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The Second Initial-Boundary Value Problem for a B-hyperbolic Equation

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Abstract—For a hyperbolic equation with a Bessel operator in a rectangular domain, we study the initial-boundary value problem in dependence of the numeric parameter that enters in the operator. We represent the solution as the Fourier–Bessel series. Using the method of integral identities, we prove the uniqueness of the problem solution. For proving the existence of the solution, we use estimates of coefficients of the series and the system of eigenfunctions; we establish them on the base of asymptotic formulas for the Bessel function and its zeros. We state sufficient conditions with respect to the initial conditions that guarantee the convergence of the constructed series in the class of regular solutions. We prove the theorem on the stability of the solution to the stated problem.

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Key words: *hyperbolic equation, Bessel operator, initial-boundary value problem, uniqueness, existence, Fourier–Bessel series, uniform convergence, stability.*

INTRODUCTION

One of the most important branches of the modern theory of partial differential equations is the theory of boundary value problems for degenerate equations. This fact is due to multiple applications of this theory in various domains of science and technology. The interest towards degenerate equations is connected not only with the necessity of solving applied problems, but also with the intensive development of the theory of mixed-type equations. The first boundary problem for degenerate partial differential equations with variable coefficients was first studied in the paper [1]. A special place in this theory belongs to the study of equations that contain the Bessel differential operator. The study of this class of equations was commenced by Euler, Poisson, and Darboux and then developed in the theory of generalized axis-symmetric potential [2–7]. The importance of this class of equations is due to their use in applications to gas dynamics and acoustics problems [4–6], to the theory of jets in hydrodynamics [8], to linearized Maxwell - Einstein equations [9, 10], and to the elasticity and plasticity theory [11]. Equations of three main classes that contain the Bessel operator, according to the terminology proposed in [12], are called B-elliptic, $B-$hyperbolic, and B-parabolic. See the paper [13] for an extensive study of B-hyperbolic equations, and see [14] for a rather complete survey of papers devoted to the study of boundary value problems for elliptic equations with singular coefficients.

In a rectangular domain $D = \{(x,t) \mid 0 < x < l,\ 0 < t < T\}$ in the coordinate plane Oxt, where $l, T > 0$ are given real values, consider the following B-hyperbolic equation:

$$\square_B u(x,t) \equiv u_{tt} - x^{-k}\frac{\partial}{\partial x}\left(x^k u_x\right) = 0, \tag{1}$$

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where $x^{-k}\frac{\partial}{\partial x}\left(x^k u_x\right)$ is the Bessel operator, while $k \neq 0$ is a given real value.

Eq. (1) with $k = 1$ occurs, for example, when studying free oscillations of a heavy homogeneous hanging thread under the action of gravity, when studying radial gas vibrations in a fixed unbounded cylindrical tube, and (with $k = 2$) when studying small gas vibrations around its equilibrium position inside an impermeable spherical shell ([15], pp. 176, 185, 191). The author of the paper [16] was first to thoroughly study Cauchy and Cauchy–Goursat problems for Eq. (1) with all $k \geq 1$ in the characteristic triangle, but authors of [17] prove that with $k < 0$ these problems are ill-posed. The paper [18] is dedicated to studying the Tricomi problem for a mixed-type equation, whose hyperbolic part coincides with Eq. (1). In papers [19] and [20] one considers Dirichlet and Keldysh problems for a mixed-type equation with the Bessel operator in a rectangular domain, and in [21] and [22] one studies the well-posedness of the initial-boundary value problems with an integral condition for Eq. (1).

Since one can reduce the initial-boundary value problem with nonlocal integral condition of the Samarskii–Ionkin type to mixed problems for Eq. (1), it is necessary to thoroughly study initial-boundary value problems for Eq. (1). The author of this paper is not aware of such research. Therefore in this paper we study the second initial-boundary value problem for the B-hyperbolic equation (1) in a rectangular domain D with all $k \neq 0$.

The problem. *Find a function $u(x,t)$ which satisfies the following conditions*:

$$u(x,t) \in C(\overline{D}) \cap C^2(D), \quad x^k u_x(x,t) \in C(\overline{D}), \tag{2}$$

$$\Box_B u(x,t) \equiv 0, \quad (x,t) \in D, \tag{3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \leq x \leq l, \tag{4}$$

$$u_x(l,t) = 0, \quad 0 \leq t \leq T, \tag{5}$$

$$\lim_{x \to 0+} x^k u_x(x,t) = 0, \quad 0 \leq t \leq T, \quad |k| < 1, \tag{6}$$

$$u(0,t) = 0, \quad 0 \leq t \leq T, \quad k \leq -1, \tag{7}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions such that $\varphi'(l) = \psi'(l) = 0$.

Problem (2)–(5) with $k \geq 1$ is a problem with incomplete boundary data. In the case of the B-elliptic equation

$$u_{tt} + x^{-k}\frac{\partial}{\partial x}\left(x^k u_x\right) = 0$$

with $k \geq 1$ in view of results obtained in papers [1, 23] in the class of bounded solutions, the segment $x = 0$ of the domain boundary is free of the boundary condition. Moreover, in papers ([23], [24], p. 68) one proves that the derivative in the normal direction, i.e., u_x, vanishes on the segment $x = 0$. An analogous case also takes place for Eq. (1). Separating variables, one can easily prove that the following equality is valid with $k \geq 1$:

$$u_x(0,t) = 0, \quad 0 \leq t \leq T. \tag{8}$$

Thus we have proved an additional property of the solution to problem (2)–(5) with $k \geq 1$. In what follows, we either use equality (8) in the proofs or not, depending of the behavior of the derivative u_x with $x \to 0$. If this derivative remains bounded as $x \to 0$, then condition (8) becomes extra.

1. THE UNIQUENESS OF THE PROBLEM SOLUTION

Theorem 1. *If problem (2)–(7) has a solution, then it is unique.*

Proof. Let u_1 and u_2 be two solutions to problem (2)–(7). Then their difference $v = u_1 - u_2$ satisfies conditions (2), (3), (5), and (6) and homogeneous initial conditions

$$v(x,0) = 0, \quad v_t(x,0) = 0, \quad 0 \le x \le l. \tag{4_0}$$

In the domain D consider the identity

$$x^k v_t \Box_B v(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left[x^k\left(v_t^2 + v_x^2\right)\right] - \frac{\partial}{\partial x}\left(x^k v_t v_x\right) \equiv 0.$$

Integrating this identity over the domain $D_{\tau,\varepsilon} = D \cap \{0 < \varepsilon < x < l,\ 0 < t < \tau \le T\}$, where ε and τ are arbitrary values in the mentioned intervals, we get the correlation

$$\int_{\partial D_{\tau,\varepsilon}} x^k\left(v_x^2 + v_t^2\right) dx + 2x^k v_x v_t dt = I_1 + I_2 + I_3 + I_4 = 0. \tag{9}$$

Let us calculate the following integrals separately:

$$I_1 = \int_0^l x^k\left(v_x^2(x,0) + v_t^2(x,0)\right) dx = 0, \quad I_2 = 2\int_0^\tau x^k v_x v_t\Big|_{x=l} dt = 0,$$

$$I_3 = -\int_0^l x^k\left(v_x^2 + v_t^2\right)\Big|_{t=\tau} dx, \quad I_4 = -\int_0^\tau 2x^k v_x v_t\Big|_{x=\varepsilon} dt.$$

Note that $\lim\limits_{\varepsilon \to 0+} I_4 = 0$, because the product $x^k v_x$ tends to zero as $x \to 0$, when $k > -1$, and this product is bounded with $x \to 0$, when $k \le -1$. Then formula (9) with $\varepsilon \to 0$ gives the correlation

$$\int_0^l x^k\left(v_x^2 + v_t^2\right)\Big|_{t=\tau} dx = 0.$$

This means that $v_x \equiv 0$ and $v_t \equiv 0$ on the segment $t = \tau$, and due to the arbitrariness of $\tau \in (0, T]$ we conclude that $v(x,t) \equiv \text{const}$ in $\overline{D}$. Then in view of zero initial conditions (4_0) we get the identity $v(x,t) \equiv 0$. Therefore, $u_1 \equiv u_2$. $\square$

2. THE EXISTENCE AND STABILITY OF THE PROBLEM SOLUTION WITH $K \ge 1$

We seek for partial solutions to Eq. (1), which differ from zero in the domain D and satisfy conditions (2) and (5), in the form $u(x,t) = X(x)T(t)$. Substituting this function in Eq. (1) and condition (5) and separating variables, we get the following spectral problem with respect to the function $X(x)$:

$$X''(x) + \frac{k}{x}X'(x) + \lambda^2 X(x) = 0, \quad 0 < x < l, \tag{10}$$

$$|X(0)| < +\infty, \quad X'(l) = 0; \tag{11}$$

here λ^2 is the separation constant.

Multiplying Eq. (10) by x^2 and changing variables as follows:

$$X(x) = x^{\frac{1-k}{2}} Z(\xi), \quad \xi = \lambda x, \tag{12}$$

we reduce Eq. (10) to the Bessel equation

$$\xi^2 \frac{d^2 Z}{d\xi^2} + \xi\frac{dZ}{d\xi} + (\xi^2 - \nu^2)Z = 0, \quad \nu = (k-1)/2,$$

whose general solution takes the form

$$Z(\xi) = P_1 J_\nu(\xi) + P_2 Y_\nu(\xi); \tag{13}$$

here $J_\nu(\xi)$ and $Y_\nu(\xi)$ are Bessel functions of the first and second kinds, respectively, of the order $\nu = (k-1)/2$, while P_1 and P_2 are arbitrary constant values.

In view of formulas (12) and (13) the general solution to Eq. (10) with $k \geq 1$ obeys the formula

$$\widetilde{X}(x) = P_1 x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x) + P_2 x^{\frac{1-k}{2}} Y_{\frac{k-1}{2}}(\lambda x). \tag{14}$$

In order to make function (14) satisfy the first condition in (11), we put $P_2 = 0$ and $P_1 = 1$, (since eigenfunctions are defined accurate to a constant factor). Then the solution takes the form

$$\widetilde{X}(x) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x). \tag{15}$$

Note that function (15) satisfies condition (8). Then by substituting function (15) in the second condition in (11), we get the correlation

$$\begin{gathered} \lambda_0 = 0, \\ \widetilde{X}'(l) = \left(x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x) \right)'_x \Big|_{x=l} = -l^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(\lambda l), \end{gathered} \tag{16}$$

whence it follows that

$$J_{\frac{k+1}{2}}(\mu) = 0, \quad \mu = \lambda l. \tag{17}$$

In the theory of Bessel functions, it is known ([25], p. 530) that the function $J_\nu(\xi)$ with $\nu > -1$ has a countable set of real zeros. Then by denoting the nth root of Eq. (17) by μ_n with given k we find eigenvalues $\lambda_n = \mu_n/l$ of problem (10), (11).

According to ([26], p. 317), zeros of Eq. (17) with large n obey the asymptotic formula

$$\mu_n = \lambda_n l = \pi n + \frac{\pi}{4} k + O\left(\frac{1}{n}\right). \tag{18}$$

Note that with $\lambda_0 = 0$ the spectral problem (10), (11) has an eigenfunction, which identically equals a constant value; we assume that it equals one. Therefore, the system of eigenfunctions of problem (10), (11) takes the form

$$\widetilde{X}_0(x) = 1, \quad \lambda_0 = 0, \tag{19}$$

$$\widetilde{X}_n(x) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n x}{l}\right) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x), \quad n \in \mathbb{N}; \tag{20}$$

here eigenvalues λ_n are zeros of Eq. (17).

Note that the system of eigenfunctions (19) and (20) of problem (10), (11) is orthogonal in the space $L_2[0,l]$ with the weight x^k and, moreover, it forms a complete system in this space ([27], p. 343).

In further calculations we use the following orthonormalized system of functions:

$$X_n(x) = \frac{1}{\|\widetilde{X}_n(x)\|} \widetilde{X}_n(x), \quad n = 0, 1, 2, \ldots, \tag{21}$$

where

$$\|\widetilde{X}_n(x)\|^2 = \int_0^l \rho(x)\, \widetilde{X}_n^2(x)\, dx, \quad \rho(x) = x^k. \tag{22}$$

According to [28], consider functions

$$u_n(t) = \int_0^l u(x,t) x^k X_n(x)\, dx, \quad n = 0, 1, 2, \ldots, \tag{23}$$

where $X_n(x)$ obey formula (21).

Basing on (23), we introduce auxiliary functions

$$u_{n,\varepsilon}(t) = \int_{\varepsilon}^{l-\varepsilon} u(x,t)x^k X_n(x)\,dx, \quad n = 1, 2, \dots, \tag{24}$$

where $\varepsilon > 0$ is a sufficiently small value.

Twice differentiating equality (24) in the variable t with $0 < t < T$, taking into account Eq. (1), we get the correlation

$$u''_{n,\varepsilon}(t) = \int_{\varepsilon}^{l-\varepsilon} u_{tt}(x,t)x^k X_n(x)\,dx = \int_{\varepsilon}^{l-\varepsilon} \left(u_{xx} + \frac{k}{x}u_x\right) x^k X_n(x)\,dx =$$

$$= \int_{\varepsilon}^{l-\varepsilon} \frac{\partial}{\partial x}(x^k u_x) X_n(x)\,dx = x^k u_x X_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \int_{\varepsilon}^{l-\varepsilon} x^k u_x X'_n(x)\,dx. \tag{25}$$

Formula (24) in view of Eq. (10) gives the correlation

$$u_{n,\varepsilon}(t) = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,t)x^k \left[X''_n(x) + \frac{k}{x}X'_n(x)\right] dx = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,t)\frac{d}{dx}\left(x^k X'_n(x)\right) dx =$$

$$= -\frac{1}{\lambda_n^2}\left[u(x,t)x^k X'_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \int\limits_{\varepsilon}^{l-\varepsilon} x^k u_x X'_n(x)\,dx\right],$$

whence we deduce that

$$\int_{\varepsilon}^{l-\varepsilon} x^k u_x X'_n(x)\,dx = \lambda_n^2 u_{n,\varepsilon}(t) + u(x,t)x^k X'_n(x)\Big|_{\varepsilon}^{l-\varepsilon}.$$

Substituting the latter equality in (25), we get the formula

$$u''_{n,\varepsilon}(t) = x^k u_x X_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(t) - u(x,t)x^k X'_n(x)\Big|_{\varepsilon}^{l-\varepsilon}. \tag{26}$$

In view of correlation (2) functions $u(x,t)$ and $x^k u_x(x,t)$ are continuous in $\overline{D}$. Then by proceeding to the limit in formula (26) as $\varepsilon \to 0$, taking into account boundary conditions (5) and (11), we get the following ordinary differential equation with respect to functions $u_n(t)$:

$$u''_n(t) + \lambda_n^2 u_n(t) = 0, \quad t \in (0, T).$$

Its general solution takes the form

$$u_n(t) = a_n \cos\lambda_n t + b_n \sin\lambda_n t, \tag{27}$$

where a_n and b_n are arbitrary constant values, which have to be defined. To this end, we subject functions (23) to initial conditions (4); we get correlations

$$u_n(0) = \int_0^l \varphi(x)x^k X_n(x)\,dx = \varphi_n, \quad u'_n(0) = \int_0^l \psi(x)x^k X_n(x)\,dx = \psi_n. \tag{28}$$

Formulas (27) and (28) give correlations $a_n = \varphi_n, b_n = \psi_n/\lambda_n$. Substituting the calculated values in (27), we find the final representation of functions

$$u_n(t) = \varphi_n \cos\lambda_n t + \frac{\psi_n}{\lambda_n}\sin\lambda_n t. \tag{29}$$

Analogously we find

$$u_0(t) = \varphi_0 + \psi_0 t, \tag{30}$$

$$u_0(0) = l^{-\frac{k+1}{2}}\sqrt{k+1}\int_0^l \varphi(x)x^k dx = \varphi_0, \quad u_0'(0) = l^{-\frac{k+1}{2}}\sqrt{k+1}\int_0^l \psi(x)x^k dx = \psi_0. \tag{31}$$

Let $\varphi(x) = \psi(x) \equiv 0$, then formulas (28) and (31) give the correlation $\varphi_n = \psi_n \equiv 0$, while formulas (29) and (30) imply that $u_n(t) = 0$ with all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then formula (23) with any $t \in [0,T]$ gives the equality $\int_0^l u(x,t)x^k X_n(x)\,dx = 0$. Hence in view of the completeness of system (21) in the space $L_2[0,l]$ with the weight x^k we conclude that $u(x,t) = 0$ almost everywhere on the segment $[0,l]$ with any $t \in [0,T]$. Since according to formula (2), the function $u(x,t) \in C(\overline{D})$, we conclude that $u(x,t) \equiv 0$ in $\overline{D}$. Thus, we have proved the uniqueness of the solution to problem (2)–(5), taking into account the completeness of the system of eigenfunctions of the scalar spectral problem.

On the base of obtained partial solutions we find a solution to problem (2)–(5) as the Fourier–Bessel series, namely,

$$u(x,t) = u_0(t)X_0(x) + \sum_{n=1}^{\infty} u_n(t)X_n(x); \tag{32}$$

here functions $u_n(t)$ obey formula (29), $X_n(x)$, $n = 0,1,2,\ldots$, do formula (21), and the function $u_0(t)$ does formula (30).

Together with series (32), let us consider the following series:

$$u_t(x,t) = \psi_0 X_0(x) + \sum_{n=1}^{\infty} u_n'(t)X_n(x), \quad u_x(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n'(x); \tag{33}$$

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} u_n''(t)X_n(x), \quad u_{xx}(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n''(x). \tag{34}$$

Let us prove the uniform convergence of series (32)–(34) in the domain $\overline{D}$, provided that functions $\varphi(x)$ and $\psi(x)$ satisfy certain additional conditions.

Formulas (29) and (18) imply the following lemma.

Lemma 1. *For sufficiently large n and any $t \in [0,T]$, bounds*

$$|u_n(t)| \le C_1\left(|\varphi_n| + |\psi_n|/n\right), \tag{35}$$

$$|u_n'(t)| \le C_2\left(n|\varphi_n| + |\psi_n|\right),$$

$$|u_n''(t)| \le C_3\left(n^2|\varphi_n| + n|\psi_n|\right)$$

are valid; hereinafter C_i are positive constant values.

Lemma 2 ([22]). *For sufficiently large n and with all $x \in [0,l]$ the following bounds are valid:*

$$|X_n(x)| \le C_4, \quad |X_n'(x)| \le C_5 n, \quad |X_n''(x)| \le C_6 n^2.$$

According to lemmas 1 and 2, with any $(x,t) \in \overline{D}$ series (32) is majorized by the series

$$C_7 \sum_{n=1}^{\infty}\left(|\varphi_n| + |\psi_n|/n\right), \tag{36}$$

while series (33) and (34) are majorized, respectively, by series

$$C_8 \sum_{n=1}^{\infty}\left(n|\varphi_n| + |\psi_n|\right), \tag{37}$$

$$C_9 \sum_{n=1}^{\infty} \left(n^2 |\varphi_n| + n |\psi_n| \right). \tag{38}$$

Let us study the convergence of series (36)–(38).

Lemma 3 ([22]). *If a function $\varphi(x) \in C^2[0, l]$ and the derivative $\varphi'''(x)$ exists and has a finite variation on $[0, l]$, a function $\psi(x) \in C^1[0, l]$ and the derivative $\psi''(x)$ exists and has a finite variation on $[0, l]$, and*

$$\varphi'(0) = \varphi''(0) = \psi'(0) = \varphi'(l) = \psi'(l) = 0,$$

then

$$|\varphi_n| \le C_{10} n^{-4}, \quad |\psi_n| \le C_{11} n^{-3}. \tag{39}$$

According to Lemma 3, series (36)–(38) are majorized by the numeric series

$$C_{14} \sum_{n=1}^{\infty} n^{-2}, \tag{40}$$

consequently, series (32)–(34) converge uniformly in the closed domain $\overline{D}$.

Therefore, the function $u(x, t)$ defined by series (32) satisfies all conditions of problem (2)–(5). Thus, the next theorem is proved.

Theorem 2. *If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions of Lemma 3, then problem (2)–(5) has a unique solution $u(x, t)$; it is defined by series (32) and satisfies the inclusion $u(x, t) \in C^2(\overline{D})$.*

Theorem 3. *The solution to problem (2)–(5) satisfies the bound*

$$\|u(x, t)\| \le C_{15}(\|\varphi(x)\| + \|\psi(x)\|), \tag{41}$$

where $\|f(x)\|^2 = \int_0^l \rho(x)|f(x)|^2 dx$, $\rho(x) = x^k$.

Proof. In view of formula (35), correlation (32) implies that

$$\|u\|^2 = \int_0^l x^k u^2(x, t)\, dx = \int_0^l x^k \sum_{n=0}^{\infty} u_n(t) X_n(x) \sum_{m=0}^{\infty} u_m(t) X_m(x)\, dx =$$

$$= \sum_{n=0}^{\infty} u_n^2(t) = u_0^2(t) + \sum_{n=1}^{\infty} u_n^2(t) \le (\varphi_0 + \psi_0 t)^2 + C_1^2 \sum_{n=1}^{\infty} \left(|\varphi_n| + \frac{|\psi_n|}{n} \right)^2 \le$$

$$\le C_0(\varphi_0^2 + \psi_0^2) + 2C_1^2 \sum_{n=1}^{\infty} \left(|\varphi_n|^2 + \frac{1}{n^2} |\psi_n|^2 \right) \le$$

$$\le C_0(\varphi_0^2 + \psi_0^2) + 2C_1^2 \left(\sum_{n=1}^{\infty} \varphi_n^2 + \sum_{n=1}^{\infty} \psi_n^2 \right) = C_{15} \left(\|\varphi\|^2 + \|\psi\|^2 \right). \quad \square$$

3. THE EXISTENCE AND STABILITY OF THE PROBLEM SOLUTION WITH $-1 < k < 1,\ k \ne 0$

Separating variables in Eq. (1), we get the following spectral problem with respect to the function $X(x)$:

$$X''(x) + \frac{k}{x} X'(x) + \lambda^2 X(x) = 0, \quad 0 < x < l, \tag{42}$$

$$\lim_{x \to 0+} x^k X'(x) = 0, \quad X'(l) = 0. \tag{43}$$

Let us define the general solution to Eq. (42) with $|k| < 1$, $k \neq 0$, by the formula

$$\widetilde{X}(x) = P_1 x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda x) + P_2 x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x); \tag{44}$$

since the value $\frac{1-k}{2}$ is not integer, $J_{\frac{1-k}{2}}(\lambda x)$ and $J_{\frac{k-1}{2}}(\lambda x)$ are linearly independent solutions to the Bessel equation.

This formula allows us to calculate

$$X'(x) = P_1 \lambda x^{\frac{1-k}{2}} J_{-\frac{k+1}{2}}(\lambda x) - P_2 \lambda x^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(\lambda x).$$

Since $x^k X'(x) = O(P_1 + P_2 x^{k+1})$ as $x \to 0$, by putting $P_1 = 0$ we make function (44) satisfy the first condition in (43). Put $P_2 = 1$.

Then the solution to Eq. (42) satisfying the first condition in (43) obeys the equality;

$$X(x) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x),$$

it formally coincides with (15), but here $-1 < k < 1$ and $k \neq 0$.

Let us now require that this function should satisfy the second boundary condition in (43), namely,

$$\left.\frac{dX(x)}{dx}\right|_{x=l} = \left.\left(x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda x)\right)'\right|_{x=l} = -\lambda l^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(\lambda l) = 0, \tag{45}$$

whence it follows that

$$\lambda_0 = 0,$$

$$J_{\frac{k+1}{2}}(\mu_n) = 0, \quad \mu_n = \lambda_n l. \tag{46}$$

Therefore, the system of eigenfunctions of problem (42), (43) takes the form

$$\widetilde{X}_0(x) = 1, \quad \lambda_0 = 0,$$

$$\widetilde{X}_n(x) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n x}{l}\right) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x), \quad n \in \mathbb{N},$$

where eigenvalues λ_n are zeros of Eq. (46).

Analogously to the case when $k \geq 1$, we get the following assertions.

Theorem 4. *If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions of Lemma* 3, *then problem* (2)–(6) *has a unique solution $u(x,t)$ which is defined by series* (32) *and satisfies the inclusion $u(x,t) \in C^2(\overline{D})$.*

Theorem 5. *The solution to problem* (2)–(6) *satisfies bound* (41).

4. THE EXISTENCE AND STABILITY OF THE PROBLEM SOLUTION WITH $k \leq -1$

Let us seek for partial solutions to Eq. (1) which differ from zero in the domain D and satisfy conditions (2), (5), and (7) in the form of the product $u(x,t) = X(x)T(t)$. Substituting this expression in Eq. (1) and conditions (5) and (7), we state the following problem with respect to the unknown function $X(x)$:

$$X''(x) + \frac{k}{x} X'(x) + \lambda^2 X(x) = 0, \quad 0 < x < l, \tag{47}$$

$$X(0) = 0, \quad X'(l) = 0. \tag{48}$$

In view of correlation (13) the general solution to Eq. (47) with $k \leq -1$ obeys the formula

$$X(x) = P_1 x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda x) + P_2 x^{\frac{1-k}{2}} Y_{\frac{1-k}{2}}(\lambda x). \tag{49}$$

Calculate

$$X'(x) = P_1 \lambda x^{\frac{1-k}{2}} J_{-\frac{k+1}{2}}(\lambda x) + P_2 \lambda x^{\frac{1-k}{2}} Y_{-\frac{k+1}{2}}(\lambda x).$$

Since with $x \to 0$ the function $X(x) = O(P_1 x^{1-k} + P_2) = O(1)$, while its derivative $X'(x) = O(P_1 x^{-k} + P_2 x) = O(x)$, for fulfilling the first condition in (48), one should choose values of constants in formula (49) as follows: $P_1 = 1$ and $P_2 = 0$. As a result, this solution takes the form

$$\widetilde{X}(x) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda x). \tag{50}$$

Substituting function (50) in the second boundary condition in formula (48), we get correlations $\lambda_0 = 0$,

$$J_{-\frac{k+1}{2}}(\mu_n) = 0, \quad \mu_n = \lambda_n l. \tag{51}$$

However with $\lambda_0 = 0$ the spectral problem (47), (48) has only the trivial solution, consequently, it is not an eigenfunction. Therefore, the system of eigenfunctions of problem (47), (48) takes the form

$$\widetilde{X}_n(x) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}\left(\frac{\mu_n x}{l}\right) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda_n x), \quad n \in \mathbb{N}, \tag{52}$$

where eigenvalues λ_n are zeros of Eq. (51). The system of functions (52), being a solution to the Sturm–Liouville problem for Eq. (47) with boundary conditions (48), is orthogonal and complete in the space $L_2[0, l]$ with the weight x^k. Note also that according to ([26], p. 317), zeros of Eq. (51) with large n satisfy the asymptotic formula

$$\mu_n = \lambda_n l = \pi n - \frac{\pi}{4} k - \frac{\pi}{2} + O\left(\frac{1}{n}\right). \tag{53}$$

In what follows, we consider the orthonormalized system of eigenfunctions

$$X_n(x) = \widetilde{X}_n(x)/\|\widetilde{X}_n\|, \quad n \in \mathbb{N}, \tag{54}$$

where the norm obeys formula (22).

Taking into account formula (54), we introduce functions

$$u_n(t) = \int_0^l u(x,t) x^k X_n(x)\, dx, \quad n = 1, 2, \ldots . \tag{55}$$

Analogously to the solution of the problem with $k \geq 1$, taking into account Eqs. (1) and (47), we get the equality

$$u''_{n,\varepsilon}(t) = x^k u_x X_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(t) - u(x,t) x^k X'_n(x)\Big|_{\varepsilon}^{l-\varepsilon}. \tag{56}$$

Formula (52) implies that $X_n(x) = O(x^{1-k})$ and $X'_n(x) = O(x^{-k})$ as $x \to 0$. Proceeding to the limit in formula (56) as $\varepsilon \to 0$, in view of conditions (2), (5), (7), and (48) we get the correlation

$$u''_n(t) + \lambda_n^2 u_n(t) = 0, \quad t \in (0, T). \tag{57}$$

Furthermore, assuming that function (55) satisfies initial conditions (4), we get equalities (28). The unique solution to problem (57), (28) obeys formula (29).

Basing on the calculated partial solutions (54) and (29), let us formally write the solution to problem (2)–(5), (7) as the Fourier–Bessel series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{58}$$

Coefficients of this series satisfy bounds established in Lemmas 1 and 2, consequently, with any $(x,t) \in \overline{D}$ series (58) and those obtained from it by termwise differentiation of the first and second orders are majorized by the series

$$C_{16} \sum_{n=1}^{\infty} \left(n^2 |\varphi_n| + n|\psi_n| \right). \tag{59}$$

Lemma 4. *Assume that a function $\varphi(x) \in C^2[0,l]$, the derivative $\varphi'''(x)$ exists and has a finite variation on $[0,l]$, there exists a function $\psi(x) \in C^1[0,l]$, the derivative $\psi''(x)$ exists and has a finite variation on $[0,l]$, and*

$$\varphi(0) = \psi(0) = \varphi(l) = \psi(l) = \varphi'(0) = \psi'(0) = \varphi'(l) = \psi'(l) = \varphi''(0) = \varphi''(l) = 0,$$

then estimates (39) are valid.

The proof is analogous to that proposed in the paper [29].

According to Lemma 4, series (59) is majorized by the converging numeric series (40), consequently, the sum of series (58) belongs to the class $C^2(\overline{D})$. The constructed function $u(x,t)$ defined by series (58) satisfies all conditions of problem (2)–(5), (7). Therefore, the following theorem is proved.

Theorem 6. *If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions of Lemma 4, then there exists a unique solution $u(x,t)$ to problem (2)–(5), (7); it is defined by series (58), while $u(x,t) \in C^2(\overline{D})$.*

Theorem 7. *The solution to problem (2)–(5), (7) satisfies the bound*

$$\|u\| \le C_{17}(\|\varphi\| + \|\psi\|), \tag{60}$$

where the constant C_{17} is independent of functions $\varphi(x)$ and $\psi(x)$.

The proof is similar to the proof of Theorem 3.

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