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Initial-Boundary Value Problem for Hyperbolic Equation with Singular Coefficient and Integral Condition of Second Kind

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Abstract—We research an initial-boundary value problem with integral condition of the second kind in a rectangular domain for a hyperbolic equation with singular coefficient. The solution is obtained in the form of the Fourier–Bessel series. There are proved theorems on uniqueness, existence and stability of the solution. In order to prove the existence of solution of the non-local problem we obtain sufficient conditions for the convergence of the series in terms of the initial values.

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1. INTRODUCTION

Let $l, T > 0$ be given real values, $D = \{(x,t)|\ 0 < x < l,\ 0 < t < T\}$ is rectangular domain. We consider hyperbolic equation

$$\Box_B u(x,t) \equiv u_{tt} - u_{xx} - \frac{k}{x} u_x = 0, \tag{1}$$

where $k \neq 0$ is given real number. Equation (1) belongs to the class of degenerated hyperbolic equations. Investigation of boundary value problems for that equations is of importance for contemporary theory of differential equations with partial derivatives. The problems have numerous applications in gas dynamics, magnet hydrodynamics, envelope theory and other fields of science and technique.

The Cauchy and Cauchy–Goursat problems for equation (1) were studied first in the work [1] for all $k \geq 1$ in characteristic triangle. As shown in the paper [2], the problems are not well-posed for $k < 0$. And the papers [3, 4] contain studies of the problems for equations of mixed type such that their hyperbolic parts coincide with equation (1). The non-local problems for equation (1) with integral conditions of the first kind and second kind are studied in the papers [5–7].

In the present paper we investigate the following initial-boundary value problem for equation (1) in domain D with non-local integral condition of the second kind for $k \leq -1$. We put in the further consideration without loss of generality $l = 1$, because equation (1) is invariant with regard to change of variables $x_1 = x/l$, $y_1 = y/l$.

Statement of the problem. It is necessary to find the function $u(x,t)$ satisfying the following restrictions:

$$u(x,t) \in C^1(\overline{D}) \cap C^2(D), \tag{2}$$

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$$\Box_B u(x,t) \equiv 0, \quad (x,t) \in D, \tag{3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le 1, \tag{4}$$

$$u(0,t) = 0, \quad 0 \le t \le T, \tag{5}$$

$$\left(x^{k-1}u(x,t)\right)'_x\Big|_{x=1} + \int\limits_0^1 u(x,t)x\,dx = 0, \quad 0 \le t \le T, \tag{6}$$

where $\varphi(x)$, $\psi(x)$ are given sufficiently smooth functions satisfying matching conditions

$$\left(x^{k-1}\varphi(x)\right)'_x\Big|_{x=1} + \int\limits_0^1 \varphi(x)x\,dx = 0, \quad \left(x^{k-1}\psi(x)\right)'_x\Big|_{x=1} + \int\limits_0^1 \psi(x)x\,dx = 0. \tag{7}$$

The problems for differential equations, where instead of classical initial and boundary value conditions are given conditions connecting meanings of desired functions or its derivatives at inner and boundary points of domains, arise in numerous branches of sciences: physics, chemistry, biology. In particular, the problems with integral conditions are encountered in mathematical modeling of the thermal conductivity, the transfer of moisture in capillary-porous media, processes in turbulent plasma. The detailed study of boundary value problems with integral conditions for hyperbolic equations can be found in the works [8–10]. The papers [11–13] contain investigations of problems with integral conditions for equations with singular coefficient.

The condition (6) contains besides an integral operator the boundary meanings of derivative of the desired function. According [8], we refer this integral condition to the second kind.

In what follows we apply the spectral analysis in proving of uniqueness, existence and stability of the solution. It is built explicitly as the Fourier–Bessel series. We check its convergence in the class of regular solutions.

2. UNIQUENESS

We seek particular solutions of equation (1), which do not vanish in domain D and satisfy restrictions (2), (5) and (6) in the form of products $u(x,t) = X(x)T(t)$. We substitute the product into equation (1), conditions (5) and (6), and obtain the following spectral problem for unknown function $X(x)$:

$$X''(x) + \frac{k}{x}X'(x) + \lambda^2 X(x) = 0, \quad 0 < x < 1, \tag{8}$$

$$X(0) = 0, \tag{9}$$

$$\left(x^{k-1}X(x)\right)'_x\Big|_{x=1} + \int\limits_0^1 X(x)x\,dx = 0, \tag{10}$$

here λ^2 is the separation constant. By virtue of equation (8) and condition (9) we obtain from integral condition (10):

$$\left(x^{k-1}X(x)\right)'_x\Big|_{x=1} + \int\limits_0^1 X(x)x\,dx = (k-1)X(1) + X'(1) - \frac{1}{\lambda^2}\int\limits_0^1 \left[xX''(x) + kX'(x)\right]dx$$

$$= (k-1)X(1) + X'(1) - \frac{1}{\lambda^2}\int\limits_0^1 \frac{\partial}{\partial x}\left[xX'(x) + (k-1)X(x)\right]dx$$

$$= (k-1)X(1) + X'(1) - \frac{1}{\lambda^2}\left[xX'(x) + (k-1)X(x)\right]\Big|_0^1 = \left(1 - \frac{1}{\lambda^2}\right)(X'(1) + (k-1)X(1)) = 0.$$

We obtain $\lambda^2 = 1$ or $X'(1) + (k-1)X(1) = 0$. Hence, the non-local condition (10) is equivalent to two local conditions. Apparently, this is inherent only in the equation (1).

The general solution of equation (8) for $k \le -1$ is determined by formula

$$\widetilde{X}(x) = K_1 x^\nu J_\nu(\lambda x) + K_2 x^\nu Y_\nu(\lambda x),$$

where $J_\nu(\xi)$, $Y_\nu(\xi)$ are Bessel functions of the first and the second kinds of order $\nu = (1-k)/2$ relatively, and K_1, K_2 are arbitrary constants. The common decision for $K_1 = 1$, $K_2 = 0$ satisfies condition (9). As a result, we obtain $\widetilde{X}(x) = x^\nu J_\nu(\lambda x)$, $\nu = (1-k)/2$.

Then eigenvalue $\lambda_0^2 = 1$ corresponds to eigenfunction $X_0(x) = x^\nu J_\nu(\lambda_0 x)$. We substitute this decision into condition of the third kind. And deduce the following equation for the eigenvalues of problem (8)–(10):

$$\lambda J'_\nu(\lambda) - \nu J_\nu(\lambda) = 0. \tag{11}$$

By virtue of formula $zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z)$ see [14] (p. 305) equation (11) is equivalent to the following one: $J_{\frac{3-k}{2}}(\lambda) = 0$. According [15] (p. 317), the zeros of this equation have asymptotic formula

$$\lambda_n = \pi n + \pi/2 - k\pi/4 + O(1/n) \tag{12}$$

for sufficiently large n. Thus, the problem (8)–(10) has the following system of eigenfunctions

$$\widetilde{X}_0(x) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda_0 x), \quad \widetilde{X}_n(x) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda_n x), \quad n \in \mathbb{N},$$

and its eigenvalues λ_n are zeros of equation (11). The obtained system of eigenfunctions is not orthogonal on segment [0, 1], because the eigenvalue λ_0 is not zero of Bessel function $J_{\frac{3-k}{2}}(\lambda x)$, i.e., it is not root of equation (11). But the subsystem of functions $\widetilde{X}_n(x)$, $n \in \mathbb{N}$, is orthogonal and complete in the space $L_2[0,1]$ with weight x^k as system of eigenfunctions of spectral problem (8), (9) and (11). The orthogonality follows from equality

$$\int_0^1 x^k \widetilde{X}_n(x)\widetilde{X}_m(x)dx = \int_0^1 xJ_{\frac{1-k}{2}}(\lambda_n x)J_{\frac{1-k}{2}}(\lambda_m x)dx = 0,$$

since λ_n and λ_m are the zeros of equation (11) and $\nu = (1-k)/2 > -1$. This system is complete in the space $L_2[0,1]$ by virtue of the Steklov theorem [14] (p. 314). Therefore, in what follows we consider the orthonormalized system of eigenfunctions $\widetilde{X}_n(x)$, $n = 1, 2, \ldots$. The equations $\widetilde{X}_n(x)$ enable us to consider below the following normalized orthogonal system of eigenfunctions

$$X_n(x) = \frac{\widetilde{X}_n(x)}{||\widetilde{X}_n||_{L_{2,\rho}(0,1)}}, \tag{13}$$

the norm is defined by formula

$$||\widetilde{X}_n||^2_{L_{2,\rho}(0,1)} = \int_0^1 \rho(x)\widetilde{X}_n^2(x)dx, \quad \rho(x) = x^k.$$

According [16], we introduce functions

$$u_n(t) = \int_0^1 u(x,t)x^k X_n(x)dx, \quad n = 1, 2, \ldots, \tag{14}$$

and auxiliary functions

$$u_{n,\varepsilon}(t) = \int\limits_{\varepsilon}^{1-\varepsilon} u(x,t)x^k X_n(x)dx, \quad n = 1, 2, \dots,$$

where $\varepsilon > 0$ is sufficiently small. We differentiate this equality twice with regard to variable t, $0 < t < T$, and obtain by means of equation (1)

$$u''_{n,\varepsilon}(t) = \int\limits_{\varepsilon}^{1-\varepsilon} u_{tt}(x,t)x^k X_n(x)dx = \int\limits_{\varepsilon}^{1-\varepsilon} \left(u_{xx} + \frac{k}{x}u_x\right) x^k X_n(x)dx$$

$$= \int\limits_{\varepsilon}^{1-\varepsilon} \frac{\partial}{\partial x}(x^k u_x) X_n(x)dx = x^k u_x X_n(x)\Big|_{\varepsilon}^{1-\varepsilon} - \int\limits_{\varepsilon}^{1-\varepsilon} x^k u_x X'_n(x)dx.$$

In addition, we obtain from this equality by virtue of equation (8):

$$\int\limits_{\varepsilon}^{1-\varepsilon} x^k u_x X'_n(x)dx = \lambda_n^2 u_{n,\varepsilon}(t) + u(x,t)x^k X'_n(x)\Big|_{\varepsilon}^{1-\varepsilon}.$$

The last two equalities imply

$$u''_{n,\varepsilon}(t) = x^k u_x X_n(x)\Big|_{\varepsilon}^{1-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(t) - u(x,t)x^k X'_n(x)\Big|_{\varepsilon}^{1-\varepsilon}.$$

It follows from formula $\widetilde{X}_n(x)$ that $X_n(x) = O(x^{1-k})$ and $X'_n(x) = O(x^{-k})$ for $x \to 0$. Then we pass in the last equality to limit for $\varepsilon \to 0$, and obtain by means of conditions (2), (5), (9), (11):

$$u''_n(t) + \lambda_n^2 u_n(t) = \left[u_x(1,t) + (k-1)u(1,t)\right] X_n(1), \quad t \in (0,T). \tag{15}$$

Then we multiply equation (1) by x and integrate the product for fixed $t \in (0,T)$ with regard to variable x from ε to $1-\varepsilon$. As a result, we obtain

$$\frac{d^2}{dt^2} \int\limits_{\varepsilon}^{1-\varepsilon} u(x,t)x\, dx - \left[xu_x + (k-1)u\right]\Big|_{\varepsilon}^{1-\varepsilon} = 0.$$

In the obtained equality we pass to the limit for $\varepsilon \to 0$, and by virtue of conditions (2), (5), (6) conclude that

$$-\frac{d^2}{dt^2}\left[\left(x^{k-1}u(x,t)\right)'_x\Big|_{x=l}\right] - \left[lu_x(l,t) + (k-1)u(l,t)\right] = 0.$$

Consequently,

$$\frac{d^2}{dt^2}\left[u_x(1,t) + (k-1)u(1,t)\right] + \left[u_x(1,t) + (k-1)u(1,t)\right] = 0.$$

We denote $Z(t) = u_x(1,t) + (k-1)u(1,t)$, and obtain ordinary differential equation $Z''(t) + Z(t) = 0$. Its general solution is

$$Z(t) = P_1 \cos t + P_2 \sin t,$$

where P_1 and P_2 are arbitrary constants. Consequently,

$$u_x(1,t) + (k-1)u(1,t) = P_1 \cos t + P_2 \sin t.$$

The initial conditions (4) enable us to find meanings of the constants from the last equality:

$$P_1 = \varphi'(1) + (k-1)\varphi(1), \quad P_2 = \psi'(1) + (k-1)\psi(1).$$

Thus,

$$u_x(1,t) + (k-1)u(1,t) = \left[\varphi'(1) + (k-1)\varphi(1)\right]\cos t + \left[\psi'(1) + (k-1)\psi(1)\right]\sin t.$$

We substitute the last equality into (15), and obtain the following equation for determination of functions $u_n(t)$: $u_n''(t) + \lambda_n^2 u_n(t) = P_3 \cos t + P_4 \sin t$, $t \in (0,T)$, where

$P_3 = P_1 X_n(1) = (\varphi'(1) + (k-1)\varphi(1)) J_{\frac{1-k}{2}}(\lambda_n)$, $\quad P_4 = P_2 X_n(1) = (\psi'(1) + (k-1)\psi(1)) J_{\frac{1-k}{2}}(\lambda_n)$.

The general solution of this ordinary equation is

$$u_n(t) = a_n \cos \lambda_n t + b_n \sin \lambda_n t + v_n(t), \tag{16}$$

where a_n and b_n are arbitrary constants, $v_n(t)$ is determined by formula

$$v_n(t) = \frac{1}{\lambda_n^2 - 1} \left[(\varphi'(1) + (k-1)\varphi(1)) \cos t + (\psi'(1) + (k-1)\psi(1)) \sin t \right] J_{\frac{1-k}{2}}(\lambda_n).$$

Note that $\lambda_n^2 \neq 1$ for any $n \in \mathbb{N}$, because ± 1 are not zeros of equation (11).

In order to determine the coefficients a_n and b_n in (14) we use the initial conditions (4):

$$u_n(0) = \int_0^1 \varphi(x) x^k X_n(x) dx = \varphi_n, \quad u_n'(0) = \int_0^1 \psi(x) x^k X_n(x) dx = \psi_n.$$

As a result, we obtain the system

$$a_n = \varphi_n - \frac{1}{\lambda_n^2 - 1} (\varphi'(1) + (k-1)\varphi(1)) J_{\frac{1-k}{2}}(\lambda_n),$$
$$b_n = \frac{\psi_n}{\lambda_n} - \frac{1}{(\lambda_n^2 - 1)\lambda_n} (\psi'(1) + (k-1)\psi(1)) J_{\frac{1-k}{2}}(\lambda_n).$$

We substitute these meanings of constants a_n and b_n into (16), and find finally

$$\begin{aligned} u_n(t) = \varphi_n \cos \lambda_n t + \frac{\psi_n}{\lambda_n} \sin \lambda_n t + \frac{1}{\lambda_n^2 - 1} \left[\varphi'(1) + (k-1)\varphi(1) \right] J_{\frac{1-k}{2}}(\lambda_n) (\cos t - \cos \lambda_n t) \\ + \frac{1}{(\lambda_n^2 - 1)\lambda_n} \left[\psi'(1) + (k-1)\psi(1) \right] J_{\frac{1-k}{2}}(\lambda_n) \left(\sin t - \frac{1}{\lambda_n} \sin \lambda_n t \right). \end{aligned} \tag{17}$$

Theorem 1. *If the problem* (2)–(7) *has a solution, then it is unique.*

Proof. Let $u(x,t)$ be a solution of homogeneous problem (2)–(7), where $\varphi(x) \equiv 0$ and $\psi(x) \equiv 0$. We multiply the equation (1) by x, and integrate it for fixed $t \in (0,T)$ in variable x from ε up to $1-\varepsilon$. As a result, we obtain

$$\int_\varepsilon^{1-\varepsilon} u_{tt} x \, dx - \int_\varepsilon^{1-\varepsilon} (x u_{xx} + k u_x) dx = 0 \quad \text{or} \quad \frac{d^2}{dt^2} \int_\varepsilon^{1-\varepsilon} u(x,t) x \, dx - \int_\varepsilon^{1-\varepsilon} \frac{\partial}{\partial x} (x u_x + (k-1)u) dx = 0.$$

We have from this

$$\frac{d^2}{dt^2} \int_\varepsilon^{1-\varepsilon} u(x,t) x dx - (x u_x + (k-1)u) \Big|_\varepsilon^{1-\varepsilon} = 0.$$

By virtue of conditions (2) and (5) we are able to pass in the last equality to the limit for $\varepsilon \to 0$, and obtain

$$\frac{d^2}{dt^2} \int_\varepsilon^{1-\varepsilon} u(x,t) x \, dx - (u_x(1,t) + (k-1)u(1,t)) = 0, \quad 0 \le t \le T.$$

We find from the integral condition (6):

$$u_x(1,t) + (k-1)u(1,t) = -\int_0^1 u(x,t) x \, dx.$$

The substitution of this expression into previous equality leads to ordinary differential equation

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)x\,dx + \int_0^1 u(x,t)x\,dx = 0,$$

whose general solution is $\int_0^1 u(x,t)x\,dx = M_1\cos t + M_2\sin t$, where M_1 and M_2 are arbitrary constants. We have from this by means of null initial conditions $\int_0^1 u(x,t)x\,dx = 0$.

Then from condition (6) we obtain $u_x(1,t) + (k-1)u(1,t) = 0$, $0 \le t \le T$. Thus, for function $u(x,t)$ we obtain homogeneous boundary condition

$$u(0,t) = 0, \quad u_x(1,t) + (k-1)u(1,t) = 0, \quad 0 \le t \le T.$$

This problem is studied above by means of method of separation of variables, i.e., we have constructed the system of eigenvalues (13), and by means of this system and the introduced functions (14) we find their explicit form (17). By assumptions we have $\varphi(x) = \psi(x) \equiv 0$, then imply that $\varphi_n = \psi_n \equiv 0$ for all $n \in \mathbb{N}$. We deduce from (17) that $u_n(t) = 0$ for all $n \in \mathbb{N}$. Then for any $t \in [0,T]$ relation (14) implies that $\int_0^1 u(x,t)x^k X_n(x)dx = 0$. System (13) is complete in the space $L_2[0,1]$ with weight x^k; hence, $u(x,t) = 0$ almost everywhere on segment $[0,1]$ for any $t \in [0,T]$. According (2), we obtain $u(x,t) \in C(\overline{D})$. Consequently, $u(x,t) \equiv 0$ in $\overline{D}$. □

3. EXISTENCE

The obtained above particular solutions (13) and (17) enable us to write a solution of problem (2)–(7) as the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x). \tag{18}$$

We assume that its term-by-term differentiation is possible, and consider the following series:

$$u_t(x,t) = \sum_{n=1}^{\infty} u_n'(t)X_n(x), \quad u_x(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n'(x),$$
$$u_{tt}(x,t) = \sum_{n=1}^{\infty} u_n''(t)X_n(x), \quad u_{xx}(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n''(x).$$

Let us show that under certain restrictions on functions $\varphi(x)$ and $\psi(x)$ (see the initial conditions (4)) these series uniformly converge in closed domain $\overline{D}$.

Lemma 1. *For sufficiently large n and any $t \in [0,T]$ there are valid bounds:*

$$|u_n(t)| \le C_1\left(|\varphi_n| + \frac{|\psi_n|}{n}\right) + \frac{|\varphi'(1)|}{n^{3/2}} + \frac{|\psi'(1)|}{n^{3/2}} + \frac{|\varphi(1)|}{n^{3/2}} + \frac{|\psi(1)|}{n^{3/2}},$$
$$|u_n'(t)| \le C_2\left(n|\varphi_n| + |\psi_n|\right) + \frac{|\varphi'(1)|}{n^{1/2}} + \frac{|\psi'(1)|}{n^{3/2}} + \frac{|\varphi(1)|}{n^{1/2}} + \frac{|\psi(1)|}{n^{3/2}},$$
$$|u_n''(t)| \le C_3\left(n^2|\varphi_n| + n|\psi_n|\right) + n^{1/2}|\varphi'(1)| + \frac{|\psi'(1)|}{n^{1/2}} + n^{1/2}|\varphi(1)| + \frac{|\psi(1)|}{n^{1/2}}.$$

Here and in what follows C_i stands for a positive constant.

Proof. Proof of the bounds follows from formulas (17) and (12). □

Lemma 2. *For sufficiently large n and any $x \in [0,1]$ there are valid bounds:*

$$|X_n(x)| \le C_4, \quad |X_n'(x)| \le C_5 n, \quad |X_n''(x)| \le C_6 n^2.$$

Proof. As known, $\widetilde{X}_n(x) = x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda_n x) \in C^2[0,1]$, and for large ξ there is valid asymptotical bound $J_\nu(\xi) = O\left(\xi^{-1/2}\right)$. We find

$$||\widetilde{X}_n||^2_{L_{2,\rho}(0,1)} = \int_0^1 x^k \widetilde{X}_n^2(\lambda_n x)dx = \int_0^1 x J^2_{\frac{1-k}{2}}(\lambda_n x)dx = \frac{1}{2} J^2_{\frac{3-k}{2}}(\lambda_n).$$

The relations (13) imply the first bound from this lemma. We evaluate derivatives of function $\widetilde{X}_n(x)$:

$$\widetilde{X}_n'(x) = \lambda_n x^{\frac{1-k}{2}} J_{-\frac{k+1}{2}}(\lambda_n x), \quad \widetilde{X}_n''(x) = -\frac{k}{x}\widetilde{X}_n'(x) - \lambda_n^2 \widetilde{X}_n(x).$$

From these equalities the remaining estimates follow. □

Lemma 3. *If function $\varphi(x)$ belongs to $C^2[0,1]$ and has third derivative $\varphi'''(x)$ with finite variation on $[0,1]$, then function $\psi(x)$ belongs to $C^1[0,1]$ and has second derivative $\psi''(x)$ with finite variation on $[0,1]$. If*

$$\varphi(0) = \psi(0) = \varphi(1) = \psi(1) = \varphi'(0) = \psi'(0) = \varphi'(1) = \psi'(1) = \varphi''(0) = \varphi''(1) = 0,$$

then

$$|\varphi_n| \le C_7 n^{-4}, \quad |\psi_n| \le C_8 n^{-3}. \tag{19}$$

Proof. We obtain by means of (8) and conditions of the lemma

$$\varphi_n = \int_0^1 \varphi(x)x^k X_n(x)dx = -\frac{1}{\lambda_n^2}\int_0^1 \varphi(x)(x^k X_n'(x))'dx = \frac{1}{\lambda_n^2}\int_0^1 \varphi'(x)x^k X_n'(x)dx$$

$$= -\frac{1}{\lambda_n^2}\int_0^1 (\varphi'(x)x^k)'X_n(x)dx = -\frac{1}{\lambda_n^2}\int_0^1 \varphi''(x)x^k X_n(x)dx - \frac{k}{\lambda_n^2}\int_0^1 \frac{\varphi'(x)}{x}x^k X_n(x)dx.$$

We denote

$$\varphi_n^{(2)} = \int_0^1 \varphi''(x)x^k X_n(x)dx, \quad \varphi_{1n} = \int_0^1 \varphi_1(x)x^k X_n(x)dx, \quad \varphi_1(x) = \varphi'(x)/x,$$

and have $\varphi_n = -\frac{1}{\lambda_n^2}\varphi_n^{(2)} - \frac{k}{\lambda_n^2}\varphi_{1n}$. We obtain from the first integral $\varphi_n^{(2)}$ by virtue of (8) that

$$\varphi_n^{(2)} = \int_0^1 \varphi''(x)x^k X_n(x)dx = -\frac{1}{\lambda_n^2}\int_0^1 \varphi''(x)\left(x^k X_n'(x)\right)'dx$$

$$= -\frac{1}{\lambda_n^2}\left[\varphi''(x)x^k X_n'(x)\Big|_0^1 - \int_0^1 \varphi'''(x)x^k X_n'(x)dx\right] = \frac{1}{\lambda_n^2}\int_0^1 \varphi'''(x)x^k X_n'(x)dx = \frac{\varphi_n^{(3)}}{\lambda_n^2},$$

where $\varphi_n^{(3)} = \int_0^1 \varphi'''(x)x^k X_n'(x)dx$.

The derivative $\varphi'''(x)$ has finite variation on segment $[0,1]$ by assumptions of the lemma. Then see [17] (p. 202) $\varphi_n^{(3)} = O(1)$ for large n, and, consequently, there is valid the bound $|\varphi_n^{(3)}| \le C_9$.

Analogously, we integrate by parts the second integral, and obtain by means of (8) and assumptions of the lemma

$$\varphi_{1n} = -\frac{1}{\lambda_n^2}\int_0^1 \varphi_1(x)\left(x^k X_n'(x)\right)'dx = \frac{1}{\lambda_n^2}\int_0^1 \varphi_1'(x)x^k X_n'(x)dx = \frac{\varphi_{1n}^{(1)}}{\lambda_n^2},$$

where $\varphi_{1n}^{(1)} = \int_0^1 \varphi_1'(x) x^k X_n'(x)dx$, and this integral converges.

We estimate the integral $\varphi_{1n}^{(1)}$ for large n by means of representation

$$\varphi_{1n}^{(1)} = \frac{\lambda_n}{||\widetilde{X}_n||_{L_{2,\rho}(0,1)}} \int\limits_0^1 \left(\frac{\varphi'(x)}{x}\right)' x^k x^{\frac{1-k}{2}} J_{-\frac{k+1}{2}}(\lambda_n x)dx$$
$$= \frac{\lambda_n}{||\widetilde{X}_n||_{L_{2,\rho}(0,1)}} \int\limits_0^1 x\left[\varphi''(x) - \frac{\varphi'(x)}{x}\right] x^{\frac{k-3}{2}} J_{-\frac{k+1}{2}}(\lambda_n x)dx.$$

By assumptions $\varphi'(0) = \varphi''(0) = 0$. Therefore, for sufficiently small $\delta > 0$ and $0 \le x \le \delta$ we have

$$\varphi'(x) = \varphi'(0) + \frac{\varphi''(0)}{1!}x + \frac{\varphi'''(\xi x)}{2!}x^2 = \frac{1}{2}\varphi'''(\xi x)x^2, \quad 0 < \xi < x,$$
$$\varphi''(x) = \varphi''(0) + \frac{\varphi'''(\theta x)}{1!}x = \varphi'''(\theta x)x, \quad 0 < \theta < x.$$

By virtue of these representations the function

$$x^{\frac{1}{2}} f(x) = x^{\frac{1}{2}}\left[\varphi''(x) - \frac{\varphi'(x)}{x}\right] x^{\frac{k-3}{2}} = \left[\varphi''(x) - \frac{\varphi'(x)}{x}\right] x^{\frac{k}{2}-1} = \left[\varphi'''(\theta x) - \frac{1}{2}\varphi'''(\xi x)\right] x^{\frac{k}{2}}$$

has bounded variation on segment $[0, \delta]$, because it is product of two functions of finite variation see [17] (p. 202).

One can show analogously that function $x^{\frac{1}{2}} f(x)$ has finite variation on segment $[\delta, 1]$. Then it has finite variation on segment $[0, 1]$. and, as in the case of integral $\varphi_n^{(3)}$, we obtain bound $|\varphi_{1n}^{(1)}| \le C_{10}$. The first estimate follows from these estimates.

We integrate by parts twice, and obtain by assumptions of the lemma

$$\psi_n = -\frac{1}{\lambda_n^2}\int\limits_0^1 \psi''(x)x^k X_n(x)dx - \frac{k}{\lambda_n^2}\int\limits_0^1 \frac{\psi'(x)}{x} x^k X_n(x)dx = -\frac{1}{\lambda_n^2}\psi_n^{(2)} - \frac{k}{\lambda_n^2}\psi_{1n},$$

where

$$\psi_n^{(2)} = \int\limits_0^1 \psi''(x)x^k X_n(x)dx, \quad \psi_{1n} = \int\limits_0^1 \frac{\psi'(x)}{x}x^k X_n(x)dx.$$

Analogously, we obtain equalities $\psi_n^{(2)} = O\left(\lambda_n^{-1}\right)$, $\psi_{1n} = O\left(\lambda_n^{-1}\right)$, which imply by means of ψ_n the second bound. □

The coefficients (17) of series (18) under assumptions of the lemma 3 turn into the following ones:

$$u_n(t) = \varphi_n \cos\lambda_n t + \frac{\psi_n}{\lambda_n}\sin\lambda_n t. \tag{20}$$

According the Lemmas 1–3, the series (18) and its derivatives up to the second order inclusive for any $(x, t) \in \overline{D}$ allow majoration by the convergent numerical series $C_{11}\sum_{n=1}^{\infty} n^{-2}$, and, consequently, uniformly converge in the closed domain $\overline{D}$. Thus, there is proved

Theorem 2. *If functions $\varphi(x)$ and $\psi(x)$ satisfy assumptions of Lemma* 3, *and conditions* (7) *hold, then there exists a unique solution of problem* (2)–(7), *it is determined by series* (18), *and* $u(x, t) \in C^2(\overline{D})$.

4. STABILITY

Theorem 3. *The solution of problem* (2)–(7) *satisfies the bound*

$$||u||_{L_{2,\rho}(0,1)} \leq C_{12}(||\varphi||_{L_{2,\rho}(0,1)} + ||\psi||_{L_{2,\rho}(0,1)}),$$

where constant C_{12} *does not depend on functions* $\varphi(x)$ *and* $\psi(x)$.

Proof. By means of (20) and Lemma 1 we have $|u_n(t)| \leq C_1\left(|\varphi_n| + |\psi_n|/n\right)$. We obtain from (18) by means of this bound

$$||u||^2_{L_{2,\rho}(0,1)} = \int\limits_0^1 x^k u^2(x,t)dx = \int\limits_0^1 x^k \sum_{n=1}^{\infty} u_n(t)X_n(x) \sum_{m=1}^{\infty} u_m(t)X_m(x)dx$$

$$= \sum_{m,n=1}^{\infty} u_n(t)u_m(t) \int\limits_0^1 x^k X_n(x)X_m(x)dx = \sum_{n=1}^{\infty} u_n^2(t) \int\limits_0^1 x^k X_n^2(x)dx = \sum_{n=1}^{\infty} u_n^2(t)$$

$$\leq 2C_1^2 \sum_{n=1}^{\infty} \left(|\varphi_n|^2 + \frac{1}{n^2}|\psi_n|^2\right) \leq 2C_1^2 \left(\sum_{n=1}^{\infty} \varphi_n^2 + \sum_{n=1}^{\infty} \psi_n^2\right) = 2C_1^2 \left(||\varphi||^2_{L_{2,\rho}(0,1)} + ||\psi||^2_{L_{2,\rho}(0,1)}\right).$$

This relation implies this bound. □

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