

A sliceness criterion for odd free knots

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Abstract. The problem of concordance and cobordism of knots is a well-known classical problem in low-dimensional topology. The purpose of this paper is to show that for odd free knots, that is, free knots with all intersections odd, the question of whether the knot is slice (concordant to a trivial knot) can be answered effectively by analysing pairing of the chords in a knot diagram.

Bibliography: 8 titles.

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§ 1. Introduction

1.1. Statement of the problem and basic definitions. In this paper, we will call a *4-graph* the following generalisation of a four-valent graph: a one-dimensional complex Γ with each connected component homeomorphic to either a circle or a four-valent graph. The *vertices* of a 4-graph are the vertices of its components homeomorphic to four-valent graphs. The *edges* are the edges of four-valent graphs as well as circle components. An edge homeomorphic to a circle will be called a *cyclic edge* of a 4-graph.

Definition 1. A 4-graph Γ is called *framed* if, for each vertex, the four half-edges incident to it are split into two pairs. The half-edges from one pair will be called *formally opposite*.

In this paper, we study concordance (see Definition 10) of 4-valent graphs. In particular, we identify when a 4-graph is *slice*, that is, concordant to a trivial graph without vertices and with a single cyclic edge. The central result of the paper is Theorem 1, which gives a criterion that leads to a solution of the slice problem for a certain class of framed four-valent graphs (odd graphs) in terms of the chord diagram of a graph (see Definition 5).

We recall the basic definitions necessary for the formulation and proof of the main result of the paper.

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Definition 2. A *classical knot* is an isotopy class of a circle embedded in $\mathbb{R}^3$.

A *diagram* of a classical knot is its image under a generic projection onto a two-dimensional subspace $\mathbb{R}^2 \subset \mathbb{R}^3$.

Different diagrams can correspond to the same classical knot. It is known that two diagrams D_1 and D_2 represent the same classical knot if and only if they can be connected by a chain of trivial isotopies and *Reidemeister moves* shown in Figure 1.

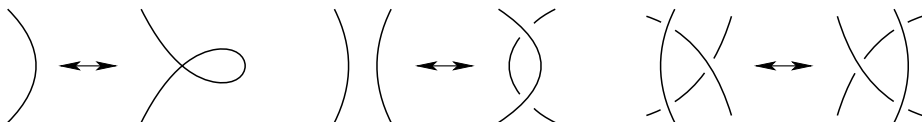


Figure 1. Reidemeister moves.

The following notion of concordance is important in the theory of classical knots.

Definition 3. Let K_1 and K_2 be two classical knots embedded in a four-dimensional space $\mathbb{R}^3 \times [0, 1]$ in such a way that $K_1 \subset \mathbb{R}^3 \times \{0\}$ and $K_2 \subset \mathbb{R}^3 \times \{1\}$. The knots K_1 and K_2 are called *concordant* if there is a smooth proper embedding f of a cylinder $C = S^1 \times [0, 1]$ in $\mathbb{R}^3 \times [0, 1]$ such that $f(S^1 \times \{0\}) = K_1$, $f(S^1 \times \{1\}) = K_2$.

Definition 4. A classical knot K is *slice* if it is concordant to a trivial knot.

Next we recall some basic definitions of the theory of virtual and free knots, following [1] and [2].

Definition 5. A *chord diagram* is a 3-valent graph of the following form. It consists of a circle, called the *base circle*, on which an even number of vertices is fixed. Each vertex is connected by an edge to exactly one other vertex. A circle with an empty set of vertices is also a (trivial) chord diagram.

Two chords in a chord diagram are called *linked* if their ends alternate in going along the base circle of the diagram.

Every diagram of a classical knot corresponds to a chord diagram. Namely, consider a knot K and its diagram D . By definition, K is the image of a circle S^1 under an embedding f , and D is the image of K under a projection π onto a generic subspace. For each crossing x of D we connect its two preimages $x_1, x_2 \in (\pi \circ f)^{-1}(x)$ by a chord. In this way S^1 is turned into a chord diagram of K .

Furthermore, each chord of the diagram is *framed*, that is, it is labelled with a sign and an arrow going from the preimage of the overcrossing to the preimage of the undercrossing.

Reidemeister moves can be reformulated in the language of chord diagrams (Figure 2). In this case two chord diagrams constructed from equivalent diagrams of a classical knot can be connected by a sequence of Reidemeister moves.

However, not every framed chord diagram corresponds to a diagram of a classical knot.

Definition 6. A *virtual knot* is an equivalence class of framed chord diagrams under Reidemeister moves.

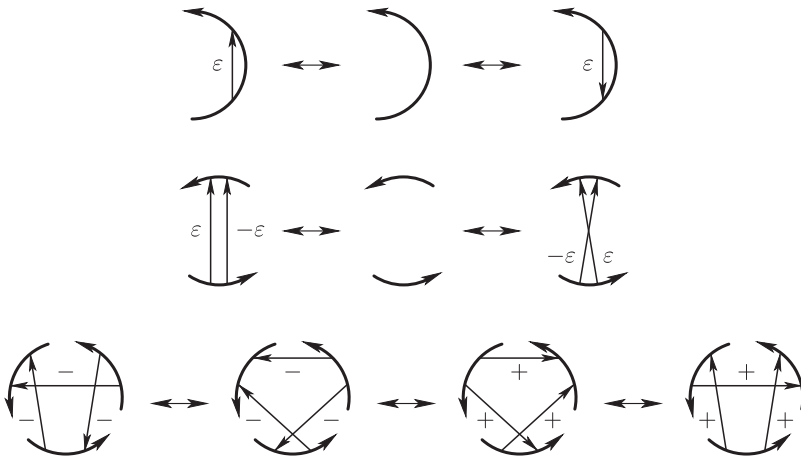


Figure 2. Reidemeister moves of chord diagrams (only the part of the diagram that changes under the move is shown).

Virtual knots can be coarsened further by forgetting the framing of chord diagrams (that is, forgetting the orientation and signs of the chords). The Reidemeister movements for such diagrams are obtained from the moves shown in Figure 2 by forgetting the arrows and signs on the chords.

Definition 7. A *free knot* is an equivalence class of chord diagrams under Reidemeister moves.

There is a one-to-one correspondence between chord diagrams and framed 4-valent graphs (see Definition 1) with one unicursal component. We can therefore introduce the following equivalence relation on framed 4-graphs.

Definition 8. Two framed 4-graphs Γ_1 and Γ_2 are said to be *equivalent* if their chord diagrams can be connected by a sequence of Reidemeister moves.

Virtual knots can be defined by diagrams on two-dimensional surfaces in which it is indicated at every vertex which pair of opposite edges forms an overcrossing; the other pair then forms an undercrossing. Forgetting this information gives a general position curve immersed in an oriented two-dimensional surface. A 4-graph embedded in a two-dimensional surface has a natural rotation structure at each vertex, a cyclic order of the outgoing half-edges. A framed 4-graph is obtained from a graph with rotation structure by forgetting the order of the half-edges and remembering only which half-edges are opposite. So (a, b, c, d) and (a, d, c, b) represent the same framing at a vertex (since the half-edge a is opposite to c in both cases), although the cyclic orders are different (since b follows a in one case, and d follows a in the other).

It follows that framed 4-graphs are obtained from diagrams of virtual knots (or knots in thickened surfaces $S_g \times [0, 1]$) by forgetting two ‘bits’ of information at each vertex, the overcrossing-undercrossing structure and the cyclic order, while keeping the information about opposite half-edges.

Equivalent virtual knots clearly define equivalent framed 4-graphs, since the Reidemeister moves for graphs are obtained from the Reidemeister moves for virtual knots by forgetting the orientation. Therefore, a sharp coarsening occurs when one passes from virtual knots to free knots.

The concordance relation for virtual knots (see [2]) generates naturally the concordance relation for free knots. In particular, a slice virtual knot (a virtual knot concordant to a trivial virtual knot) gives rise to a slice-free knot. More precisely, the concepts of concordance and slice carry over to framed graphs as follows.

Definition 9. A finite two-dimensional complex K is called *standard* if each of its points has a boundary of one of the types shown in Figures 3 and 4.

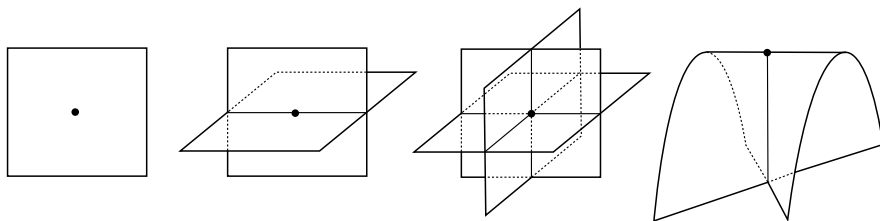


Figure 3. Types of interior points of a standard two-dimensional complex.

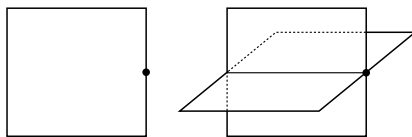


Figure 4. Types of boundary points of a standard two-dimensional complex.

Definition 10. Two framed 4-graphs Γ_1 and Γ_2 are called *concordant* if there exists a standard two-dimensional complex K , called the *slicing complex*, satisfying the following conditions:

- (1) there is a continuous map $f: S^1 \times [0, 1] \rightarrow K$ such that $f(S^1 \times [0, 1]) = K$;
- (2) $f(S^1 \times \{0\}) = \Gamma_1$ and $f(S^1 \times \{1\}) = \Gamma_2$;
- (3) in a neighbourhood of the preimage of each vertex v of the graph Γ_i , $i = 1, 2$, the boundary component of the cylinder is mapped to the union of two opposite half-edges according to the framing of the graph.

A framed 4-graph Γ is *slice* if it is concordant to a trivial graph (given by a circle without vertices).

If Γ is slice, then by gluing the trivial circle with a disc, we get a complex that is the image of a disc, called a *slicing disc* for Γ .

It is easy to see that any two graphs which differ by a Reidemeister move of their chord diagrams are concordant. Therefore, concordance is an equivalence relation for free knots.

It follows that any invariant of free knots and their concordance lifts tautologically to an invariant of virtual knots. The same can be said about invariants of

slice virtual knots (and knots in thickened surfaces). Thus, all invariants of free knots (and their concordance) can be ‘lifted’ to invariants of virtual knots (and their concordance).

The question of the existence of nontrivial free knots had been open for five years; see [3] and [4]. Thanks to the parity theory [1], new subtle invariants of virtual knots have appeared; see [2] and [5]. These invariants take values in the pictures and allow one to describe properties of all diagrams of a given virtual knot by looking at a single diagram.

Concordance of virtual knots is a coarser equivalence relation than their isotopy as circles embedded in a thickened surface $S_g \times [0, 1]$. Likewise, concordance of framed 4-graphs is a coarser equivalence relation than the equivalence generating free knots.

The first nontrivial invariant of concordance for free knots (and an obstruction to being slice) was obtained in [2]. Although this invariant was introduced as an element of a certain group, it is essentially an integer invariant.

The purpose of this paper is to obtain a subtle obstruction to a rough equivalence of rough objects.

It turns out that for a special class of free knots defined below (see Definition 11), a slice can be recognised from its chord diagram by a finite verification procedure (Theorem 1).

The obstruction for a free knot to be slice can be understood as follows. For any framing-preserving embedding of a given framed 4-graph Γ in a surface S_g , and for any three-dimensional manifold M with boundary S_g , there is no smooth map of a disc to M such that the boundary of the disc is mapped onto Γ preserving the orientation.

Nontriviality of virtual knots implies nontriviality of knots and links in different three-dimensional manifolds. In particular, if there is a multicomponent link $L = K' \sqcup K$ in S^3 such that one of its components K' is a fibred link, then the nontriviality of K in the complement of K' implies the non-triviality of L . Nontriviality of knots in the complement of K' can be deduced from nontriviality of virtual (or free) nodes. Similarly, one can raise the question of concordance of classical links of a special type, and obstructions to this concordance can be obtained from obstructions to concordance of free knots.

In a recent work by Chrisman and the first author [6], a programme was initiated to use virtual knots to study classical links whose components are not necessarily fibred knots. Questions of concordance and being slice arise naturally for the corresponding classes of links.

1.2. Statement of the main results. The concept of Gaussian parity is defined as follows for free knots (and therefore for framed 4-valent graphs).

Definition 11. Consider a framed 4-graph Γ and its chord diagram $D(\Gamma)$. Each vertex v of Γ corresponds to a chord $d(v)$. The *Gaussian parity* of the vertex v is the parity of the number of chords linked with $d(v)$.

A graph is called *odd* if all its vertices are odd.

We need one more concept before stating the main result of the paper. Let Γ be a framed 4-graph and let $D(\Gamma)$ be the corresponding chord diagram.

Definition 12. A *pairing* $\mathcal{P}$ of chords in the diagram $D(\Gamma)$ is a partition of all chords $\{d_i\}$ into pairwise disjoint sets $\mathcal{P}_i$ consisting of one or two elements, such that whenever $\mathcal{P}_i$ consists of two chords c_i and d_i , each endpoint of c_i corresponds uniquely to one of the endpoints of d_i .

Define the chord diagram $C(\mathcal{P})$ consisting of chords whose endpoints coincide with the endpoints of the chords of the diagram D , and each chord of $C(\mathcal{P})$ either coincides with a chord of D forming a singleton family, or connects two corresponding endpoints of two different chords from a two-element family.

We see that a pairing $\mathcal{P}$ of chords in a chord diagram D leads to a new diagram $C(\mathcal{P})$. There may be different pairing of chords in the same diagram.

Definition 13. We say that a pairing $\mathcal{P}$ is *intersection-free* if the chords of the diagram $C(\mathcal{P})$ are not pairwise linked.

Example 1. The pairing in Figure 5 has the form

$$\{\{c_1\}, \{c_2, d_2\}, \{c_3, d_3\}\};$$

the figure on the right shows a fold point on the chord c_1 , which is paired with itself.

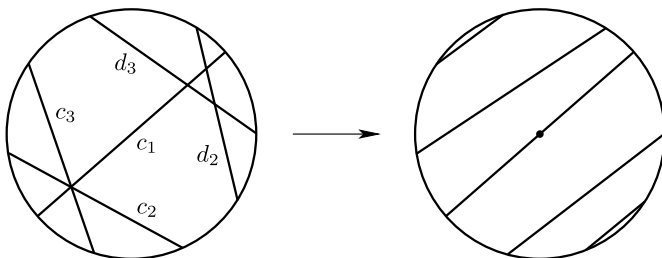


Figure 5. The existence of a suitable pairing implies that a knot is slice.

Here is the main theorem in this paper.

Theorem 1. *If a diagram K of a free knot is odd, then K is slice if and only if its chords admit an intersection-free pairing.*

Example 2. Consider a free knot with chord diagram shown in Figure 6. It has 10 chords: five ‘long’ ones, a_1, a_2, a_3, a_4 and a_5 , and five ‘short’ ones, b_1, b_2, b_3, b_4 and b_5 . We show below that this chord diagram does not admit an intersection-free pairing.

First observe that in an intersection-free pairing, an odd chord must be paired with another odd chord. Indeed, orient the circle of the chord diagram arbitrarily and consider an even chord e with endpoints e_1 and e_2 and an odd chord o with endpoints o_1 and o_2 . We can assume without loss of generality that the arc $a_1 = \widehat{e_1 o_1}$ does not contain the points e_2 and o_2 (the endpoints of the arc are specified in the order defined by the orientation). Similarly, the arc $a_2 = \widehat{e_2 o_2}$ does not contain the points e_1 and o_1 . Then the numbers of ends of the chords on a_1 and a_2 have different parity. Therefore, the pairing of e with o must have intersections.

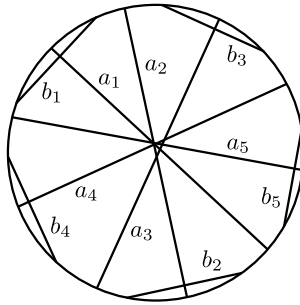


Figure 6. A free knot which is not slice.

Analysing different cases of pairing, one can easily notice that two short chords cannot be paired with each other. It follows that each long chord must be paired with a short chord.

Further, no chord can be paired with itself, since each of the arcs subtended by each of the chords contains an odd number of endpoints of the other chords. Therefore, if a chord λ were paired with itself, then after pairing one could find a chord μ with endpoints on the two different arcs subtended by λ , so the pairing would have intersections.

If b_1 is paired with a_1 , then each of the ‘long’ chords a_2, a_3, a_4 and a_5 must be paired with a long chord, leading to a contradictory pairing of short chords.

Pairing of a_1 with b_3 (or with b_4) makes it impossible to pair b_5 (respectively, b_2) with any other chord. Similarly, pairing a_1 with b_2 (or with b_5) makes it impossible to pair b_4 (respectively, b_3).

Remark 1. The example shown in Figure 5 illustrates the situation when a chord paired with itself (a chord generating a cusp) is required to construct a slicing disc.

It is easy to construct an example of a slicing disc with triple points. Such an example is shown in Figure 7.

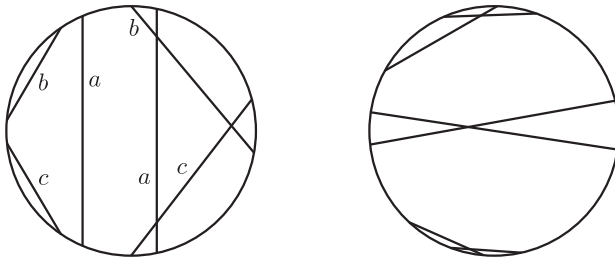


Figure 7. A slicing disc with triple points and cusps.

In the figure on the left, the paired chords are denoted by the same letters a, b and c . The pairing shown in the figure on the right has three intersections, but the corresponding double lines can be identified in such a way that the triple point is at the intersection of three double lines. As a result, we obtain a slicing disc for the original knot.

The knot with the chord diagram shown in the figure on the left is slice, since its chord diagram is obtained by applying the third Reidemeister move to a diagram of the connected sum $K \# \overline{K}$ of two trivial free knots with obvious pairing, and the Reidemeister moves preserve slice knots. Moreover, the third Reidemeister move on a chord diagram creates a triple point on the slicing complex.

The following construction allows one to obtain families of nontrivial slice knots. Let K be a free knot and let $\overline{K}$ be its mirror image. Take a point $x \in K$ and let $f(x) \in \overline{K}$ be its image under the mirror map. Consider the connected sum $K \# \overline{K}$ at the points x and $f(x)$; see Figure 8.

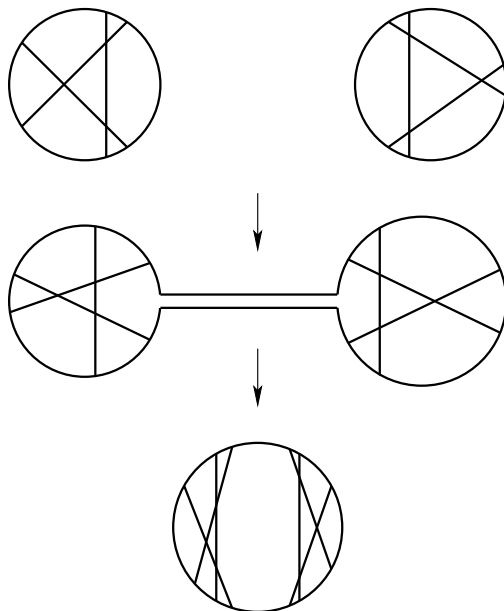


Figure 8. Connected sum of chord diagrams of a knot and its mirror image.

The following statement is trivial.

Statement. *The free knot $K \# \overline{K}$ is slice.*

The structure of this paper is as follows. In §2 we give some facts from the theory of two-dimensional knots required in the proof of the main theorem. In §3 we define the operation of resolution on two-dimensional knots. In §4 we prove the main theorem (Theorem 1) of this paper.

§ 2. Two-dimensional knots

Two-dimensional knots are analogues of one-dimensional knots. A two-dimensional knot is a knotted sphere in four-dimensional space. We recall the basic definitions and facts of this theory, following [7] and [8].

Definition 14. A *two-dimensional knot* (respectively, a *two-dimensional n -component link*) is a smooth embedding of a 2-sphere S^2 (respectively, of a disjoint union of n spheres) in general position in $\mathbb{R}^4$ or S^4 up to isotopy.

As in the case of one-dimensional knots, 2-knots can be represented by diagrams.

Definition 15. A 3-dimensional *diagram* of a 2-knot K is a generic projection of K onto a subspace $\mathbb{R}^3$ of $\mathbb{R}^4$. Here a projection is *generic* if each point in its image is either a regular point, or a transverse double point, or a transverse triple point, or a cusp point (of the Whitney umbrella).

Two diagrams represent the same knot if and only if one diagram can be transformed into the other by a sequence of *Roseman moves*, shown in Figure 9, and trivial isotopies. The Roseman moves are the two-dimensional analogues of the Reidemeister moves.

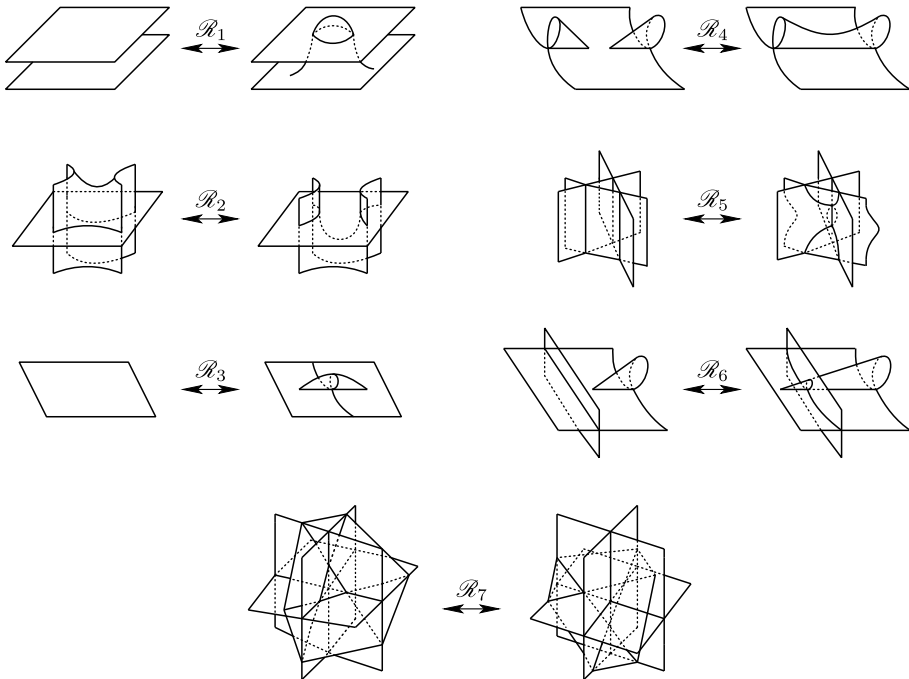


Figure 9. Roseman moves.

Another way to represent 2-knots is spherical diagrams. These diagrams are two-dimensional analogues of chord (Gaussian) diagrams of one-dimensional knots.

Definition 16. A *spherical (Gaussian) diagram* of a 2-knot is a two-dimensional complex consisting of the sphere S^2 and a set D of labelled curves on the sphere with the following properties:

(1) each curve is either closed or ends with two cusps; the number of cusps is finite;

(2) each curve in D is paired with exactly one other curve from the set; one curve in the pair is labelled as the upper one, both curves are oriented (equipped with arrows), and the pairing is pointwise, continuous and consistent with the orientation of the curves;

(3) two curves ending at the same cusp are paired, and both their arrows point either towards the cusp or away from it;

(4) if two curves intersect, then the curves paired with them also intersect (therefore, a triple point appears on the sphere S^2 three times);

(5) all triple points are geometric (see Figure 10 and Remark 2).

Remark 2. We comment on the notion of a geometric triple point.

A triple point appears three times on a spherical diagram according to property (4) in Definition 16. Neighbourhoods of the three preimages of a triple point are shown in Figure 10. Paired curves are denoted by the same letters a, b and c , and the signs indicate the top (sign ‘+’) and bottom (sign ‘−’) curves in the pair. The arrows indicate the orientation of the paired curves.

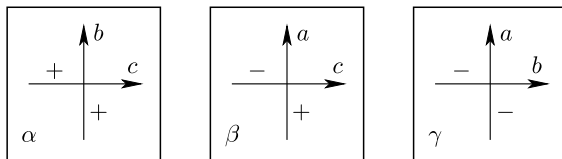


Figure 10. Three local leaves of a spherical diagram in a neighbourhood of a geometric triple point (the ‘+’ sign indicates the top curve in a pair, and ‘−’ indicates the bottom curve).

A triple point is geometric if its preimages have neighbourhoods which look exactly as shown in Figure 10. Only geometric triple points are allowed.

Roseman moves can be easily translated into the language of spherical diagrams.

Definition 17. Consider the set of spherical diagrams and the following operations:

- A. Roseman moves 1–7;
- B. simultaneous change of orientation of two paired curves;
- C. swapping the ‘top’ curve with the ‘bottom’ one in a pair.

A *free two-dimensional knot* is an equivalence class of spherical diagrams under operations A, B and C; see [7].

Along with two-dimensional knots and free two-dimensional knots, we can consider *two-dimensional (free) knots with boundary*.

Definition 18. A *two-dimensional (classical) knot with boundary* is a generic smooth embedding of a 2-disc D^2 in $\mathbb{R}^4$ or S^4 up to isotopy.

Definition 19. A *diagram* of a 2-knot K with boundary is a generic projection of K onto a subspace $\mathbb{R}^3$ in $\mathbb{R}^4$. Here a projection is *generic* if each point in its image is either a regular point, or a transverse double point, or a transverse triple point, or a cusp point (of a Whitney umbrella). In addition, it is required that the

image of the boundary ∂D of the disc does not intersect the image of the interior of the disc.

Equivalence of diagrams of two-dimensional knots with boundary is defined by means of Roseman moves, as in the case of knots without boundary. It is required that Roseman moves do not involve the boundary of the knot.

Definition 20. A *spherical diagram* of a 2-knot with boundary is a two-dimensional complex consisting of the disc D^2 and a set $\mathcal{D}$ of labelled curves on the disc with the following properties:

- (1) each curve is either closed, or ends with two cusps, or ends at the boundary of the disc; the number of cusps is finite;
- (2) each curve in $\mathcal{D}$ is paired with exactly one other curve from the set; one curve in the pair is labelled as the upper one, both curves are oriented (equipped with arrows), and the pairing is pointwise, continuous and consistent with the orientation of the curves;
- (3) two curves ending at the same cusp are paired, and both their arrows point either towards the cusp or away from it;
- (4) if a curve ends at the boundary of the disc, then the curve paired with it also ends at the boundary, and both arrows of the paired curves point either towards the boundary or away from it;
- (5) if two curves intersect, then the curves paired with them also intersect (therefore, a triple point appears three times on D^2);
- (6) all triple points are geometric.

A *free two-dimensional knot with boundary* is an equivalence class of spherical diagrams with boundary under operations A, B and C from Definition 17.

In what follows, we will call a double line of a diagram of a two-dimensional knot with boundary (and the curves corresponding to it on the spherical diagram) *internal* if it is either closed or ends with two cusps.

It follows from Definition 10 that if a framed 4-graph Γ is slice, then its slicing complex corresponds to a spherical diagram with boundary. Conversely, let Γ be a graph. Note that any spherical diagram with boundary defines a pairing of the ends of the curves lying on its boundary. Suppose there is a chord diagram of Γ that can be obtained from a spherical diagram D with boundary by removing all double lines from D and connecting the corresponding points on the boundary by chords. (In this case we say that the boundary of D *corresponds* to the graph Γ .) Then Γ is slice.

This observation allows us to use the technique of two-dimensional knots to study slice framed 4-graphs and will be used in the proof of Theorem 1.

In [2] and [7] the *Gaussian parity* of double lines of (free) two-dimensional knots with or without boundary was defined as follows.

Consider a double line of a 2-knot and the corresponding curves η_1 and η_2 on the spherical diagram. For each point on the double line, we connect the corresponding points $x_1 \in \eta_1$ and $x_2 \in \eta_2$ on the spherical diagram by a curve γ that intersects the curves of the diagram transversely at a finite number of points and does not pass through triple points and cusps. We also impose a compatibility condition near the endpoints of γ . Namely, consider two vectors v_1 and v_2 tangent to γ at its endpoints x_1 and x_2 , respectively. We assume that the bases (η_1, v_1) and (η_2, v_2)

define the opposite orientations of the sphere S ; here η_i denotes the unit tangent vector to η_i .

Now we calculate the number of intersections of γ with the set of curves of the spherical diagram modulo two. The resulting number is called the *parity* of the point on the double line under consideration. It is easy to verify that the parity is well defined along the entire double line, and therefore we have actually defined the parity of the double line. It is also easy to verify that a double line approaching a cusp is even, and at a triple point, either all three double lines are even, or two are odd and one is even.

The definition of Gaussian parity for two-dimensional knots with boundary is the same as in the case of knots without boundary. It is important to note that the parity of a double line ending at the boundary coincides with the Gaussian parity of its endpoint as a vertex of the corresponding chord diagram, because we can take γ to be an arc of the boundary circle of the diagram.

§ 3. Resolution of two-dimensional knots

In one-dimensional knot theory, *resolution* of a crossing means cutting out this crossing together with a small neighbourhood and gluing two pairs of half-edges in one of the two possible ways:

$$\times \rightarrow \rangle \langle \quad \text{or} \quad \times \rightarrow \times$$

Resolution is an important operation used to construct knot invariants. In particular, different polynomial invariants, such as the Kauffman bracket, are constructed using the resolution of crossings of a knot diagram.

In the two-dimensional case, not intersections, but double lines are subject to resolution. Locally, a two-dimensional resolution can be viewed as a one-dimensional resolution on a transverse section multiplied by a straight-line interval. The difficulty lies in defining the resolution of a double line (or a family of double lines) in a consistent way. In this paper, we propose an approach to defining resolutions based on spherical diagrams.

Consider paired curves γ and γ' on a spherical diagram of a knot. The procedure of resolving the double line corresponding to this pair of curves consists of the following steps.

1. Cut the diagram along the curves γ and γ' . The resulting multicomponent complex has four boundary components: $\gamma_1, \gamma'_1, \gamma_2, \gamma'_2$.
2. Glue the components along the boundary components in a way compatible with the orientation and the curves intersecting the boundary using the following rule: a curve with a dash is glued to a curve without a dash; see Figure 11.
3. To complete the process, we need to understand what happens to the triple points on the double line being resolved.

It is easy to see that the resolution of the curve γ induces a natural one-dimensional resolution of the third preimage of the encountered triple point; see Figure 12. The type of this resolution depends on the choice of pairing of the glued curves: γ_1 with γ'_1 , or γ_1 with γ'_2 .

Note that the resolution of a self-intersecting double line is well defined. In this case, new cusps appear; see Figure 13.

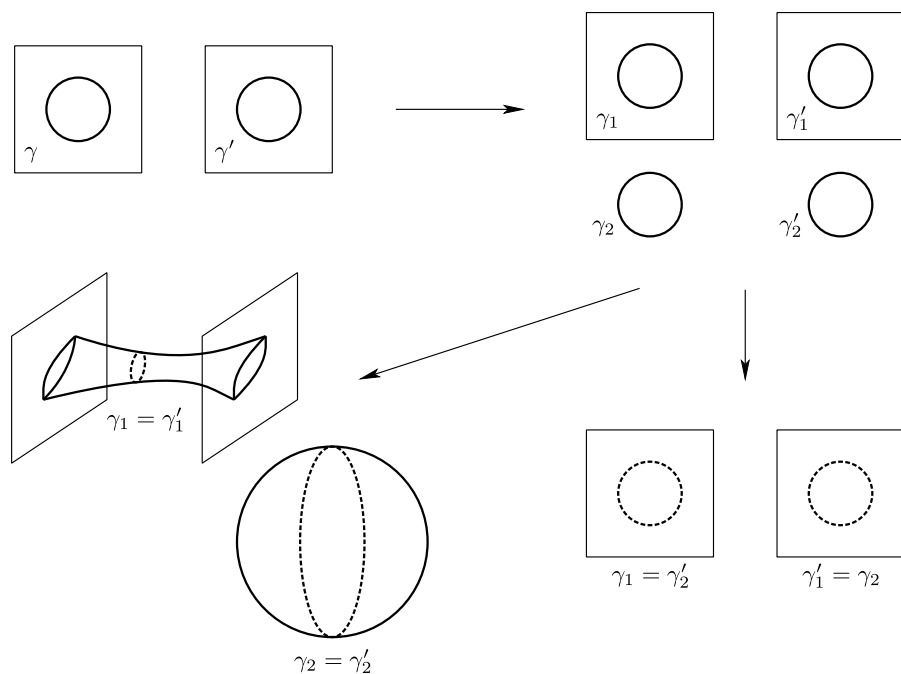


Figure 11. Resolution of a double line: the curve γ is paired with γ' ; the spherical diagram is cut along these curves and then glued together in one of the two possible ways.

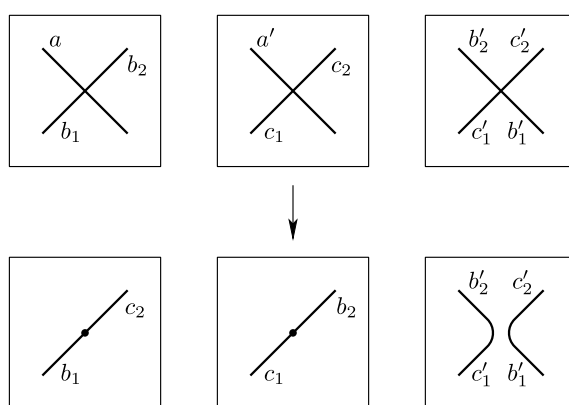


Figure 12. Resolution of a double line a removes the curves a and a' from the diagram and induces a one-dimensional resolution of the crossing of the curves b' and c' .

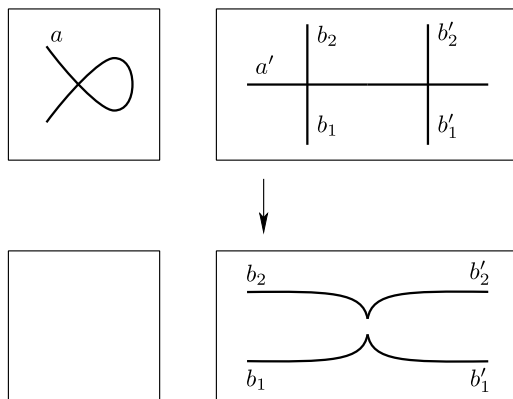


Figure 13. Resolution of a self-intersecting double line a removes the curves a and a' from the diagram and creates cusps.

The procedure described above defines a *resolution* of a double line. Resolution of a set of double lines is performed in succession, in any order. We introduce the notation: if a diagram D_2 is obtained from a diagram D_1 by resolving one double line, then we write $D_1 \rightsquigarrow D_2$.

To prove Theorem 1, we need the following lemma.

Lemma 1. *For any diagram of a free 2-knot without boundary there exists a resolution giving a trivial free knot.*

Proof. A double line can be resolved in two ways. One of them creates an additional connected component (thereby transforming the knot into a link), and the other does not change the number of connected components. This follows from the fact that any closed curve on a sphere bounds a set of discs. Therefore, the cutting procedure can be understood as removing a certain set of open discs from the sphere. Furthermore, one of the two possible ways of gluing the obtained boundaries is to paste these discs back, possibly with a flip. This operation does not create additional connected components.

Let us take a closer look at what happens to the complex after the resolution that preserves the number of connected components.

1. If the resolved double line corresponds to two paired curves with common endpoints, then the resolution consists in cutting out a disc from the sphere and pasting it back with a flip. Hence the surface obtained as a result of resolution is still a sphere.

2. If the resolved double line corresponds to two disjoint closed curves, then the resolution consists in cutting out two discs from the sphere and pasting each of them into the hole left by the other. Hence the resulting surface is still a sphere.

3. If the resolved double line corresponds to two intersecting closed curves, then the resolution consists in cutting out a disc from the sphere and pasting it back with a flip, as in the first case. Hence the resulting surface is still a sphere.

Therefore, the diagram remains spherical after a resolution that preserves the number of connected components. We denote a resolution preserving the number of connected components by $D_1 \rightsquigarrow_0 D_2$.

Given a diagram D of a 2-knot, we refer to the triple (the number of triple points, the number of double lines, the number of cusps) as the *complexity* of D and denote it by $\Psi(D)$. We order such triples lexicographically. Note that a resolution of a double line reduces the complexity, that is, $D_1 \rightsquigarrow_0 D_2$ implies $\Psi(D_2) < \Psi(D_1)$. Indeed, if the resolved double line does not pass through triple points, then the resolution does not change the number of triple points and reduces the number of double lines, and therefore reduces the complexity. If the resolved double line passes through triple points, then the resolution reduces the number of triple points, and therefore the complexity.

Let D be a diagram of a free two-dimensional knot. Consider the chain $D_1 = D \rightsquigarrow_0 D_2 \rightsquigarrow_0 \cdots$, where the double line resolved at each step is chosen arbitrarily. Since the complexity decreases, that is, $\Psi(D) = \Psi(D_1) > \Psi(D_2) > \cdots$, we obtain $\Psi(D_n) = 0$ for some n . A diagram of complexity zero does not have cusps, triple points and double lines, which means that it is a diagram of a trivial knot. The lemma is proved.

The operation of resolution can be defined in the same way for internal double lines of (free) two-dimensional knots with boundary. The following statement is proved in the same way as Lemma 1.

Lemma 2. *For any diagram D of a free two-dimensional knot with boundary, there exists a resolution giving a diagram D' which coincides with D in a small neighbourhood of the boundary and does not have internal double lines.*

These lemmas, which allow one to eliminate interior double lines of free two-dimensional knots using resolution, are central to the proof of the main result of this paper.

§ 4. Proof of the main Theorem 1

Consider a framed 4-graph Γ and its diagram $D(\Gamma)$. Assume that $D(\Gamma)$ is an odd diagram (that is, all its chords are odd). We shall prove that Γ is slice if and only if $D(\Gamma)$ admits an intersection-free pairing.

1. Suppose the diagram $D = D(\Gamma)$ admits an intersection-free pairing $\mathcal{P}$. This means that there exists a chord diagram $D(\mathcal{P})$ whose circle looks the same as the circle of the diagram D and the chords of $D(\mathcal{P})$ are pairwise unlinked (in particular, no chord of D is paired with itself). A chord diagram can be viewed as a disc with a set of curves connecting points on its boundary. Note that the chords of $D(\mathcal{P})$ are naturally split into pairs. Indeed, the pairing $\mathcal{P}$ consists in splitting the chords of D into disjoint two-element subsets $\mathcal{P}_i$. We say that two chords c and d of $D(\mathcal{P})$ form a pair if the set of their ends $\{c_1, c_2, d_1, d_2\}$ coincides with the set of ends $\{\tilde{c}_1, \tilde{c}_2, \tilde{d}_1, \tilde{d}_2\}$ of chords of some subset $\mathcal{P}_i = \{\tilde{c}, \tilde{d}\}$. Without loss of generality, we can assume that $c_1 = \tilde{c}_1$, $c_2 = \tilde{d}_1$, $d_1 = \tilde{c}_2$ and $d_2 = \tilde{d}_2$. Then we orient c and d as follows: $c_1 \rightarrow c_2$, $d_1 \rightarrow d_2$.

Therefore, the chord diagram $D(\mathcal{P})$ defines a spherical diagram of a two-dimensional knot with boundary corresponding to the graph Γ . Thus, Γ is slice.

We also note that all the curves of the obtained spherical diagram of the slicing complex do not intersect and their endpoints lie on the boundary of the disc. Therefore, the resulting complex has no cusps and triple points.

2. We prove the reverse implication. Suppose the graph Γ is slice, that is, there exists a slicing complex K . We shall show that then there exists a slicing complex $\tilde{K}$ without triple points and cusps.

Consider the complex K . By Lemma 2 there exists a complex $\tilde{K}$ coinciding with K in a small neighbourhood of the boundary and containing no internal lines. Since small neighborhoods of the boundaries of K and $\tilde{K}$ are identical, the complex $\tilde{K}$ is also a slicing complex for Γ . We show that it has all the necessary properties.

Indeed, the Gaussian parity of any double line of a complex for which an endpoint lies on the boundary coincides with the Gaussian parity of its endpoint, which is a vertex of Γ . Since Γ is odd by assumption, all double lines of the complex $\tilde{\Gamma}$ are odd. However, the properties of Gaussian parity imply that any double line ending with a cusp is even, and among the three double lines incident to one triple point, either one or three are even. Therefore, the complex $\tilde{\Gamma}$ has no cusps and triple points.

Now we consider a spherical diagram (with boundary) of the complex $\tilde{K}$. It consists of a disc whose boundary coincides with the base circle of the chord diagram of Γ and disjoint curves connecting points on the boundary of the disc. Obviously, the spherical diagram defines an intersection-free pairing of the chord diagram of Γ (see Definition 13). Theorem 1 is proved.

It is important to note that the existence of an intersection-free pairing for a given chord diagram is verified in a finite algorithmic way (by explicit enumeration of the ends of the chords). Thus, Theorem 1 gives a constructive criterion for a free odd knot to be slice.

Finally, note that we have actually proved that any slice framed 4-graph is *elementary* slice, that is, does not have cusps or triple points.

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