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PARTIAL DIFFERENTIAL EQUATIONS

Initial Value Problem for B-Hyperbolic Equation with Integral Condition of the Second Kind

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Abstract—For the hyperbolic equation with Bessel operator, we study the initial boundary-value problem with integral nonlocal condition of the second kind in a rectangular domain. The integral identity method is used to prove the uniqueness of the solution to the posed problem. The solution is constructed as a Fourier–Bessel series. To justify the existence of the solution to the nonlocal problem, we obtain sufficient conditions to be imposed on the initial conditions to ensure the convergence of the constructed series in the class of regular solutions.

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1. INTRODUCTION

We consider the hyperbolic equation with Bessel operator

$$\Box_B u(x,t) \equiv u_{tt} - x^{-k}\frac{\partial}{\partial x}(x^k u_x) = 0, \tag{1}$$

where $x > 0$ and k is a given real number, $k > -1$, $k \neq 0$. Equation (1), which, after Kipriyanov [1, p. 7], will be called a B-hyperbolic equation, arises, for example, when switching from Cartesian to cylindrical coordinates in the wave equation while considering radial gas vibrations in a stationary infinite cylindrical pipe or when switching to spherical coordinates while considering small vibrations of a gas near its equilibrium position inside an impermeable spherical shell [2, pp. 185, 191]. Pul'kin [3] studied the Cauchy and Cauchy–Goursat problems for Eq. (1) with $k \geq 1$. The Tricomi problem for a mixed-type equation with a hyperbolic part that coincides with Eq. (1) was considered in [4, 5].

Let $D = \{(x,t) : 0 < x < l,\ 0 < t < T\}$ be a rectangular domain in the Oxt coordinate plane, where $l, T > 0$ are given real numbers. In the present paper, for Eq. (1) in the domain D, we study the following initial value problems with a nonlocal boundary condition of the second kind for $k \geq 1$ and $-1 < k < 1$, $k \neq 0$.

Problem 1. Let $k \geq 1$. To determine such a solution $u(x,t)$ of the equation

$$\Box_B u(x,t) \equiv 0, \qquad (x,t) \in D, \tag{2}$$

that

$$u(x,t) \in C^1(\overline{D}) \bigcap C^2(D) \tag{3}$$

and it satisfies the initial conditions

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad 0 \leq x \leq l, \tag{4}$$

and the integral condition of the second kind

$$u_x(l,t) + \int\limits_0^l u(x,t)\, x^k\, dx = 0, \qquad 0 \le t \le T. \tag{5}$$

Problem 2. Let $-1 < k < 1$, and let $k \neq 0$. To determine a solution $u(x,t)$ to Eq. (2) satisfying conditions (3)–(5) and the condition

$$\lim_{x\to 0+} x^k u_x(x,t) = 0, \qquad 0 \le t \le T. \tag{6}$$

Here $\varphi(x)$, $\psi(x)$ are given sufficiently smooth functions satisfying the relations

$$\varphi'(l) + \int\limits_0^l \varphi(x)\, x^k\, dx = 0, \qquad \psi'(l) + \int\limits_0^l \psi(x)\, x^k\, dx = 0. \tag{7}$$

Pul'kina [6] coined the term "integral condition of the second kind". In [7] and the monograph [8], she was also the first to use functional-analysis methods to study the boundary value problems with integral conditions for Eq. (1) at $k = 0$, for telegraph equation, and for more general hyperbolic-type equations with smooth coefficients

$$u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t).$$

Mixed problems with integral conditions of the first kind for hyperbolic equations with Bessel operator were considered in [9, 10], and the boundary value problems equipped with such a condition for mixed-type equations were studied in [11–13].

In the present paper, based on [11–13], we prove the existence theorem for the solution to the problem in Eqs. (2)–(7) for all $k > -1$, with the solution being constructed in the form of a Fourier–Bessel series and the convergence of the series being substantiated in the class of regular solutions (2) and (3).

We note that, for the B-elliptic equation

$$u_{tt} + x^{-k}\frac{\partial}{\partial x}(x^k u_x) = 0, \qquad (x,t) \in D,$$

at $k \ge 1$ by virtue of the results in [14, 3] in the class of bounded solutions, we need not pose the Dirichlet condition at the boundary $x = 0$ of the rectangle D. In this case, it was shown in [3; 15, p. 68] that the derivative along the normal to this interval, i.e., u_x, is zero on the interval $x = 0$.

Equation (2) possesses the same property for $k \ge 1$. Separating the variables, we can easily show that

$$u_x(0,t) = 0, \qquad 0 \le t \le T, \tag{8}$$

thereby proving an additional property of the solution to Problem 1. In what follows, the relation in Eq. (8) can also be taken advantage of in our reasoning. However, with the derivative u_x remaining bounded as $x \to 0$, no need arises to use this relation, as will be shown below.

2. UNIQUENESS OF SOLUTIONS TO PROBLEMS 1 AND 2

Theorem 1. *If solutions to Problems* 1 *and* 2 *exist then they are unique.*

Proof. Let u_1 and u_2 be two solutions of the problem in Eq. (2)–(5). Then their difference $u = u_1 - u_2$ satisfies Eq. (2), inclusion (3), the homogeneous initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 \le x \le l, \tag{4_0}$$

the integral condition in Eq. (5), and condition (8).

We substitute the function u into Eq. (1), multiply the result by x^k, and integrate it for a fixed $t \in (0,T)$ over variable x from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number, to obtain

$$\int\limits_{\varepsilon}^{l-\varepsilon} u_{tt} x^k\,dx - \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\partial}{\partial x}(x^k u_x)\,dx = 0$$

or

$$\int\limits_{\varepsilon}^{l-\varepsilon} u_{tt} x^k\,dx - \left(x^k \frac{\partial u}{\partial x}\right)\Bigg|_{\varepsilon}^{l-\varepsilon} = 0. \tag{9}$$

Passing to the limit in (9) as $\varepsilon \to 0$, due to condition (3), we have

$$\frac{d^2}{dt^2}\int\limits_0^l u(x,t)x^k\,dx = l^k u_x(l,t). \tag{10}$$

Substituting the value of the derivative from the integral condition in Eq. (5) into (10), we obtain

$$\frac{d^2}{dt^2}\int\limits_0^l u(x,t)x^k\,dx + l^k\int\limits_0^l u(x,t)\,x^k\,dx = 0. \tag{11}$$

We introduce the notation

$$Z(t) = \int\limits_0^l u(x,t)\,x^k\,dx$$

to write Eq. (11) as the equation

$$Z''(t) + l^k Z(t) = 0$$

that has a general solution of the form

$$Z(t) = \int\limits_0^l u(x,t)\,x^k\,dx = C_1\cos(\sqrt{l^k}\,t) + C_2\sin(\sqrt{l^k}\,t), \tag{12}$$

where C_1 and C_2 are arbitrary constants.

With regard to the initial conditions (4_0), we determine the constants $C_1 = 0$ and $C_2 = 0$ from Eq. (12). As a result, we have

$$\int\limits_0^l u(x,t)\,x^k\,dx = 0. \tag{13}$$

Substituting (13) into condition (5), we obtain

$$u_x(l,t) = 0. \tag{14}$$

Further, we consider the following identity which can easily be verified by direct differentiation:

$$x^k u_t \Box_B u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}[x^k(u_t^2 + u_x^2)] - \frac{\partial}{\partial x}(x^k u_t u_x),$$

where $u(x,t)$ is an arbitrary function with continuous second derivatives. Since the function u satisfies Eq. (1), the last expression can be written as

$$\frac{1}{2}\frac{\partial}{\partial t}[x^k(u_t^2 + u_x^2)] = \frac{\partial}{\partial x}(x^k u_t u_x). \tag{15}$$

Now we integrate identity (15) for a fixed $t \in (0, T)$ over the variable x from ε to $l - \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number, and then pass to the limit as $\varepsilon \to 0$. By virtue of condition (3), we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_0^l (u_t^2 + u_x^2)x^k\,dx = l^k u_t(l,t)u_x(l,t)$$

or, with regard to (14),

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_0^l (u_t^2 + u_x^2)x^k\,dx = 0,$$

which implies

$$\int\limits_0^l (u_t^2 + u_x^2)x^k\,dx = C(x), \tag{16}$$

where $C(x)$ is a function depending only on the variable x. Now we set $t = 0$ in Eq. (16), and, with regard to conditions (4_0), obtain the identity

$$\int\limits_0^l (u_t^2(x,0) + u_x^2(x,0))x^k\,dx = C(x) \equiv 0,$$

and, hence, $u_t(x,t) \equiv 0$ and $u_x(x,t) \equiv 0$. Therefore, we have $u(x,t) \equiv \text{const}$, which again implies the identity $u(x,t) \equiv 0$ in view of the homogeneous initial conditions (4_0). Thus, $u_1 \equiv u_2$.

The uniqueness of the solution of Problem 2 can be proved similarly. The theorem is proved.

3. EXISTENCE OF SOLUTION TO PROBLEM 1

Particular solutions of Eq. (1) that are nonzero in the domain D and satisfy conditions (3) and (5) will be sought as products $u(x,t) = X(x)T(t)$. We substitute this expression into Eq. (1) and into conditions (3) and (8) to obtain the following spectral problem for the unknown $X(x)$:

$$X''(x) + \frac{k}{x}X'(x) + \lambda^2 X(x) = 0, \qquad 0 < x < l, \tag{17}$$

$$|X(0)| < +\infty, \tag{18}$$

$$X'(l) + \int\limits_0^l X(x)x^k\,dx = 0, \tag{19}$$

where λ^2 is the separation constant.

With regard to Eq. (17), from the boundary condition (19) we obtain the relations

$$\begin{aligned} X'(l) + \int\limits_0^l X(x)x^k\,dx &= X'(l) - \frac{1}{\lambda^2}\int\limits_0^l \left[X''(x) + \frac{k}{x}X'(x)\right]x^k\,dx \\ &= X'(l) - \frac{1}{\lambda^2}\int\limits_0^l \frac{d}{dx}(x^k X'(x))\,dx = \left(1 - \frac{l^k}{\lambda^2}\right)X'(l) = 0. \end{aligned}$$

Hence it follows that, with condition (3) taken into account, the nonlocal integral condition in Eq. (19) for $\lambda \neq l^{k/2}$ is equivalent to the local boundary condition of the second kind

$$X'(l) = 0. \tag{20}$$

The solution of Eq. (17) satisfying condition (18) has the form

$$\widetilde{X}(x) = x^{(1-k)/2} J_{(k-1)/2}(\lambda x), \tag{21}$$

where $J_\nu(\xi)$ is the Bessel function of the first kind of order $\nu = (k-1)/2$. In this case, it is easy to see that the function in Eq. (21) satisfies the condition $\widetilde{X}'(0) = 0$, as one should expect due to relation (8).

Substituting the function in Eq. (21) into the boundary condition in Eq. (20), we obtain

$$J_{(k+1)/2}(\mu) = 0, \qquad \mu = \lambda l. \tag{22}$$

By the Lommel theorem [17, p. 530], Eq. (22) has countably many zeros for $(k+1)/2 > -1$, with all zeros being real. Denoting the nth root of Eq. (22) as μ_n, $n \in \mathbb{N}$, we obtain the eigenvalues of the problem in Eqs. (17)–(19). It is also well known [18, p. 317] that the zeros μ_n of Eq. (22) satisfy, for large n, the asymptotic representation

$$\mu_n = \lambda_n l = \pi n + \frac{\pi}{4} k + O\left(\frac{1}{n}\right). \tag{23}$$

Then the corresponding system of the eigenfunctions of problem (17), (18), and (20) becomes

$$\widetilde{X}_n(x) = x^{(1-k)/2} J_{(k-1)/2}(\lambda_n x), \qquad n \in \mathbb{N}. \tag{24}$$

For ease of further calculations, we normalize the system of functions (24) as follows:

$$X_n(x) = \frac{1}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}} \widetilde{X}_n(x), \tag{25}$$

where

$$\|\widetilde{X}_n\|^2_{L_{2,\rho}(0,l)} = \int_0^l \rho(x) \widetilde{X}_n^2(x)\,dx, \qquad \rho(x) = x^k. \tag{26}$$

Let $u(x,t)$ be a solution of problem (2)–(5). Following [19, 20], we consider the functions

$$u_n(t) = \int_0^l u(x,t) x^k X_n(x)\,dx, \qquad n = 1, 2, \ldots, \tag{27}$$

where $X_n(x)$ are determined by Eqs. (25) and (26). Based on the functions in Eq. (27), we introduce the auxiliary functions

$$u_{n,\varepsilon}(t) = \int_\varepsilon^{l-\varepsilon} u(x,t) x^k X_n(x)\,dx, \qquad n = 1, 2, \ldots, \tag{28}$$

where $\varepsilon > 0$ is a sufficiently small number.

We differentiate identity (28) twice with respect to the variable t for $0 < t < T$ and, with regard to Eq. (1), obtain

$$\begin{aligned} u''_{n,\varepsilon}(t) &= \int_\varepsilon^{l-\varepsilon} u_{tt}(x,t) x^k X_n(x)\,dx = \int_\varepsilon^{l-\varepsilon} \left(u_{xx} + \frac{k}{x} u_x\right) x^k X_n(x)\,dx \\ &= \int_\varepsilon^{l-\varepsilon} \frac{\partial}{\partial x}(x^k u_x) X_n(x)\,dx = x^k u_x X_n(x)\big|_\varepsilon^{l-\varepsilon} - \int_\varepsilon^{l-\varepsilon} x^k u_x X_n'(x)\,dx. \end{aligned} \tag{29}$$

From Eq. (28), by virtue of Eq. (17), we have

$$u_{n,\varepsilon}(t) = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,t)x^k\left[X_n''(x) + \frac{k}{x}X_n'(x)\right]dx = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,t)\frac{d}{dx}(x^k X_n'(x))\,dx$$

$$= -\frac{1}{\lambda_n^2}\left[u(x,t)x^k X_n'(x)\big|_{\varepsilon}^{l-\varepsilon} - \int\limits_{\varepsilon}^{l-\varepsilon} x^k u_x X_n'(x)\,dx\right],$$

which implies

$$\int\limits_{\varepsilon}^{l-\varepsilon} x^k u_x X_n'(x)\,dx = \lambda_n^2 u_{n,\varepsilon}(t) + u(x,t)x^k X_n'(x)\big|_{\varepsilon}^{l-\varepsilon}. \tag{30}$$

Taking relation (30) into account in representation (29), we obtain

$$u_{n,\varepsilon}''(t) = x^k u_x X_n(x)\big|_{\varepsilon}^{l-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(t) - u(x,t)x^k X_n'(x)\big|_{\varepsilon}^{l-\varepsilon}. \tag{31}$$

Now, as in the proof of the uniqueness of the solution of the problem, we multiply Eq. (1) by x^k and integrate it for a fixed $t \in (0,T)$ over the variable x from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. As a result, we obtain

$$\frac{d^2}{dt^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,t)x^k\,dx - \left(x^k\frac{\partial u}{\partial x}\right)\bigg|_{\varepsilon}^{l-\varepsilon} = 0.$$

Now we pass to the limit as $\varepsilon \to 0$ in the obtained relation and, by virtue of conditions (3) and (5), arrive at the equation

$$\frac{d^2}{dt^2}(u_x(l,t)) + l^k u_x(l,t) = 0.$$

We introduce the notation $Z(t) = u_x(l,t)$ and, as a result, obtain the ordinary differential equation

$$Z''(t) + l^k Z(t) = 0,$$

which has a general solution of the form $Z(t) = \widetilde{C}_1\cos(\sqrt{l^k}\,t) + \widetilde{C}_2\sin(\sqrt{l^k}\,t)$, where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. Then

$$u_x(l,t) = \widetilde{C}_1\cos(\sqrt{l^k}\,t) + \widetilde{C}_2\sin(\sqrt{l^k}\,t). \tag{32}$$

With regard to the initial conditions in Eq. (4), from Eq. (32) we obtain the values of the constants $\widetilde{C}_1 = \varphi'(l)$ and $\widetilde{C}_2 = l^{-k/2}\psi'(l)$.

It follows from Eq. (25) that $X_n(x) = O(1)$ and $X_n'(x) = O(x)$ as $x \to 0$. Then, with regard to conditions (3), (20) and relation (32), we pass in (31) to the limit as $\varepsilon \to 0$ and derive the following equation for determining the functions $u_n(t)$:

$$u_n''(t) + \lambda_n^2 u_n(t) = C_1\cos(\sqrt{l^k}\,t) + C_2\sin(\sqrt{l^k}\,t), \qquad t \in (0,T), \tag{33}$$

where

$$C_1 = \widetilde{C}_1 l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l) = \varphi'(l) l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l),$$
$$C_2 = \widetilde{C}_2 l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l) = \psi'(l)\sqrt{l} J_{(k-1)/2}(\lambda_n l).$$

The general solution of a homogeneous equation that corresponds to Eq. (33) has the form

$$u_n(t) = a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t),$$

where a_n and b_n are arbitrary constants.

We seek a particular solution of Eq. (33) in the form

$$v_n(t) = A\cos(\sqrt{l^k}\,t) + B\sin(\sqrt{l^k}\,t), \tag{34}$$

where A and B are the coefficients to be determined. Substituting the sought-for solution (34) into Eq. (33), we obtain

$$A = \frac{C_1}{\lambda_n^2 - l^k} = \frac{\varphi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l), \qquad B = \frac{C_2}{\lambda_n^2 - l^k} = \frac{\psi'(l)}{\lambda_n^2 - l^k}\sqrt{l} J_{(k-1)/2}(\lambda_n l).$$

Then the partial solution (34) of Eq. (33) is determined by the relation

$$v_n(t) = \frac{\sqrt{l}}{\lambda_n^2 - l^k}(\sqrt{l^k}\,\varphi'(l)\cos(\sqrt{l^k}\,t) + \psi'(l)\sin(\sqrt{l^k}\,t))J_{(k-1)/2}(\lambda_n l), \tag{35}$$

and its general solution becomes

$$u_n(t) = a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t) + v_n(t). \tag{36}$$

To determine arbitrary constants a_n and b_n, we impose the initial conditions (4) on the functions (27), i.e.,

$$u_n(0) = \int_0^l u(x,0)x^k X_n(x)\,dx = \int_0^l \varphi(x)x^k X_n(x)\,dx = \varphi_n, \tag{37}$$

$$u_n'(0) = \int_0^l u_t(x,0)x^k X_n(x)\,dx = \int_0^l \psi(x)x^k X_n(x)\,dx = \psi_n. \tag{38}$$

With regard to conditions (37) and (38), from Eqs. (36) and (35) we obtain

$$u_n(0) = a_n + \frac{\varphi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l) = \varphi_n,$$
$$u_n'(0) = b_n\lambda_n + \frac{\psi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l) = \psi_n,$$

which implies

$$a_n = \varphi_n - \frac{\varphi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l), \qquad b_n = \frac{\psi_n}{\lambda_n} - \frac{\psi'(l)}{(\lambda_n^2 - l^k)\lambda_n} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l). \tag{39}$$

Substituting the values (39) into (36), we find the ultimate form of the functions

$$\begin{aligned} u_n(t) = \varphi_n\cos(\lambda_n t) + \frac{\psi_n}{\lambda_n}\sin(\lambda_n t) + \frac{\varphi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l)(\cos(\sqrt{l^k}\,t) - \cos(\lambda_n t)) \\ + \frac{\psi'(l)}{\lambda_n^2 - l^k} l^{(k+1)/2} J_{(k-1)/2}(\lambda_n l)\left(l^{-k/2}\sin(\sqrt{l^k}\,t) - \frac{1}{\lambda_n}\sin(\lambda_n t)\right). \end{aligned} \tag{40}$$

We use the obtained partial solutions in Eqs. (25) and (40) to write the solution of problem (2)–(5) formally as a series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x), \tag{41}$$

where the functions $X_n(x)$ are determined by formula (25) and the functions $u_n(t)$ by formula (40).

Along with the series in Eq. (41), we consider the series

$$u_t(x,t)=\sum_{n=1}^{\infty}u_n'(t)X_n(x),\qquad u_x(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n'(x),\tag{42}$$

$$u_{tt}(x,t)=\sum_{n=1}^{\infty}u_n''(t)X_n(x),\qquad u_{xx}(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n''(x).\tag{43}$$

Now we prove the uniform convergence of the series in Eqs. (41)–(43) in the domain $\overline{D}$ under some additional conditions imposed on the functions $\varphi(x)$ and $\psi(x)$.

Lemma 1. *The following estimates hold for sufficiently large n and for any $t\in[0,T]$:*

$$|u_n(t)|\le C_1\left(|\varphi_n|+\frac{|\psi_n|}{n}\right)+\frac{|\varphi'(l)|}{n^{5/2}}+\frac{|\psi'(l)|}{n^{5/2}},\tag{44}$$

$$|u_n'(t)|\le C_2(n|\varphi_n|+|\psi_n|)+\frac{|\varphi'(l)|}{n^{3/2}}+\frac{|\psi'(l)|}{n^{5/2}},\tag{45}$$

$$|u_n''(t)|\le C_3(n^2|\varphi_n|+n|\psi_n|)+\frac{|\varphi'(l)|}{n^{1/2}}+\frac{|\psi'(l)|}{n^{3/2}};\tag{46}$$

here and below, C_i are positive constants.

Proof. The proof of the estimates in Eqs. (44)–(46) follows from formulas (40) and (23).

Lemma 2. *The following estimates hold for sufficiently large n and for all $x\in[0,l]$:*

$$|X_n(x)|\le C_4,\qquad |X_n'(x)|\le C_5 n,\qquad |X_n''(x)|\le C_6 n^2.\tag{47}$$

Proof. It is known that, for large ξ,

$$J_\nu(\xi)=O(\xi^{-1/2}).\tag{48}$$

From (26) we obtain

$$\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}=\frac{l}{\sqrt{2}}|J_{(k+1)/2}(\mu_n)|.\tag{49}$$

Then it follows from Eqs. (48) and (49) that

$$\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}=O(n^{-1/2})\quad\text{as}\quad n\to\infty.\tag{50}$$

With regard to Eq. (49), formula (25) becomes

$$X_n(x)=\frac{1}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}}\widetilde{X}_n(x)=\frac{\sqrt{2}x^{(1-k)/2}J_{(k-1)/2}(\lambda_n x)}{l|J_{(k+1)/2}(\mu_n)|}.\tag{51}$$

Then the first estimate in Eq. (47) follows from relations (48), (50), and (51).

Now we calculate

$$\widetilde{X}_n'(x)=-\lambda_n x^{(1-k)/2}J_{(k+1)/2}(\lambda_n x).\tag{52}$$

Then the second estimate in Eq. (4.7) follows from Eqs. (48), (50), and (52).

Equation (17) implies the relation

$$\widetilde{X}_n''(x)=-\frac{k}{x}\widetilde{X}_n'(x)-\lambda_n^2\widetilde{X}_n(x).$$

By virtue of the first two estimates, this implies the third estimate in Eq. (47). The lemma is proved.

Lemma 3. *If a function $\varphi(x)$ belongs to the space $C^2[0,l]$ and its derivative $\varphi'''(x)$ exists and is of finite variation on the interval $[0,l]$; a function $\psi(x)$ belongs to the space $C^1[0,l]$ and its derivative $\psi''(x)$ exists and is of finite variation on the interval $[0,l]$; and the relations*

$$\varphi'(0)=\varphi''(0)=\psi'(0)=\varphi'(l)=\psi'(l)=0$$

are satisfied, then the following estimates hold true:

$$|\varphi_n|\le\frac{C_7}{n^4},\qquad |\psi_n|\le\frac{C_8}{n^3}. \tag{53}$$

Proof. With regard to Eqs. (17) and (20) and the conditions of the lemma, we twice apply integration by parts to Eq. (37) to obtain

$$\begin{aligned}\varphi_n&=\int_0^l\varphi(x)x^kX_n(x)\,dx=-\frac{1}{\lambda_n^2}\int_0^l\varphi(x)x^k\left[X_n''(x)+\frac{k}{x}X_n'(x)\right]dx=-\frac{1}{\lambda_n^2}\int_0^l\varphi(x)(x^kX_n'(x))'\,dx\\&=-\frac{1}{\lambda_n^2}\left[\varphi(x)x^kX_n'(x)|_0^l-\int_0^l\varphi'(x)x^kX_n'(x)\,dx\right]=\frac{1}{\lambda_n^2}\int_0^l\varphi'(x)x^kX_n'(x)\,dx\\&=\frac{1}{\lambda_n^2}\left[\varphi'(x)x^kX_n(x)|_0^l-\int_0^l(\varphi'(x)x^k)'X_n(x)\,dx\right]=-\frac{1}{\lambda_n^2}\int_0^l(\varphi'(x)x^k)'X_n(x)\,dx\\&=-\frac{1}{\lambda_n^2}\int_0^l\varphi''(x)x^kX_n(x)\,dx-\frac{k}{\lambda_n^2}\int_0^l\frac{\varphi'(x)}{x}x^kX_n(x)\,dx.\end{aligned}$$

We introduce the notation

$$\varphi_n^{(2)}=\int_0^l\varphi''(x)x^kX_n(x)\,dx,\qquad \varphi_{1n}=\int_0^l\varphi_1(x)x^kX_n(x)\,dx,\qquad \varphi_1(x)=\frac{\varphi'(x)}{x}, \tag{54}$$

and, as a result, obtain

$$\varphi_n=-\frac{1}{\lambda_n^2}\varphi_n^{(2)}-\frac{k}{\lambda_n^2}\varphi_{1n}. \tag{55}$$

By virtue of Eqs. (17) and (20), from the first integral in Eq. (54) we obtain the relations

$$\begin{aligned}\varphi_n^{(2)}&=\int_0^l\varphi''(x)x^kX_n(x)\,dx=-\frac{1}{\lambda_n^2}\int_0^l\varphi''(x)(x^kX_n'(x))'\,dx\\&=-\frac{1}{\lambda_n^2}\left[\varphi''(x)x^kX_n'(x)|_0^l-\int_0^l\varphi'''(x)x^kX_n'(x)\,dx\right]=\frac{1}{\lambda_n^2}\int_0^l\varphi'''(x)x^kX_n'(x)\,dx=\frac{\varphi_n^{(3)}}{\lambda_n^2},\end{aligned} \tag{56}$$

where

$$\varphi_n^{(3)}=\int_0^l\varphi'''(x)x^kX_n'(x)\,dx.$$

It follows from Eq. (52) that

$$\varphi_n^{(3)} = -\frac{\lambda_n}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}} \int\limits_0^l \varphi'''(x) x^{1+k} J_{(k+1)/2}(\lambda_n x)\, dx.$$

Since the function $x^{1/2} f(x) = x^{1/2}\varphi'''(x)x^k = \varphi'''(x)x^{k+1/2}$ is of bounded variation on the interval $[0, l]$ [21, p. 202], it follows, based on the theorem in [17, p. 653], that

$$\int\limits_0^l \varphi'''(x) x^{1+k} J_{(k+1)/2}(\lambda_n x)\, dx = O\left(\frac{1}{\lambda_n^{3/2}}\right)$$

as $n \to \infty$. Then, by virtue of the asymptotic relation in Eq. (50), we have $\varphi_n^{(3)} = O(1)$ for large n, and hence, the following estimate holds true:

$$|\varphi_n^{(3)}| \le C_9. \tag{57}$$

Similarly, from the second relation in Eq. (54), based on Eqs. (17), (20) and the conditions of the lemma, we obtain the integral

$$\varphi_{1n} = \int\limits_0^l \varphi_1(x) x^k X_n(x)\, dx = -\frac{1}{\lambda_n^2} \int\limits_0^l \varphi_1(x)(x^k X_n'(x))'\, dx$$

$$= -\frac{1}{\lambda_n^2}\left[\varphi_1(x) x^k X_n'(x)\big|_0^l - \int\limits_0^l \varphi_1'(x) x^k X_n'(x)\, dx\right] = \frac{1}{\lambda_n^2} \int\limits_0^l \varphi_1'(x) x^k X_n'(x)\, dx = \frac{\varphi_{1n}^{(1)}}{\lambda_n^2}, \tag{58}$$

where

$$\varphi_{1n}^{(1)} = \int\limits_0^l \varphi_1'(x) x^k X_n'(x)\, dx,$$

and, in view of Eq. (52), this integral converges.

Now we estimate the integral $\varphi_{1n}^{(1)}$ for large n. We represent it as

$$\varphi_{1n}^{(1)} = -\frac{\lambda_n}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}} \int\limits_0^l \left(\frac{\varphi'(x)}{x}\right)' x^k x^{(1-k)/2} J_{(k+1)/2}(\lambda_n x)\, dx$$

$$= -\frac{\lambda_n}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}} \int\limits_0^l \left[\frac{\varphi''(x)}{x} - \frac{\varphi'(x)}{x^2}\right] x^{(k+1)/2} J_{(k+1)/2}(\lambda_n x)\, dx$$

$$= -\frac{\lambda_n}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}} \int\limits_0^l x\left[\varphi''(x) - \frac{\varphi'(x)}{x}\right] x^{(k-3)/2} J_{(k+1)/2}(\lambda_n x)\, dx.$$

Since $\varphi'(0) = \varphi''(0) = 0$ by the condition of the lemma, for x such that $0 \le x \le \delta$, where $\delta > 0$ is a sufficiently small number, we obtain

$$\varphi'(x) = \varphi'(0) + \frac{\varphi''(0)}{1!}x + \frac{\varphi'''(\xi x)}{2!}x^2 = \frac{1}{2}\varphi'''(\xi x)x^2, \qquad 0 < \xi < x,$$

$$\varphi''(x) = \varphi''(0) + \frac{\varphi'''(\theta x)}{1!}x = \varphi'''(\theta x)x, \qquad 0 < \theta < x.$$

In consequence of these representations, the function

$$x^{1/2}f(x) = x^{1/2}\left[\varphi''(x) - \frac{\varphi'(x)}{x}\right]x^{(k-3)/2} = \left[\varphi''(x) - \frac{\varphi'(x)}{x}\right]x^{k/2-1} = \left[\varphi'''(\theta x) - \frac{1}{2}\varphi'''(\xi x)\right]x^{k/2}$$

is of bounded variation on the interval $[0, \delta]$ as the product of two functions with a finite variation [21, p. 202].

On the interval $[\delta, l]$, we can similarly show that the function $x^{1/2}f(x)$ is of finite variation. Then it is of finite variation on the entire interval $[0, l]$. Therefore, as in the case of the integral $\varphi_n^{(3)}$, we have the estimate

$$|\varphi_{1n}^{(1)}| \le C_{10}. \tag{59}$$

Substituting expressions (56) and (58) into Eq. (55), we arrive the relation

$$\varphi_n = -\frac{1}{\lambda_n^4}\varphi_n^{(3)} - \frac{k}{\lambda_n^4}\varphi_{1n}^{(1)},$$

which together with inequalities (57) and (59) imply the first estimate in Eq. (53).

Based on the conditions in Eq. (38) for $\psi'(l) = 0$, after similar calculations, we obtain

$$\psi_n = \int_0^l \psi(x)x^k X_n(x)\,dx$$

$$= -\frac{1}{\lambda_n^2}\int_0^l \psi''(x)x^k X_n(x)\,dx - \frac{k}{\lambda_n^2}\int_0^l \frac{\psi'(x)}{x}x^k X_n(x)\,dx = -\frac{1}{\lambda_n^2}\psi_n^{(2)} - \frac{k}{\lambda_n^2}\psi_{1n}, \tag{60}$$

where

$$\psi_n^{(2)} = \int_0^l \psi''(x)x^k X_n(x)\,dx, \qquad \psi_{1n} = \int_0^l \frac{\psi'(x)}{x}x^k X_n(x)\,dx.$$

Now we estimate the integrals $\psi_n^{(2)}$ and ψ_{1n} in a similar way. Based on formula (25), we have

$$\psi_n^{(2)} = \frac{1}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}}\int_0^l x\psi''(x)x^{(k-1)/2}J_{(k-1)/2}(\lambda_n x)\,dx.$$

The latter and the fact that the function $x^{1/2}f(x) = \psi''(x)x^{k/2}$ is of finite variation entail the estimate

$$\psi_n^{(2)} = O(\lambda_n^{-1}) \tag{61}$$

as $n \to \infty$. Similarly, we have

$$\psi_{1n} = \frac{1}{\|\widetilde{X}_n\|_{L_{2,\rho}(0,l)}}\int_0^l x\psi'(x)x^{(k-3)/2}J_{(k-1)/2}(\lambda_n x)\,dx.$$

By the condition of the lemma, $\psi'(0) = 0$, and hence, the function

$$x^{1/2}f(x) = \psi'(x)x^{k/2-1} = \psi''(\theta x)x^{k/2}, \qquad 0 < \theta < x \le \delta,$$

where $\delta > 0$ is a small number, is of finite variation on the interval $[0, \delta]$. Similarly, the function $x^{1/2}f(x)$ is of finite variation on the interval $[\delta, l]$, and hence, on the entire interval $[0, l]$. Then, for large n we obtain the estimate

$$\psi_{1n} = O(\lambda_n^{-1}). \tag{62}$$

Thus, Eq. (60) and relations (61), (62) imply the second estimate in Eq. (53). The lemma is proved.

With regard to the conditions of Lemma 3, the coefficients $u_n(t)$ of the series in Eq. (41) become

$$u_n(t) = \varphi_n \cos(\lambda_n t) + \frac{\psi_n}{\lambda_n} \sin(\lambda_n t). \tag{63}$$

According to Lemmas 1–3, for any $(x,t) \in \overline{D}$, the series in Eqs. (41)–(43) can be majorized by the numerical series $C_{11} \sum_{n=1}^{\infty} n^{-2}$, and hence, converge uniformly in a closed domain $\overline{D}$. Thus, we have proved the following theorem.

Theorem 2. *If functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma* 3, *then there exists a unique solution to problem* (2)–(5). *This solution can be represented by the series in Eq.* (41), *with the sum of the series belonging to the space $C^2(\overline{D})$.*

4. EXISTENCE OF SOLUTION TO PROBLEM 2

As in Problem 1, we substitute the product $u(x,t) = X(x)T(t)$ into Eq. (1) and in conditions (5) and (6) to obtain the following spectral problem for $X(x)$:

$$X''(x) + \frac{k}{x} X'(x) + \lambda^2 X(x) = 0, \qquad 0 < x < l, \tag{64}$$

$$\lim_{x \to 0+} x^k X'(x) = 0, \qquad X'(l) + \int_0^l X(x)\, x^k \, dx = 0, \qquad 0 \le t \le T. \tag{65}$$

The system of eigenfunctions $\widetilde{X}_n(x)$ of problem (64), (65) has the form in Eq. (24), and the eigenvalues $\lambda_n = \mu_n / l$ $(n = 1, 2, \ldots)$ are determined as zeros of Eq. (22), with the asymptotic formula in Eq. (23) holding true for them for large n.

We introduce the norm by formula (26) and then consider the functions in Eq. (25).

As in the preceding problem, we construct the solution of problem (2)–(6) as the sum of the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{66}$$

where the functions $X_n(x)$ are determined by formula (25) and the functions $u_n(t)$ by formula (40), where the coefficients φ_n and ψ_n are given by formulas (37) and (38). The functions $u_n(t)$ and $X_n(t)$ satisfy, respectively, the estimates in Lemmas 1 and 2.

We impose the conditions of Lemma 3 on the functions $\varphi(x)$ and $\psi(x)$. Then the functions $u_n(t)$ in the series in Eq. (66) take on the form in Eq. (63). According to Lemmas 1–3, for any $(x,t) \in \overline{D}$, the series in Eq. (66) and its derivatives up to the second order inclusively can be majorized by the numerical series $C_{12} \sum_{n=1}^{\infty} n^{-2}$. Therefore, the sum of the series in Eq. (66) satisfies conditions (2) and (3). We have thus proved the following theorem.

Theorem 3. *If functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma* 3, *then there exists a unique solution of problem* (2)–(6). *This solution can be represented by the series in Eq.* (66), *with the sum of the series belonging to the space $C^2(\overline{D})$.*

REFERENCES

1. Kipriyanov, I.A., *Singulyarnye ellipticheskie kraevye zadachi* (Singular Elliptic Boundary Value Problems), Moscow: Fizmatlit, 1997.
2. Koshlyakov, N.S., Gliner, E.B., and Smirnov, M.M., *Uravneniya v chastnykh proizvodnykh matematicheskoi fiziki* (Partial Differential Equations of Mathematical Physics), Moscow: Vysshaya Shkola, 1970.

3. Pul'kin, S.P., On the uniqueness of solution of the singular Gellerstedt problem, *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1960, no. 6 (19), pp. 214–225.
4. Sabitov, K.B. and Il'yasov, R.R., On ill-posedness of boundary-value problems for a class of hyperbolic equations, *Russ. Math. (Izv. Vyssh. Uchebn. Zaved. Mat.)*, 2001, vol. 45, no. 5, pp. 56–60.
5. Sabitov, K.B. and Il'yasov, R.R., Solution of the Tricomi problem for an equation of mixed type with a singular coefficient by the spectral method, *Russ. Math. (Izv. Vyssh. Uchebn. Zaved. Mat.)*, 2004, vol. 48, no. 2, pp. 61–68.
6. Pul'kina, L.S., A nonlocal problem with integral conditions for a hyperbolic equation, *Differ. Equations*, 2004, vol. 40, no. 7, pp. 947–953.
7. Pul'kina, L.S., Boundary-value problems for a hyperbolic equation with nonlocal conditions of I and II kind, *Russ. Math. (Izv. Vyssh. Uchebn. Zaved. Mat.)*, 2012, vol. 56, no. 4, pp. 62–69.
8. Pul'kina, L.S., *Zadachi s neklassicheskimi usloviyami dlya giperbolicheskikh uravnenii* (Problems with Nonclassical Conditions for Hyperbolic Equations), Samara: Samara Univ., 2012.
9. Benuar, N.E. and Yurchuk, N.I., Mixed problem with integral conditions for hyperbolic equations with Bessel operator, *Differ. Equations*, 1991, vol. 27, no. 12, pp. 1482–1487.
10. Sabitova, Yu.K., Nonlocal initial boundary-value problems for a degenerate hyperbolic equation, *Russ. Math. (Izv. Vyssh. Uchebn. Zaved. Mat.)*, 2009, vol. 53, no. 12, pp. 41-49.
11. Sabitov, K.B., Boundary-value problem for a parabolic-hyperbolic equation with a nonlocal integral condition, *Differ. Equations*, 2010, vol. 46, no. 10, pp. 1472–1481.
12. Sabitov, K.B., Nonlocal problem for a parabolic-hyperbolic equation in a rectangular domain, *Math. Notes*, 2011, vol. 89, nos. 3–4, pp. 562–567.
13. Sabitov, K.B., Boundary-value problem with nonlocal integral condition for mixed-type equations with degeneracy on the transition line, *Math. Notes*, 2015, vol. 98, nos. 3–4, pp. 454–465.
14. Keldysh, M.V., On some cases of degeneracy of elliptic-type equations on the boundary of a domain, *Dokl. Akad. Nauk SSSR*, 1951, vol. 77, no. 2, pp. 181–183.
15. Sabitov, K.B., *K teorii uravnenii smeshannogo tipa* (To the Theory of Mixed-Type Equations), Moscow: Fizmalit, 2014.
16. Bateman, H. and Erdelyi, A., *Higher Transcendental Functions. Tables of Integral Transforms*, McGraw-Hill, 1954.
17. Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd.ed., Cambridge: Cambridge University, 1966, Pt. 1.
18. Olver, F.W.J., *Asymptotics and Special Functions*, New York: Academic, 1974.
19. Sabitov, K.B. and Vagapova, E.V., Dirichlet problem for an equation of mixed type with two degeneration lines in a rectangular domain, *Differ. Equations*, 2013, vol. 49, no. 1, pp. 68–78.
20. Safina, R.M., Keldysh problem for a mixed-type equation of the second kind with the Bessel operator, *Differ. Equations*, 2015, vol. 51, no. 10, pp. 1347–1359.
21. Natanson, I.P., *Teoriya funktsii veshchestvennoi peremennoi* (Theory of Functions of a Real Variable), Moscow: Nauka, 1974.