q \le 1$, investments only increase the debt and divert budget funds to debt service (rB).

GDP grows faster than without debt if

$$(q-1)\gamma Q > rB. \tag{16}$$

Inequality (16) holds in the presence of high-return investments $(q > 1)$ and when the interest expense is less than the inflow of new investments multiplied by the rate of return. If the reverse inequality holds, the GDP growth rate is reduced despite the investments.

Without a Ponzi game (condition (6)), we obtain

$$(q-1)\gamma Q \;>\; rB \;>\; (\gamma+\delta)Q - wB. \tag{17}$$

Assume that the borrowing for social needs is equal to debt repayment $(\delta Q = wB)$. Then from inequalities (14) we have

$$(q-1)\gamma Q \;>\; \gamma Q \qquad \text{for} \quad rB > \gamma Q. \tag{18}$$

Hence it follows that the return on investing external loans in the economy should be at least 2 and interest expense should exceed external loans. If, however, the borrowing for social needs is greater than loan repayment $(\delta Q > wB)$, the return on investment should be even greater.

Summarizing, we assert that new borrowing accelerates GDP growth if the investment return is high, i.e., every dollar invested produces several dollars (at least more than 2). All this assumes the absence of a Ponzi game.

(iii) *Conditions for GDP growth driven by national debt*. Assume that without investments $(\alpha_1 = 0)$ GDP does not grow. Let us solve the system for this case and elucidate at what point state borrowing triggers GDP growth.

Assume that interest expense $r(t)$ and principal repayment $w(t)$ are related by

$$r \;=\; \rho w, \qquad \rho > 0.$$

For simplicity we first assume that all new borrowing is directed only to investment. In this case, $\delta = 0$ and the rate of return on investment is

$$q \;=\; 1 + \rho \;>\; 1.$$

Then system (2.4) takes the form

$$\frac{dQ}{dt} \;=\; q(\gamma Q - wB), \tag{19.1}$$

$$\frac{dB}{dt} \;=\; \gamma Q - wB. \tag{19.2}$$

It is easy to solve system (19) with initial values (3):

$$B(t) \;=\; \left(B_0 + \frac{\gamma C}{\lambda}\right)e^{\lambda t} - \frac{\gamma C}{\lambda}, \tag{20.1}$$

$$Q(t) \;=\; qB(t) + C, \qquad C \;=\; Q_0 - qB_0, \qquad \lambda \;=\; q\gamma - w. \tag{20.2}$$

Hence it follows that GDP and the national debt either both grow exponentially (when $q\gamma > w$) or both decrease (when $q\gamma < w$). By (19), $\frac{dQ}{dt} = q\frac{dB}{dt}$, which implies that GDP grows or decreases q times faster than the national debt.

This is precisely the case when GDP may grow due to new borrowing and it may grow faster than the debt. State borrowing plays such a positive role only if the investments are economically efficient $(q > 1)$.

If the Ponzi game is ruled out by (6), we obtain the inequalities

$$qwB > \gamma Q > wB. \tag{21}$$

Hence, as in the previous section, increases of national depth produce GDP growth only with high-return investments $(q > 2)$.

To find an analytical solution, we have simplified the system and omitted some of the expenses. In reality, the GDP growth factor q should probably be still higher.

4. Numerical Calculations of GDP and National Debt Dynamics for Various Parameter Values

We will illustrate the possible types of GDP and national debt dynamics described by system (4) with initial values (3). As the basic parameter set we take

$$\alpha_1 = 0.01; \quad q = 3; \quad \gamma = 0.02; \quad \delta = 0.01; \quad w = 0.02; \quad r = 0.05. \tag{22}$$

Then the model coefficients take the following values:

$$a_{11} = \alpha_1 + q\gamma + \delta = 0.06; \quad a_{12} = -(r+w) = 0.07; \tag{23}$$

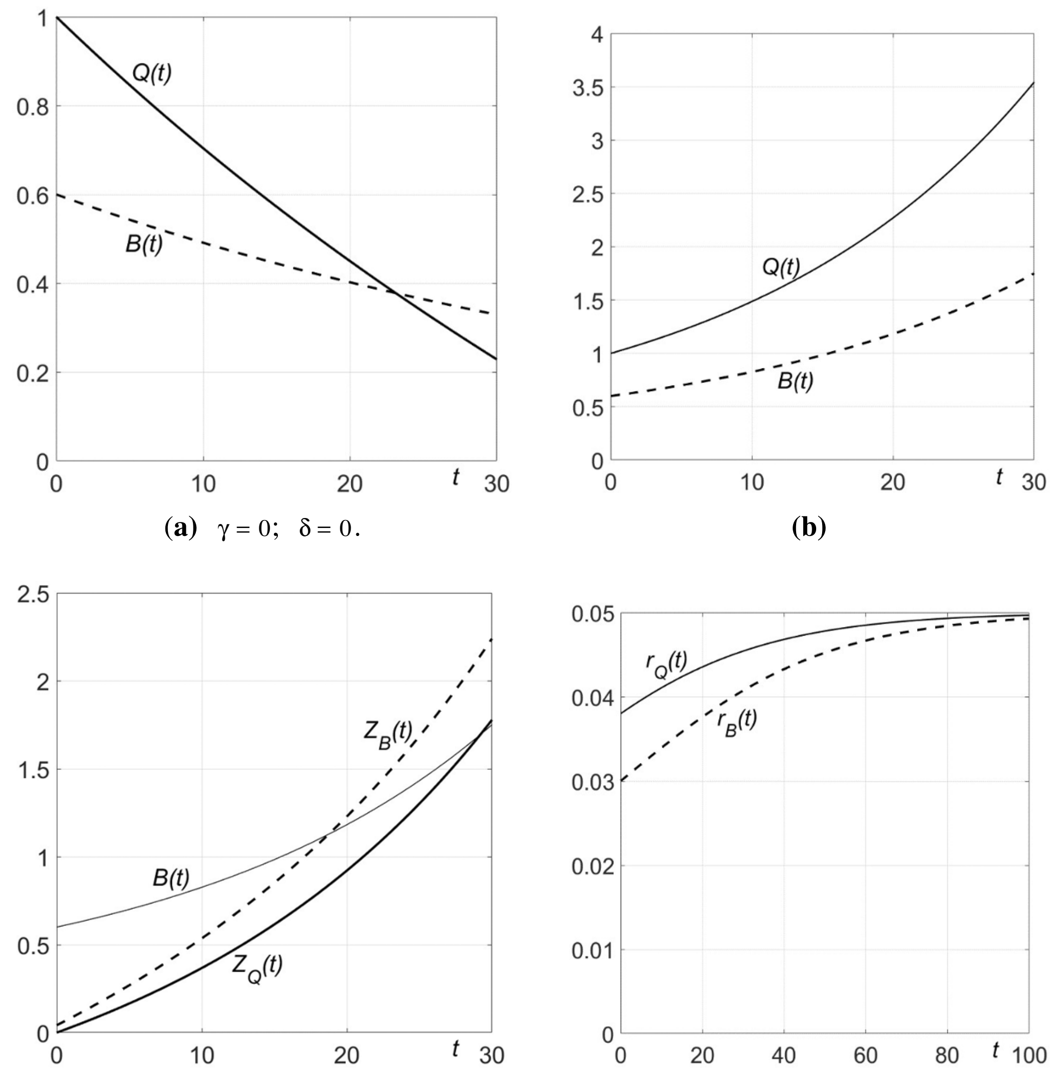
$$a_{21} = \gamma + \delta = 0.03; \qquad a_{22} = -w = 0.02.$$

For the initial GDP we set $Q(0) = 1$. The national debt is measured in units of GDP. Regarding the parameter values in (22), without debt $(B(0) = 0)$ and without new borrowing, the GDP would grow annually by 1% $(\alpha_1 = 0.01)$. But even a small amount of debt siphons off budget funds and the GDP decreases.

In our example, the interest rate is 5% and the repayment rate is 2% $(r = 0.05,\ w = 0.02)$. Total payments from the budget are thus 7% of the amount of debt. We will show that external investments in the economy can accelerate GDP growth. Assume that the investment rate is 2% of GDP $(\gamma = 0.02)$ and the investment return is quite high $(q = 3)$. External borrowing is also needed for social and other needs $(\delta = 0.01)$.

1. Accelerating GDP growth by external borrowing. The effect of initial debt. We first consider the effect of the initial debt on GDP growth. The parameters (22) allow stable economic growth because the relative debt has a stable stationary state: $b_1 \approx 0.43$, $b_2 = 1$. For $b(0) < b_2 = 1$ the relative debt approaches b_1 over time. Total debt service Z_B approaches total borrowing Z_Q, because $b_2 = 1$.

We numerically calculate model (4) with the parameters (22) for various initial debt levels. Figures 1–3 plot the growth dynamics of GDP and national debt for $B(0) = 0.6,\ 0.3,$ and $1.1,$ respectively. Panels a) show



(a) $\gamma = 0$; $\delta = 0$. **(b)** **(c)** **(d)**

Fig. 1. Growth dynamics of GDP and national depth with $B(0) = 0.6$, $b(t) > b_1$, $b(t) \to b_1 \approx 0.43$, $b_2 = 1$.

how GDP and national debt vary over time if there is no new borrowing $(\gamma = 0;\ \delta = 0)$ and only the previously accrued debt is paid down. Panels (b) show the growth dynamics of GDP and national debt in case of constant loans. Panels (c) plot the growth of total debt service Z_B and total borrowing Z_Q. For comparison, this panel also shows the growth of national debt. Panels (d) plot the growth rates of GDP r_Q and national debt r_B versus time:

$$r_Q = \frac{dQ}{dt}/Q, \qquad r_B = \frac{dB}{dt}/B.$$

We see from Fig. 1a that for $B(0) = 0.6$ and without new borrowing, the GDP rapidly decreases (by about 30% over 10 years). With new borrowing, the GDP grows at a fast rate, about 4% annually in the first 10 years; the GDP growth rates rise gradually to 5% annually (Fig. 1b, 1d). The national debt grows more slowly than GDP;

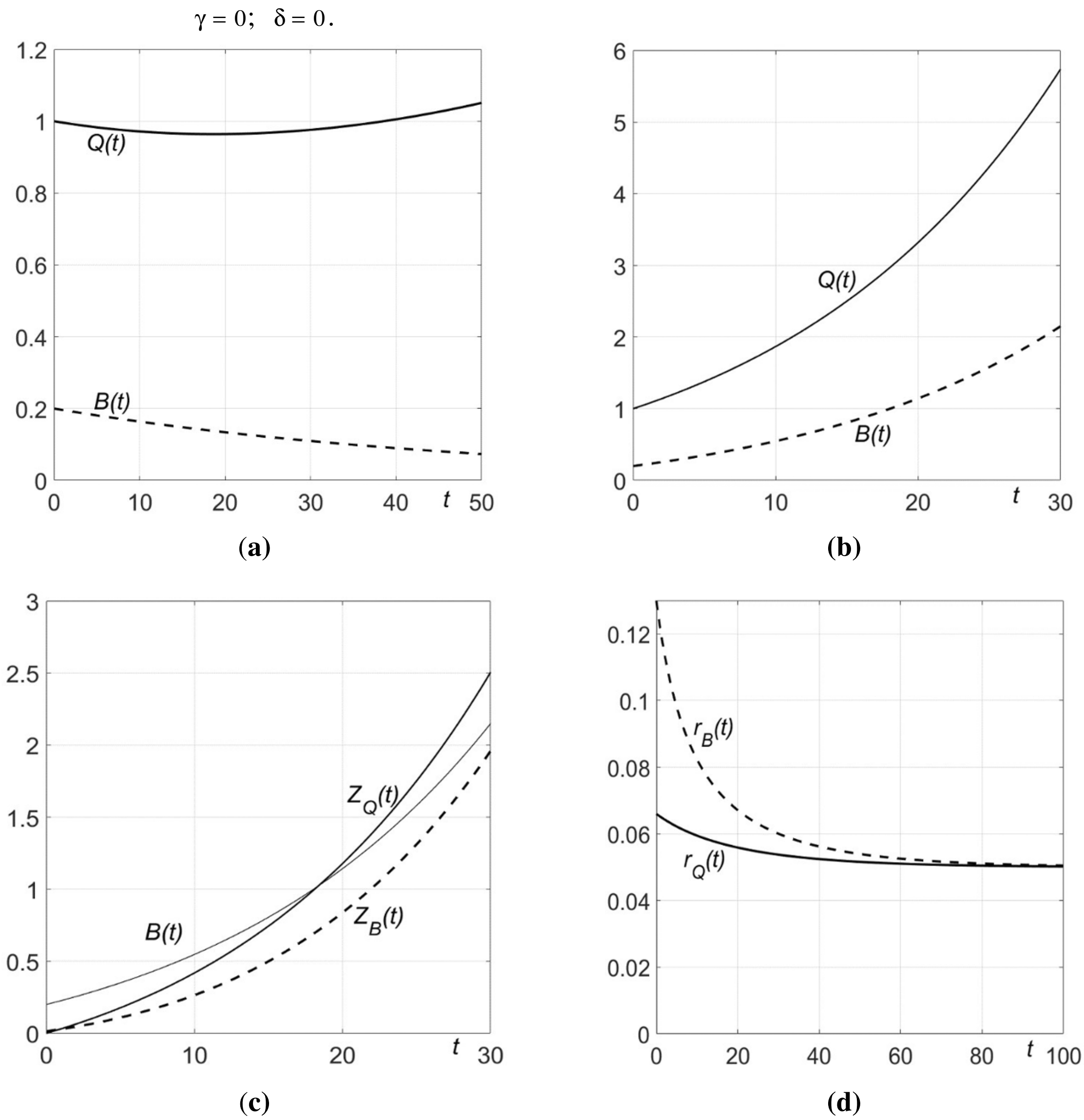


Fig. 2. Growth dynamics of GDP and national debt with $B(0) = 0.2$, $b(t) < b_1$, $b(t) \to b_1 \approx 0.43$, $b_2 = 1$.

the relative debt decreases and stabilizes at $b_1 \approx 0.43$. Total debt service is substantially greater than total borrowing over a fairly long period, and after about 18 years it begins to exceed the total volume of debt (Fig. 1c). It is only after a very long time (more than 100 years) the total debt service approaches total borrowing. No Ponzi game is possible in this case.

With a small initial debt $B(0) = 0.2$ (Fig. 2), the GDP growth dynamics improves substantially. Even without new borrowing, with $\gamma = 0$ and $\delta = 0$, the GDP does not drop to zero (contrary to the case with $B(0) = 0.6$), and remains almost constant until eventually (after about 40 years) it starts to increase very slowly (Fig. 2a). The use of constant state loans produces a rapid GDP growth (at first by more than 6% annually, but the growth rate subsequently drops to 5% as the national debt increases — Fig. 2d). The national debt also grows, but at a slower rate than GDP. The relative debt at first increases and then stabilizes at $b_1 \approx 0.43$. Total borrowing exceeds total debt service and after about 18 years they begin to exceed the total national

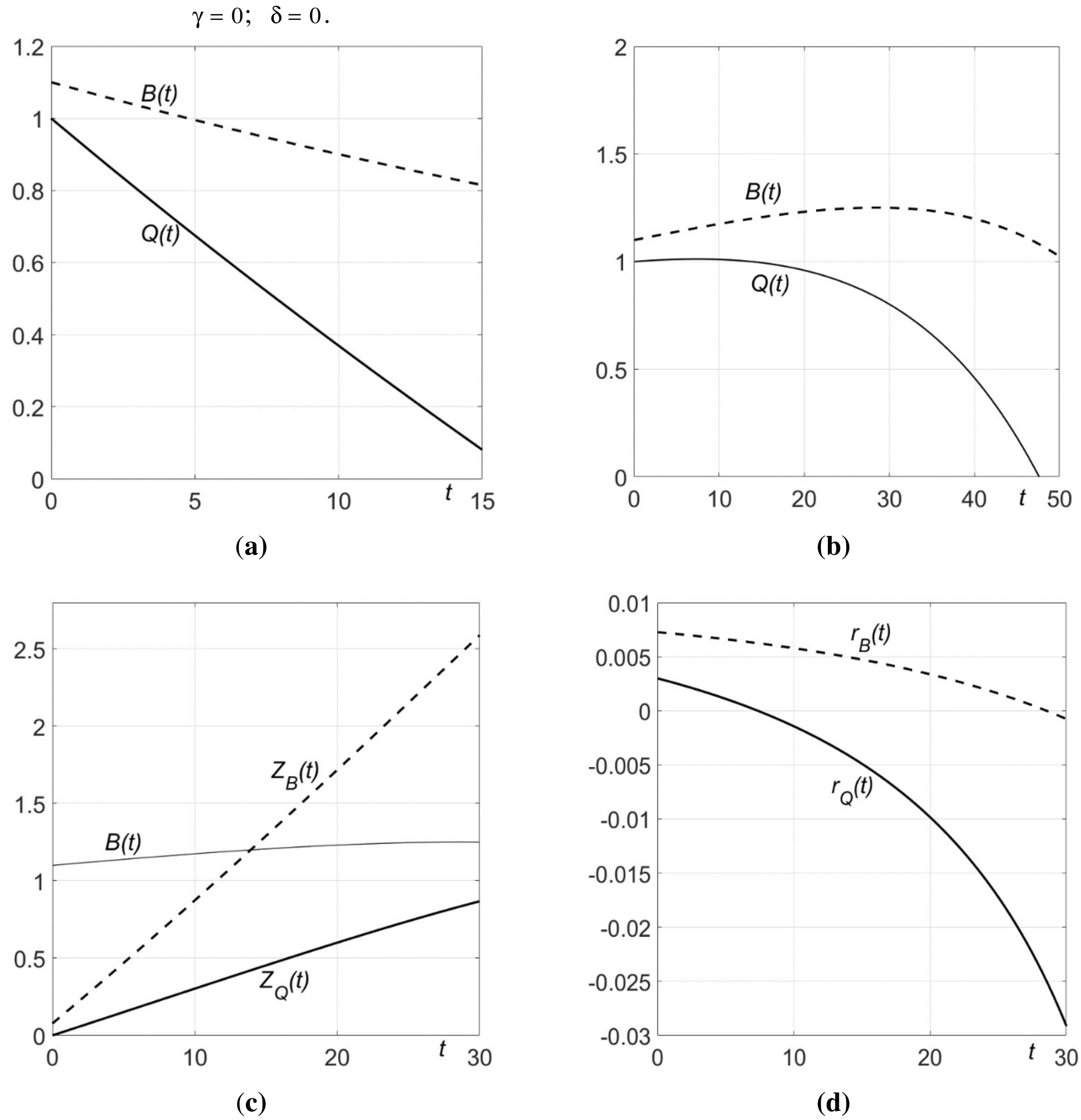
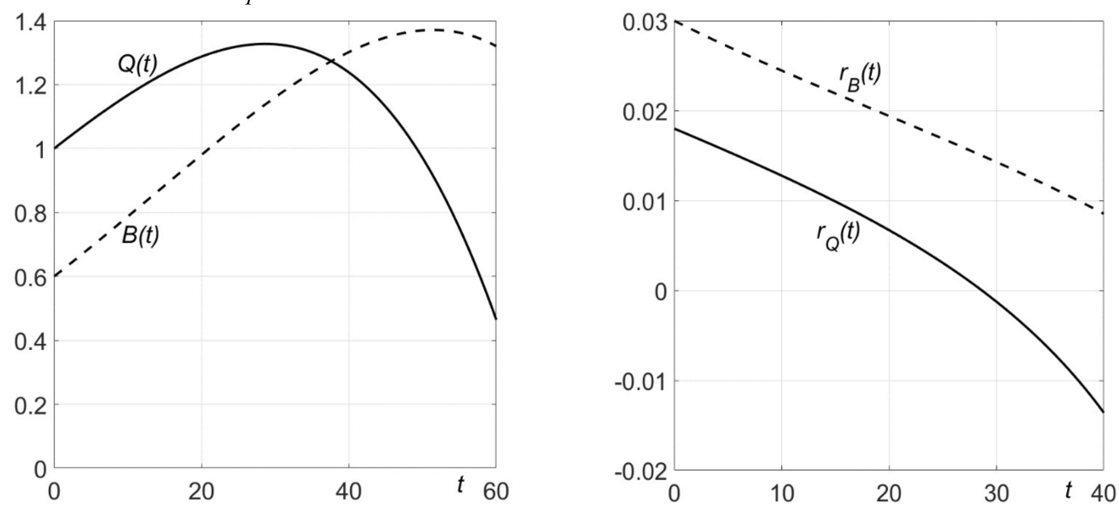


Fig. 3. Growth dynamics of GDP and national debt for $B(0) = 1.1$, $b(t) > b_2 = 1$, $b(t) \to \infty$.

debt (Fig. 2c). This situation is qualitatively different from the previous case. It is only after a very long period (more than 100 years) that the total debt service approaches total borrowing. Here a Ponzi game is possible. *In successfully growing economies, there is always an opportunity for a Ponzi game over a finite time interval!*

Let us now consider the GDP dynamics when the initial debt is above the critical level. Let $B(0) = 1.1$. We see from Fig. 3a that without new borrowing GDP rapidly decreases, by 50% in 8 years. New borrowing during the same 8 years arrests the GDP decline, but the national debt grows (Fig. 3b). Since the initial debt is above the critical level, GDP decline and rapid growth of relative debt are necessarily observed over time: GDP falls by 50% in approximately 38 years. Thus, economic reforms should be instituted during the first 8 years, while GDP remains almost constant, steering the economy onto a stable growth trajectory. The difference between total borrowing and total debt service is huge (Fig. 3c). Total debt service is much higher than total borrowing, which is a consequence of the huge debt, with the lion's share of GDP going to debt service.

$q = 2$.



(a) **(b)**

Fig. 4. Growth dynamics of GDP and national debt with $q = 2$; $b(t) \to \approx \infty$.

2. *The effect of investment return rate* q.

We have theoretically demonstrated the importance of the investment return q. Let us examine the situation in reality. We compare the GDP growth dynamics with $q = 2$ and $q = 3$ for the basic parameter set (22). As previously, let $Q(0) = 1$ and $B(0) = 0.6$. Figure 4a plots the growth of GDP and national debt for $q = 2$; the same curves for $q = 3$ are shown in Fig. 2a. We see that with $q = 3$ the GDP grows faster than the national debt and its growth rate is much higher than with $q = 2$. With $q = 2$, GDP and the national debt initially increase almost at the same rate, but in the long term GDP reaches a maximum and starts declining, while the national debt overtakes GDP. Figure 4b shows the growth rates of GDP and national debt. We see that with $q = 2$ the GDP growth rates start high, but then decrease and become negative. With $q = 3$, the GDP growth rates remain high at all times. With $q = 2$, the relative debt increases without bound in the long term $(b(t) \to \approx \infty)$ and with $q = 3$ it stabilizes. Note that although with $q = 2$ the system inevitably hits a crisis, during the first 10–25 years it demonstrates high rates of economic growth. This is the time to implement economic reforms, boosting budget revenues and moving onto a stable growth trajectory. In our model, this requires increasing the investment return rate.

3. *The effect of investment volume*.

GDP growth is also sensitive to the volume of investment. We double the investment volume compared with the previous case to $\gamma = 0.04$ ($q = 2$, $B(0) = 0.6$, the other parameters are as in (22)).

Figure 5 shows the growth dynamics for this case. It is qualitatively different from before. Here, stable economic growth is observed. GDP initially grows faster than debt, and then both grow at the same rate. The relative debt slightly increases, and then stabilizes at $b_1 \approx 0.714$. The GDP growth rates are always high, more than 5% (Fig. 5b).

It is not only the investment volume but also the structure of investments that is important. In this example, loans used for technological innovation are four times the loans used for social or military expenditures $(\gamma = 0.04,\ \delta = 0.01)$. With the same volume of debt and the reverse ratio of expenditures $(\gamma = 0.01,\ \delta = 0.04)$,

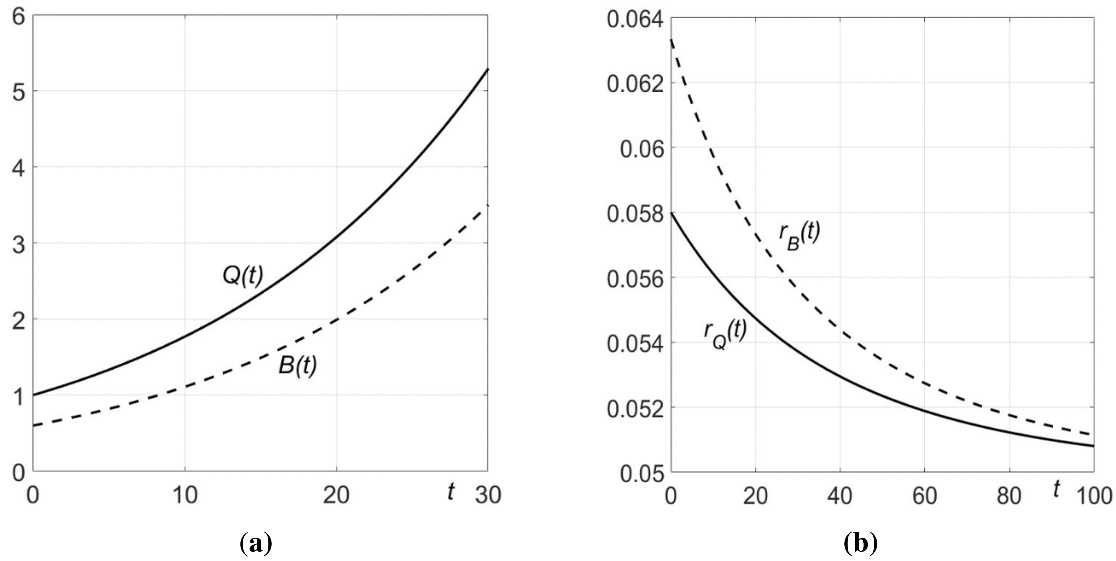


Fig. 5. Growth dynamics of GDP and national depth with $\gamma = 0.04$, $b(t) <\!\!\rightarrow b_1 \approx 0.714$, $b_2 = 1$.

stable growth is unreachable. In this case, the national debt increases much faster than GDP and starts exceeding GDP after about 17 years. The system hits a crisis, payments substantially exceed borrowing and withdraw significant budget funds.

5. Using Statistical Data to Investigate the Model of Growth Dynamics of GDP and National Debt

The mathematical model (4), (3) with constant parameters adequately describes the joint dynamics of GDP and national debt in periods of stable growth, when the country's economic strategy remains unchanged. During periods of economic instability, when the GDP and national debt curves show fluctuations, the model describes the main trends.

Consider a time interval when the model coefficients a_{11}, a_{22}, a_{12}, a_{21} are constant and let us fit their values using statistical data. We use the ordinary least squares method. Given the coefficients and the initial values Q_0 and B_0 (also estimated by OLS), we find the solutions $Q(t)$ and $B(t)$ of model (4), (3). These solutions describe a certain averaged dynamic of GDP and national debt development over the chosen time interval.

To find the coefficients a_{11}, a_{22}, a_{12}, a_{21} and the initial values Q_0 and B_0 we proceed as follows. The solution of system (4), (3) is written in the form

$$Q(t_j) = Q_0 + a_{11}\int_{t_0}^{t_j} Q(\tau)\,d\tau + a_{12}\int_{t_0}^{t_j} B(\tau)\,d\tau, \tag{24.1}$$

$$B(t_j) = B_0 + a_{21}\int_{t_0}^{t_j} Q(\tau)\,d\tau + a_{22}\int_{t_0}^{t_j} B(\tau)\,d\tau. \tag{24.2}$$

Assume that at times $t_0, t_1, \dots, t_N$ we have the statistical data for GDP and national debt, G_j and D_j, $j = 0, 1, \dots, N$, respectively. To find the solution at times t_j the integrals over unknowns in (24) are replaced with integrals over the statistical data. Evaluating the integrals by the trapezoid method, we obtain

$$IQ_j = IQ(t_j) = \int_{t_0}^{t_j} Q(\tau)\,d\tau \approx \int_{t_0}^{t_j} G(\tau)\,d\tau \approx \sum_{j=1}^{N} 0.5(G_j + G_{j-1})\,\Delta t, \tag{25.1}$$

$$IB_j = IB(t_j) = \int_{t_0}^{t_j} B(\tau)\,d\tau \approx \int_{t_0}^{t_j} D(\tau)\,d\tau \approx \sum_{j=1}^{N} 0.5(D_j + D_{j-1})\,\Delta t. \tag{25.2}$$

The statistical data for GDP and national debt are usually published annually as of the end of the year, and we accordingly take $\Delta t = 1$ year.

For the sum of squared deviations of the solution from the statistical data we have

$$SQ = \sum_{j=0}^{N} (G_j - Q_j)^2 = \sum_{j=0}^{N} \left(G_j - a_{11} IQ_j - a_{12} IB_j - Q_0\right)^2, \tag{26.1}$$

$$SB = \sum_{j=0}^{N} (D_j - B_j)^2 = \sum_{j=0}^{N} \left(D_j - a_{21} IQ_j - a_{22} IB_j - B_0\right)^2. \tag{26.2}$$

Differentiating SQ and SB with respect to the parameters a_{11}, a_{22}, a_{12}, a_{21}, Q_0, and B_0 and setting the partial derivatives equal to zero, we obtain two linear systems of third order with the same coefficient matrix and different right-hand sides. Solving these systems, we find the sought parameters a_{11}, a_{22}, a_{12}, a_{21}, Q_0, and B_0. However, the inverse problem of finding the parameters of a linear system of ordinary differential equations from the system solution is ill-posed; systems with wildly different parameters may give very close solutions. As a result, this parameter-estimation technique may produce unrealistic parameter values. To avoid the difficulty, we introduce certain constraints. Note that the coefficients a_{12} and a_{22} in system (4) are negative and $|a_{12}| > |a_{22}|$. The coefficient $a_{21} \ge 0$; it vanishes in the absence of new borrowing. The coefficient a_{11} may be positive or negative depending on whether GDP increases or decreases (positive or negative difference between aggregate revenues and aggregate expenditures) in case of zero debt service.

We will now simplify the problem of estimating the parameters of system (3), (4). Assume that the averaged coefficients a_{12} and a_{22} are known (they describe the share of payments to national debt). Substitute the required negative values for these coefficients. Then only four unknowns remain: a_{11}, a_{22}, Q_0, and B_0. We find them by OLS using expressions (24) and (25) and construct trend curves for GDP and national debt. The using the estimated parameters to calculate the solution of model (3), (4) over the long term, we can apply the GDP and national debt trends to make predictions. If they are unfavorable, we can check how many years remain to introduce economic reforms and correct the situation.

We use the proposed method to find the model parameters from statistical data and demonstrate the model features by examining the potential economic growth in three countries. Figure 6 shows the growth dynamics of GDP and national debt in USA, Japan, and Singapore. Figure 6a plots the USA economic growth trends in

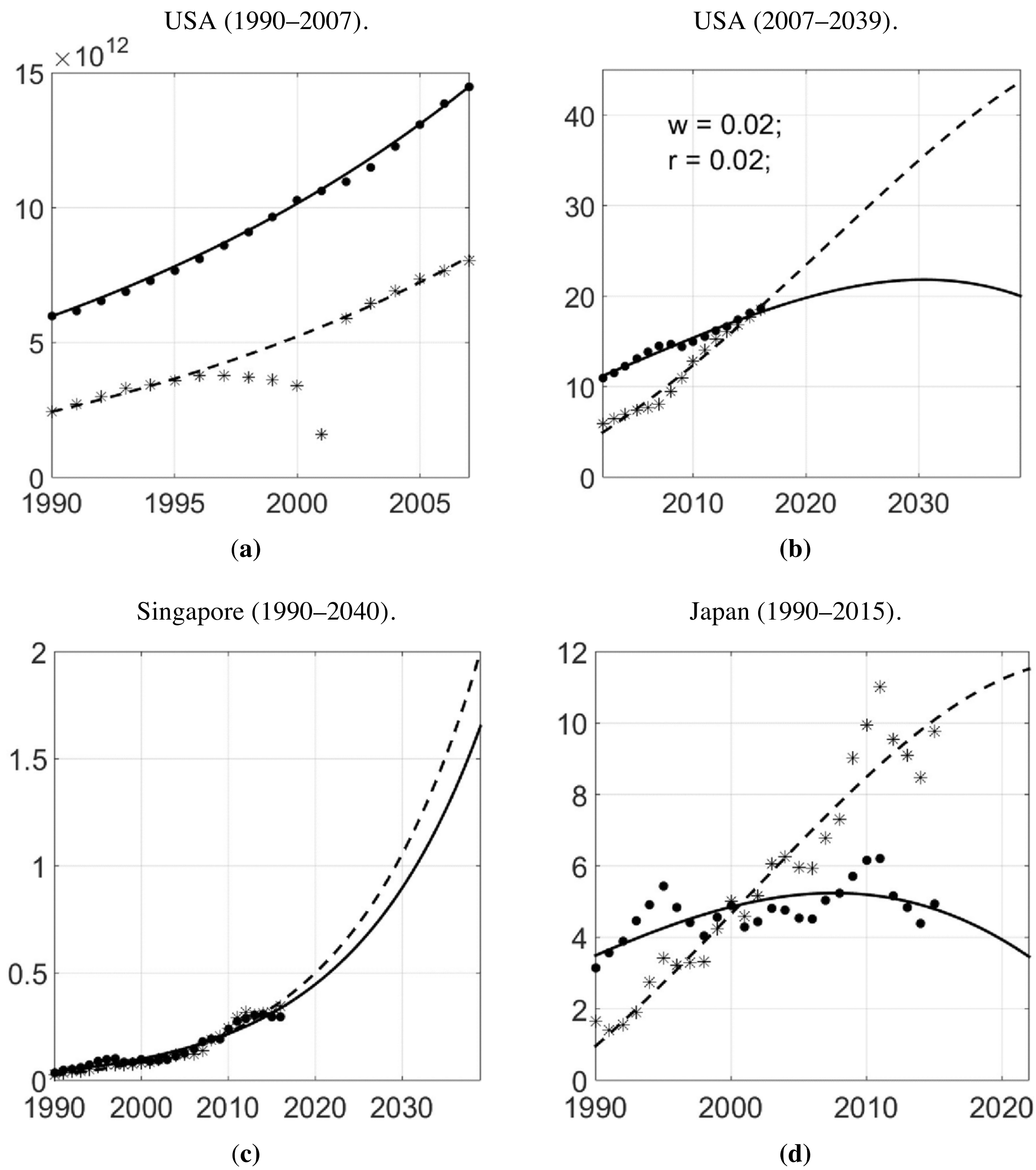


Fig. 6. GDP and national debt trends for three countries over limited time intervals as calculated from the model (GDP — solid curves, national debt — broken curves). Statistical data: • for GDP, ∗ for national debt.

the pre-crisis years (1990–2007); Fig. 6b plots the trends for the crisis years and the post-crisis years. We see that the pre-crisis years are characterized by stable economic growth, with GDP growing faster than national debt. At the same time, a large financial bubble forms, and it produces a crisis when it bursts. In the crisis years, the USA government tried to solve the problems and maintain growth by sharply increasing the national debt, which altered the development trends. GDP continues to grow as before, but the national debt now grows faster than GDP. In 2016, the national debt catches up to GDP by volume and continues to grow faster than GDP.

Figure 6b shows the development of GDP and national debt in the USA in the nearest 20 years as calculated from the model. We see that if the economic policy remains unchanged, the service of the growing national debt will slow down GDP growth and eventually lead to GDP decline. The model also shows that the government has enough time to restructure its expenditures and alter the economic development trends.

A similar dynamic is observed for Singapore (Fig. 6c). Before the crisis, the GDP grows faster than the national debt; during the crisis years the trends change, and although GDP continues to grow, the national debt starts growing faster. The model shows that eventually the growth will be arrested and GDP will start decreasing, but the government has plenty of time to find a solution for expenditure reform and debt restructuring.

Figure 6d shows the dynamics of GDP and national debt of Japan since 1990. The GDP and national debt growth curves fluctuate as a sign of unstable economic development. The model in this case describes development trends. The GDP trend is declining, while the national debt shows an increasing trend. The economic situation in Japan on the whole deteriorates.

Conclusion

We have proposed a mathematical model linking GDP growth with changes in national debt. The model is based on two linear ordinary differential equations. The rate of change of GDP in the model is defined by the difference between aggregate revenues and aggregate expenditures, including national debt service. The model specifically highlights external investments that stimulate growth. This effect is incorporated through the investment parameters q that represents the investment rate of return. In the first, approximation, new external borrowing is assumed proportional to GDP.

The model is analyzed analytically and numerically, different GDP and national debt dynamics are investigated as a function of parameter values. External investments increasing national debt may become a powerful factor of economic growth. They accelerate GDP growth, whereas without new external borrowing GDP may not grow or grow slowly. We have derived and investigated an equation that describes the variation of external debt. We show that the external debt is never fully repaid, a result consistent with other models [11].

Conditions of stable economic growth are determined, when GDP grows faster than or at the same rate as the national debt, while the relative debt approaches a constant value.

We have investigated the conditions that preclude the possibility of a Ponzi game, so that a debtor country cannot construct a financial pyramid to repay its outstanding debt. Numerical calculations show that in successful economies total borrowing may be much greater than total debt service over the long term, which implies that a Ponzi game operates in these economies over a finite time interval.

The model features have been demonstrated in application to statistical data of several countries. The model parameters have been estimated and the growth trajectories for GDP and national debt have been calculated over certain time intervals; appropriate predictions have been made.

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