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A new algorithm for solving the shallow water equations on the sphere based on the cabaret scheme

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Abstract. A new approach to the construction of computational algorithms for atmospheric and ocean dynamics problems is considered. The approach is based on the integral form of recording the laws of conservation of mass and angular momentum, geodetic grids with quadrangular cells and the cabaret scheme guaranteeing the absence of computational dissipation in flows in which the characteristics of one family are not crossed. The quality of the new algorithm is demonstrated on test and model problems.

1. Introduction

One of the basic problems of large-scale dynamics of the atmosphere and the ocean is described by a system of two-dimensional (single-layer) shallow water equations (SWE) on the sphere [1]. It is one of the main blocks in solving complete baroclinic systems of prognostic equations and is used to assess the accuracy and efficiency of computational algorithms for solving direct and indirect problems. Important questions arising in the construction of methods for solving SWE are the choice of an appropriate form of recording differential equations and the search for effective algorithms for their numerical solution. Until now, they remain the subject of numerous studies of meteorologists and oceanographers. One of the difficulties that arise here is related to the system of coordinates on the sphere, which affects the form of writing equations and, as a consequence, the quality of discrete models.

The work is devoted to a new approach to the construction of computational algorithms for atmospheric and ocean dynamics problems, based on the integral form of recording the laws of conservation of mass and angular momentum, geodesic computational meshes with quadrangular cells and the cabaret scheme providing time reversibility in the case, when the characteristics of one family do not cross.

2. Integral form of the shallow water equation on the sphere

Consider a sphere of radius R with center at the origin of the Cartesian coordinate system. Let $H(\vec{r},t) \ll R$ is the thickness of the liquid layer at the point of the sphere $\vec{r}$, g is the gravitational acceleration. Equations describing the dynamics of such a thin layer are derived from the fundamental conservation laws. On the sphere, such laws are the law of conservation of mass and angular momentum.

The mass balance of an arbitrary region G on the surface of a sphere is described by the integral equation:

$$\iint_G \frac{\partial H}{\partial t} dS + \oint_{\partial G} H\left(\vec{w}\cdot\vec{n}\right) dL = 0, \tag{1}$$

where $\vec{w}$ is the velocity vector, $\vec{n}$ is the unit vector of the outward normal, dS is the element of the area of the sphere, and dL is the element of the length of the boundary on the sphere. The integral equation of momentum balance for the same region has the form:

$$\rho_0 \iint_G \frac{\partial}{\partial t}\left[H\left(\vec{r}\times\vec{w}\right)\right] dS + \rho_0 \oint_{\partial G} H\left(\vec{r}\times\vec{w}\right)\left(\vec{w}\cdot\vec{n}\right) dL + \frac{\rho_0 g}{2}\oint_{\partial G} H^2\left(\vec{r}\times\vec{n}\right) dL = 0; \tag{2}$$

where ρ_0 is the constant density of the liquid.

3. Features of the method of control volume on the sphere

Let us choose four points on the sphere that do not lie on one arc of a large circle. Let us connect those points by disjoint geodesic arcs $L_m, m = 1...4$ in counter-clockwise direction so that they form a spherical quadrangle G_C (a computational cell). For the cell G_C, the law of conservation of mass takes the form:

$$\iint_{G_C} \frac{\partial H}{\partial t} ds + \sum_{m=1}^{4} \int_{m-1/2}^{m+1/2} H\left(\vec{w}\cdot\vec{n}\right) dl = 0 \tag{3}$$

where $\vec{n}$ is an outward unit vector perpendicular to the large circle L_m. Below $\vec{\tau}$ is a unit vector lying in the plane of a large circle and tangent to the geodesic circle $\vec{L}_m$. For each side L_m, we assume $H = H_m = const$, $w^n = w_m^n = \left(\vec{w}\cdot\vec{n}\right)_m = const$, $w^\tau = w_m^\tau = const$, where $\vec{w} = w^n \cdot \vec{n} + w^\tau \cdot \vec{\tau}$, which corresponds to the Riemann sum method of the second order of accuracy when integrating by the cell boundaries. We approximate (3) as follows:

$$\frac{\partial H_C}{\partial t} + \frac{1}{\Delta S}\sum_{m=1}^{4} H_m \left(\vec{w}\cdot\vec{n}\right)_m \Delta L_m = \frac{\partial H_C}{\partial t} + \frac{1}{\Delta S}\sum_{m=1}^{4}\left(H_m w_m^n \Delta L_m\right) = 0 \tag{4}$$

where H_C is the mean value of the depth in the cell G_c, $\left(H_m w_m^n \Delta L_m\right)$, $m = 1,..,4$ are the approximations of the integrals along the sides of the cell, and ΔL_m is the length of the geodetic arc.

The approximation of the law of conservation of angular momentum on the basis of the integral law (2) can be carried out in several stages. We take into account equality

$$\int_{m-1/2}^{m+1/2} \left(\vec{r}\times\vec{n}\right) dL = \left(\vec{r}_m \times \vec{n}_m\right) \Delta l_m, \tag{5}$$

where $\vec{r}_m$ is the coordinate of the middle of the arc L_m, and $\Delta l_m = \left|\vec{r}_{m+1/2} - \vec{r}_{m-1/2}\right|$ is the length of the chord of the sphere. Then the moment of the pressure force (the third term in (2)) is approximated as follows:

$$\frac{g}{2}\oint_{\partial G} H^2\left(\vec{r}\times\vec{n}\right) dL \simeq \frac{g}{2}\sum_{m=1}^{4} H_m^2 \int_{m-1/2}^{m+1/2}\left(\vec{r}\times\vec{n}\right) dL = \frac{g}{2}\sum_{m=1}^{4} H_m^2 \left(\vec{r}_m \times \vec{n}_m\right)\Delta l_m \tag{6}$$

The second term in equation (2) can be approximated in two stages:

$$\oint_{\partial G} H\left(\vec{r}\times\vec{w}\right)\left(\vec{w}\cdot\vec{n}\right)dL=\sum_{m=1}^{4}\left[\int_{m-1/2}^{m+1/2} Hw^{n}\left(\vec{r}\times\vec{n}\right)\left(\vec{w}\cdot\vec{n}\right)dL+\int_{m-1/2}^{m+1/2} Hw^{\tau}\left(\vec{r}\times\vec{\tau}\right)\left(\vec{w}\cdot\vec{n}\right)dL\right]\simeq$$
$$\simeq\sum_{m=1}^{4} H_m\left[w_m^n w_m^n\int_{m-1/2}^{m+1/2}\left(\vec{r}\times\vec{n}\right)dL+w_m^{\tau}w_m^n\int_{m-1/2}^{m+1/2}\left(\vec{r}\times\vec{\tau}\right)dL\right] \tag{7}$$

For the first integral in (7) equation (5) is valid. For the second one:

$$\int_{m-1/2}^{m+1/2}\left(\vec{r}\times\vec{\tau}\right)dL=\left(\vec{r}_m\times\vec{\tau}_m\right)\Delta L_m \tag{8}$$

Finally, after substitution (5) and (8) in (7):

$$\oint_{\partial G} H\left(\vec{r}\times\vec{w}\right)\left(\vec{w}\cdot\vec{n}\right)dL\simeq\sum_{m=1}^{4}H_m w_m^n\left[w_m^n\left(\vec{r}_m\times\vec{n}_m\right)\Delta l_m+w_m^{\tau}\left(\vec{r}_m\times\vec{\tau}_m\right)\Delta L_m\right] \tag{9}$$

The first integral in (2) is replaced by a quadrature:

$$\iint_{G_C}\frac{\partial}{\partial t}\left[H\left(\vec{r}\times\vec{w}\right)\right]dS\simeq\frac{\partial}{\partial t}\left[H\left(\vec{r}\times\vec{w}\right)\right]_C\Delta S_C=\left[\left(\vec{r}\times\frac{\partial H\vec{w}}{\partial t}\right)_C+H_C\left(\frac{\partial\vec{r}}{\partial t}\times\vec{w}\right)_C\right]\Delta S_C\,, \tag{10}$$

where the values marked with a subscript C refer to the center of the cell. If the sphere rotates with angular velocity Ω then $\frac{\partial\vec{r}}{\partial t}=\vec{r}\,\Omega\sin\theta$:

$$H_C\left(\frac{\partial\vec{r}}{\partial t}\times\vec{w}\right)_C=H_C\Omega\sin\theta_C\left(\vec{r}\times\vec{w}\right)_C \tag{11}$$

Finally, the approximation of equation (2) has the following form:

$$\left(\vec{r}\times\frac{\partial H\vec{w}}{\partial t}\right)_C S_C+\sum_{m=1}^{4}H_m w_m^n\left[w_m^n\left(\vec{r}_m\times\vec{n}_m\right)\Delta l_m+w_m^{\tau}\left(\vec{r}_m\times\vec{\tau}_m\right)\Delta L_m\right]+$$
$$+\frac{g}{2}\sum_{m=1}^{4}H_m^2\left(\vec{r}_m\times\vec{n}_m\right)\Delta l_m=-H_C\Omega\sin\theta_C\left(\vec{r}\times\vec{w}\right)_C \tag{12}$$

Formulas (4), (12) form a closed system of difference-differential equations that approximates the laws of conservation of mass and angular momentum for single-layer shallow water on a smooth rotating sphere. Here we must take into account that the angular momentum in (4), (10) is described by only two independent components.

4. Cabaret scheme

The cabaret scheme is based on quadrangular geodetic calculating grids. The initial data for the velocities and the height of the free surface are given both in the centers of the calculated cells and in the centers of their faces. Thus, the number of variables in the cabaret scheme is three times greater than in the classical methods. Variables related to cell centers are called conservative and represent the mean values of the corresponding cell values. The variables given in the midpoints of the faces are called flux ones and define convective fluxes. Calculated grids with the described distribution of variables have not been previously encountered in the problems of atmospheric and ocean dynamics and are not included in the well-known classification of Arakawa [2]. We call them G-grids.

The computational procedure in the cabaret scheme includes three phases. In the first phase, the values of the conservative variables on the intermediate time layer are found from the explicit difference schemes obtained above by the control volume method (4) and (12):

$$\frac{H_C^{t+1/2}-H_C^t}{\tau/2}+\frac{1}{\Delta S}\left\{\sum_{m=1}^{4}H_m\,w_m^n\,\Delta l_m\right\}^t=0$$

$$\vec{r}_C \times \frac{[H\,\vec{w}]_C^{t+1/2} - [H\,\vec{w}]_C^{t}}{\tau/2}\Delta S_C + \left\{\sum_{m=1}^{4} H_m w_m^n \left[w_m^n (\vec{r}_m \times \vec{n}_m)\Delta l_m + w_m^\tau (\vec{r}_m \times \vec{\tau}_m)\Delta L_m \right]\right\}^t +$$
$$+\frac{g}{2}\left\{\sum_{m=1}^{4} H_m^2 (\vec{r}_m \times \vec{n}_m)\Delta l_m\right\}^t = -H_C^{t+1/2}\sin\theta_C\,\Omega\,(\vec{r}\times\vec{w})_C^{t+1/2}\,\Delta S_C$$

In the second phase flux variables are calculated for the next time layer $t+1$. This part of the algorithm is the most specific, and distinguishes the cabaret scheme from all other schemes. Its detailed description does not fit into the format of this publication. We only note here that the second phase involves the procedure of linear extrapolation of the local Riemann invariants and the procedure of nonlinear correction of their new values on the basis of the maximum principle. This procedure is described in detail in the monograph [3].

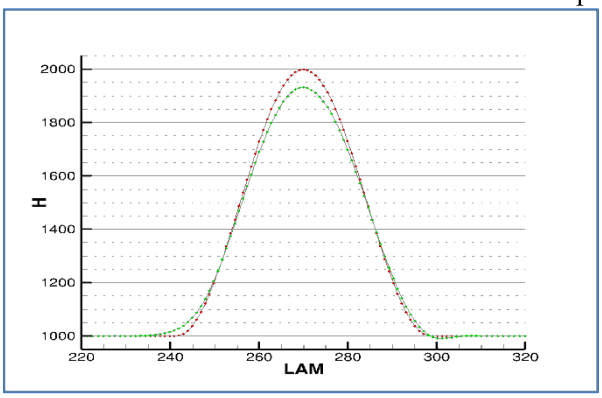
In the third and last phase, new values of conservative variables are determined from the new flux variables found in the second phase:

$$\frac{H_C^{t+1} - H_C^{t+1/2}}{\tau/2} + \frac{1}{\Delta S}\left\{\sum_{m=1}^{4} H_m\, w_m^n\, \Delta l_m\right\}^{t+1} = 0$$

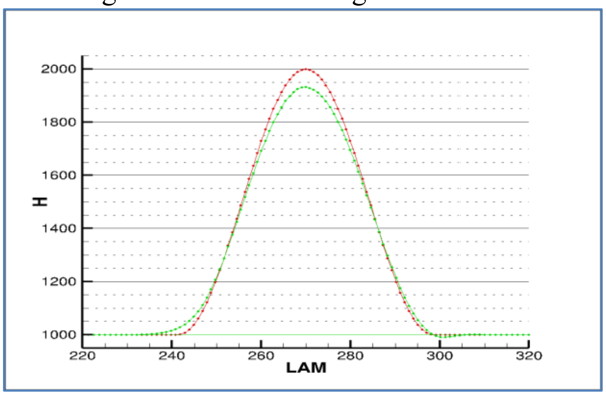
$$\vec{r}_C \times \frac{[H\,\vec{w}]_C^{t+1} - [H\,\vec{w}]_C^{t+1/2}}{\tau/2}\Delta S_C + \left\{\sum_{m=1}^{4} H_m w_m^n \left[w_m^n (\vec{r}_m \times \vec{n}_m)\Delta l_m + w_m^\tau (\vec{r}_m \times \vec{\tau}_m)\Delta L_m \right]\right\}^{t+1} +$$
$$+\frac{g}{2}\left\{\sum_{m=1}^{4} H_m^2 (\vec{r}_m \times \vec{n}_m)\Delta l_m\right\}^{t+1} = -\sin\theta_C\,\Omega\left[H_C^t(\vec{r}\times\vec{w})_C^t + H_C^{t+1}(\vec{r}\times\vec{w})_C^{t+1} - H_C^{t+1/2}(\vec{r}\times\vec{w})_C^{t+1/2}\right]\Delta S$$

5. Examples of test calculations

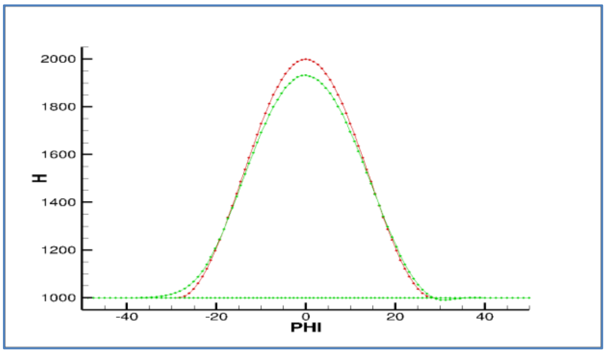
Consider the first task from the test set [4]. On a full sphere of radius of 6.37122×10^6 meters only the equation of continuity is solved, and the velocity field is fixed in time. The initial perturbation of depth is at the equator, and the velocity field is chosen so that transported profile is not distorted in the analytical case. After one circle around the sphere, which occurs over 12 days, the form of the perturbation is compared with the initial one. Several angles of inclination of the flow to the equator are considered. The calculation was carried out on a quad sphere grid C90 with a number of cells 90x90x6. Below the sections the initial and final profiles are given for different angles:



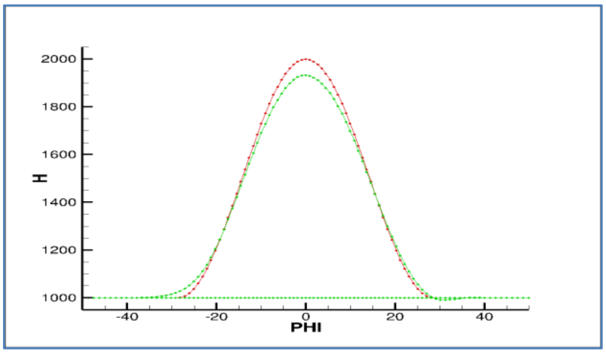
$\alpha = 0$



$\alpha = 0.05$



$\alpha = \pi / 2 - 0.05$



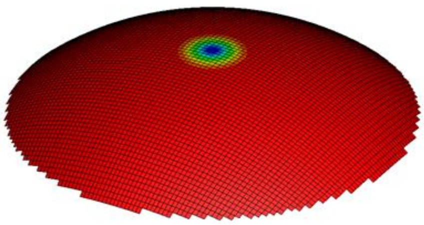
$\alpha = \pi / 2$

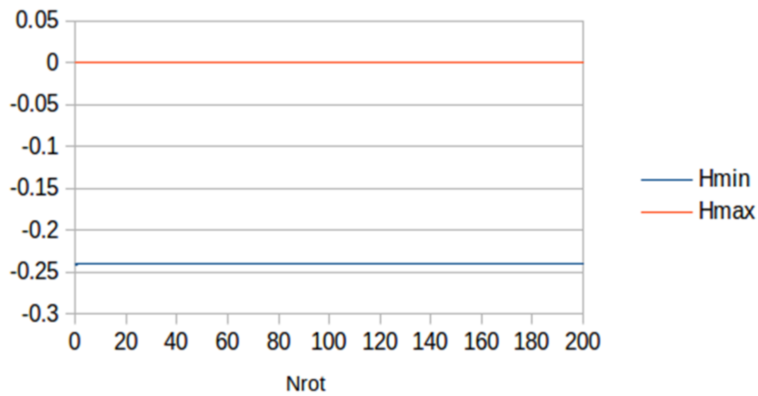
The loss in the amplitude of the disturbance is 7-8 percent for all directions of the flow.

The second problem uses the complete system of equations. On a sector of a non-rotating sphere, a circular flow is defined by formulas:

$$H = H_0 - \frac{\Theta^2}{4g\beta}, u = \Theta\frac{\phi - \phi_0}{r_0}, v = -\Theta\frac{\lambda - \lambda_0}{r_0}, \Theta = \alpha \exp\left[\beta\left(1 - \frac{r^2}{r_0^2}\right)\right],$$

where (λ_0, ϕ_0) is the center of the vortex, r_0 is the radius of the vortex, r is the distance to the center, and $\alpha = 40$, $\beta = 0{,}3$ are the parameters of the vortex. Such a flow is stationary. The flow is calculated up to 200 circles of the vortex. The figure shows the grid (part of the quad sphere grid) and the initial distribution of the parameter H on it, as well as the graphic of the dependence $H_{\min}$ and $H_{\max}$ (in the figure, this is the deviation from the background value H_0) from the calculated time given in the term of the number of circles of the rotation.





It can be seen that the intensity of the vortex does not change.

6. Conclusion

A system of differential-difference equations approximating on the sphere a system of integral laws of conservation of mass and angular momentum for single-layer shallow water is obtained by a finite volume method on the G-grid. The approach used here is of independent interest, since it can be used to construct various computational algorithms, including triangular grids.

To describe vector variables, it is not necessary to introduce a global parametrization on the whole sphere, which solves the problem of singular points at the poles. The velocities are characterized by two components; the appearance of a component normal to the sphere surface is excluded. Balance equations are written through flows defined on the faces. The obtained DDEs are conservative and were used as a basis for implementing the cabaret scheme on the sphere.

The cabaret scheme on the sphere retains all its distinctive features: it is defined on the minimal possible computational template, is explicit, nondissipative, and conditionally stable for $CFL \leq 0.5$. The nondissipation of the scheme leads to the fact that the acoustic perturbations do not decay and stationary vortices do not dissipate when the radius is greater than three cells.

The results of test and model calculations illustrating the properties of the scheme on the sphere are presented.

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