Izvestiya, Physics of the Solid Earth, Vol. 40, No. 4, 2004, pp. 276–292. Translated from Fizika Zemli, No. 4, 2004, pp. 12–29.

Inverse Problems of Magnetotellurics: A Modern Formulation

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Received October 23, 2003

Abstract—Inverse magnetotelluric (MT) problems are considered in a formulation reflecting the methodological and technological progress in modern magnetotellurics. Relations between MT and magnetovariational (MV) problems are determined. The theorem of uniqueness is proven for 2-D MV inversion. Features inherent in the multidimensional geoelectric interpretation are examined and its rigorous mathematical formulation is proposed. The notion of a model error is introduced and principles of constructing interpretational models are developed.

INTRODUCTION

Progress in modern magnetotellurics is related to the striking technological and methodological advances that took place in this field of exploration and deep geophysics over the past decade. Field instruments ensuring a stable determination of magnetotelluric (MT) and magnetovariational (MV) characteristics have been created. Effective programs have been developed for automated 2- and 3-D inversion of impedances and tippers. MV sounding, for many years taking a back seat to magnetotellurics, became a basic method of deep geoelectric studies that is free from the distorting effect of local near-surface heterogeneities (geoelectric noise). New approaches to the analysis and interpretation of MT and MV data widening the geological and geophysical informativeness of geoelectrics are proposed. Field investigations that have been conducted in many tectonic provinces of the world have provided basically new information on the structure of the sedimentary cover, solid crust, and upper mantle.

Presently, it is evident that all these results need to be generalized and the development of a theory providing a methodological basis for modern magnetotellurics is a challenge of current research. In this work, we attempt to answer some of the relevant questions.

1. GENERAL DEFINITIONS

The main MT characteristic (response function) is the impedance tensor [**Z**], determined from the relation between horizontal components of electric and magnetic fields

$$\mathbf{E}_\tau = [\mathbf{Z}]\mathbf{H}_\tau, \tag{1}$$

where

$$\mathbf{E}_\tau = \mathbf{E}_\tau(E_x, E_y)$$

$$[\mathbf{Z}] = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \quad \mathbf{H}_\tau = \mathbf{H}_\tau(H_x, H_y).$$

The main MV characteristic (response function) is the tipper [**W**] (the Wiese–Parkinson vector), determined from the relation between the vertical component of the magnetic field and its horizontal components:

$$H_z = [\mathbf{W}]\mathbf{H}_\tau, \tag{2}$$

where

$$[\mathbf{W}] = \begin{bmatrix} W_{zx} & W_{zy} \end{bmatrix}.$$

The inverse MT problem consists in the determination of the geoelectric structure of the Earth from a known dependence of the MT and MV response functions on the coordinates of the surface observation point and the frequency of the observed electromagnetic field.

The electrical conductivity $\sigma(x, y, z)$ is found from the conditions

$$\|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\| \le \delta_Z, \tag{3a}$$

$$\|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\| \le \delta_W, \tag{3b}$$

where $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$ are the impedance tensor and tipper determined on the sets of surface points $M(x, y)$ and frequencies ω; δ_Z and δ_W are determination errors of $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$; and [**Z**] and **W** are operators of the forward problem that calculate the impedance tensor and tipper, depending parametrically on x, y, and ω, from a given electrical conductivity $\sigma(x, y, z)$.

Inverse problem (3) includes MT inversion (3a) and MV inversion (3b) and is solved in the class of piecewise-homogeneous and piecewise-continuous models excited by a plane wave vertically incident on the

Earth's surface. Inversions (3a) and (3b) should be mutually consistent. They result in a distribution $\sigma(x, y, z)$ such that misfits of the impedance tensor and tipper do not exceed errors in the initial information δ_Z and δ_W. This distribution generates the set of equivalent inversion solutions $\Sigma_{\bar{\delta}}$.

Errors in the initial information δ_Z and δ_W consist of measurement and model errors. The measurement errors are commonly random. They arise due to instrumental noise, external interferences, and uncertainties involved in the calculation of $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$. Improvements in instrumentation and data processing methods decrease these errors. Presently, due to progress in MT technologies, measurement uncertainties are, as a rule, fairly small (at least, far from sources of intense industrial noise). A main difficulty is related to model errors that arise due to the inevitable deviation of inversion models from real geoelectric structures and the real MT field. As an illustrative example, we consider uncertainties arising in the 2-D inversion of data obtained above 3-D structures and uncertainties typical of polar zones, where the magnetic field of ionospheric currents has a vertical component and cannot be approximated by a plane wave. Model errors are systematic and can be estimated with the use of mathematical modeling. Model uncertainties are usually larger than measurement uncertainties.

The strategy and informativeness of the inverse problems depend on the dimensionality of models in use.

The simplest inverse problem is 1-D inversion, carried out in the class of 1-D models. It provides the local determination of the electrical conductivity along vertical profiles passing through observation points. The 1-D inversion evidently ignores distortions produced by horizontal geoelectric inhomogeneities and is justified if horizontal variations in the conductivity are fairly small. Otherwise, it can miss real structures and provide false structures (artifacts).

The transition to 2- and 3-D inversions, carried out in the classes of 2- and 3-D models, enables the due regard for the effects of horizontal geoelectric inhomogeneities but substantially complicates the inverse problem.

(1) A contradiction arises between the finite region of MT and MV observations and the infinite region of the inverse problem. In forward problems, this contradiction is easily removed through the introduction of an infinite normal layered structure outside the observation region. In the inverse problem, the normal structure of the medium is unknown and should be specified as a mathematical abstraction consistent with data of observations. Such a structure can be constructed by the extrapolation of scalar invariants of the impedance tensor, for example, the invariant Z_B (the Berdichevsky impedance). Let values of the impedance tensor $[\mathbf{Z}]$ be determined in an observation region S_0 bounded by a contour C_0 and let $[\mathbf{Z}^{(m)}]$, $m = 1, 2, \ldots, M$ be specified at M points of C_0 (Fig. 1). The average value of the invariant Z_B on the contour C_0, i.e., on the boundary of the observation region, is found as

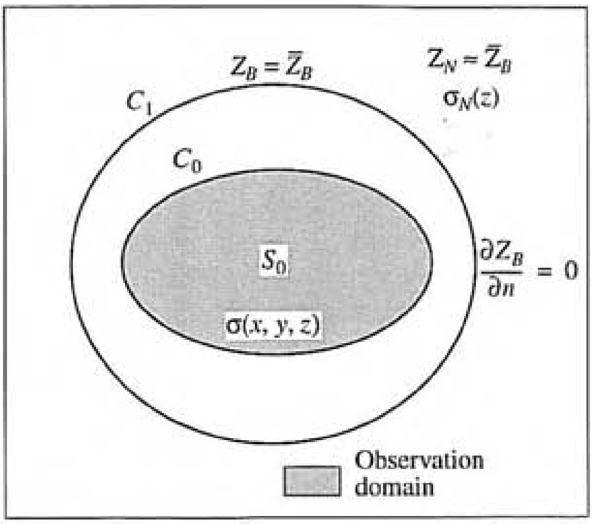

Fig. 1.

$$\bar{Z}_B = \frac{1}{M}\sum_{m=1}^{M} Z_B^{(m)} = \frac{1}{M}\sum_{m=1}^{M}\frac{Z_{xy}^{(m)} - Z_{yx}^{(m)}}{2}.$$

Using a spline approximation, the values Z_B are extrapolated in such a way that the condition $Z_B = \bar{Z}_B$ is valid on a new boundary contour C_1 and the derivative of Z_B along the normal to C_1 vanishes. Given these conditions, we assume that the impedance $\bar{Z}_B$ is close to the normal impedance Z_N of a horizontally layered medium in the region external with respect to C_1 and determine the normal conductivity $\sigma_N(z)$ by the 1-D inversion of the impedance $\bar{Z}_B$. To test this algorithm, one should make sure that an increase in the distance between the boundaries C_0 and C_1 has no significant effect on the results of MT and MV inversions in the central part of the observation region S_0.

A similar algorithm based on the averaging and extrapolation of longitudinal and transverse components of the impedance tensor can be applied in a 2-D approximation of elongated structures. Let observations be performed on a transverse profile S_0 from $y = -c_0$ to $y = c_0$ (Fig. 2). The average of the invariant Z_B at the ends of the profile is determined as

$$\bar{Z}_B = \frac{1}{2}\{Z_B(y = -c_0) + Z_B(y = c_0)\}$$

$$= \frac{Z^{\parallel}(-c_0) + Z^{\perp}(-c_0) + Z^{\parallel}(c_0) + Z^{\perp}(c_0)}{4}.$$

Applying a spline approximation, the values Z_B are then extrapolated beyond the profile in such a way that the

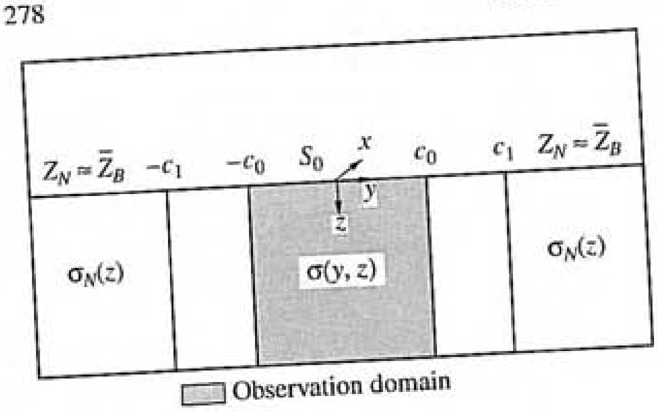

Fig. 2.

conditions $Z_B = \bar{Z}_B$ and $\partial Z_B/\partial y = 0$ are valid at the points $y = -c_1$ and $y = c_1$. This extrapolation yields a model in which the observation region is symmetrically surrounded by a horizontally homogeneous medium that has the normal impedance $Z_N = \bar{Z}_B$ and a normal conductivity $\sigma_N(z)$ determined from the 1-D inversion of the impedance $\bar{Z}_B$:

$$\sigma(y, z) = \begin{cases} \sigma_N(z) & y < -c_1 \\ \sigma(y, z) & -c_1 < y < c_1 \\ \sigma_N(z) & y > c_1. \end{cases}$$

Evidently, the introduction of a symmetric normal surrounding medium is quite justified if the values of Z_B at the ends of the profile are similar. Otherwise, one should introduce an asymmetric normal model of the surrounding medium. Such models do not require special analysis, because any asymmetric model can be reduced by its mirror reflection to a symmetric model.

(2) Multidimensional inversions are less stable. This is explained by the fact that, compared to a 1-D model, a much greater number of parameters is required for an adequate description of 2- and 3-D models. It is evident that multidimensional inverse problems are distinguished by a stronger contradiction between the detailedness of their solutions and their stability, controlling the resolution of the inversion [Berdichevsky and Dmitriev, 2002]. The detailedness of inversion should be consistent with its resolution. Therefore, in solving 2-D and particularly 3-D problems, one should smooth or schematize models of the geoelectric medium. This complies with the nature of the electromagnetic field, which provides information on smoothed structures and their integral characteristics.

(3) The third property of multidimensional inversions is the redundancy of experimental data. In the general case, the real scalar function of conductivity $\sigma(x, y, z)$ is found from four complex-valued components of the impedance tensor $[\mathbf{Z}(x, y, \omega)]$ and two complex-valued components of the tipper $\mathbf{W}(x, y, \omega)$, i.e., from 12 scalar functions. However, solving a 2-D inverse problem, one separates the galvanic and induction effects associated with the TM and TE modes and reduces experimental data to two complex-valued components of the impedance tensor and one complex-valued component of the tipper. These components differ in both stability with respect to near-surface distortions and sensitivity to various target structures. From the standpoint of informativeness, they complement each other, and their successive separate focused inversions can give the most complete information on sought-for geoelectric structures. The galvanic and induction effects are nonseparable in 3-D inverse problems, and separate inversions of all six components of the impedance tensor and tipper are hardly effective in this case, because their informativeness is poorly known (not to mention the laboriousness and instability of such an interpretation). The best approach appears to be the "scalarization" of a 3-D inverse problem, i.e., the determination of conductivity from scalar invariants of the impedance tensor (e.g., using the invariant $Z_{\text{eff}} = \sqrt{Z_{xx}Z_{yy} - Z_{xy}Z_{yx}}$ or $Z_B = (Z_{xy} - Z_{yx})/2)$ and the tipper (e.g., using the invariant $W = \sqrt{W_{zx}^2 + W_{zy}^2}$). This approach includes two informativeness levels: (1) MV inversion (i.e., inversion of the scalar invariant of the tipper), which provides information on deep structures that is free from near-surface distortions, and (2) MT inversion (i.e., inversion of the scalar invariant of the impedance tensor), which can contain errors due to near-surface distortions but provides information on structures producing strong galvanic anomalies. Note that the scalarization of the 3-D inverse problem (notwithstanding a substantial simplification of the interpretation procedure) requires significant computational resources, because two forward problems for two different polarizations of the primary field should be solved at each iteration step in order to determine the impedance tensor and tipper. The required computational resources can be substantially reduced by using the method of synthetic fields. In this method, only one forward problem is solved at each iteration step, because the conductivity is found directly from the magnetic (or electromagnetic) field synthesized at the Earth's surface from a known distribution of the impedance tensor or tipper.

2. INVERSE PROBLEM IN ONE-, TWO-, AND THREE-DIMENSIONAL MODELS

Here, we address inverse problem (3) and determine the operators of the forward problem $[\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]$ and $\mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}$; at each iteration step, they calculate the impedance tensor and tipper using an approximate distribution of conductivity $\sigma(x, y, z)$. Obviously, these operators depend on the dimensionality of models used.

Inversion in the Class of 1-D Models

We consider a 1-D model in which the electrical conductivity $\sigma(z)$ is a piecewise-constant function of depth z:

$$\sigma(z) = \sigma_n \text{ at } z_{n-1} < z < z_n,$$
$$n \in [1, N], z_0 = 0, \ z_N = \infty, \ h_n = z_n - z_{n-1},$$

where σ_n and h_n are the conductivity and thickness of the nth layer, respectively. The model rests at the depth $z = z_{N-1}$ on an infinite homogeneous basement having a conductivity $\sigma_N = \text{const}$. The scalar impedance Z of this model can be directly determined from the Riccati equation. Therefore, determining $\sigma(z)$ from the impedance $\tilde{Z}$, we construct the operator $Z\{z = 0, \omega, \sigma(z)\}$ with the use of the Riccati equation:

$$\frac{dZ(z, \omega)}{dz} - \sigma(z)Z^2(z, \omega) = i\omega\mu_0, \quad z \in [0, z_{N-1}], \tag{4}$$

with $Z(z, \omega)$ being continuous at layer boundaries and with the boundary condition

$$Z(z_{N-1}, \omega) = (1 - i)\sqrt{\frac{\omega\mu_0}{2\sigma_N}}. \tag{5}$$

Inversion in the Class of 2-D Models

Let a 2-D model striking along the x axis contain an anomalous region $|y| \le l$ the conductivity of which is a piecewise-constant function of the horizontal coordinate y and the depth z and let this region border infinite normal regions $y < -l$ and $y > l$ in which the conductivity $\sigma_N(z)$ depends solely on the depth z (Fig. 3). Then, we have

$$\sigma = \begin{cases} \sigma_N(z) & y < -l \\ \sigma(y, z) & -l \le y \le l \\ \sigma_N(z) & y > l. \end{cases} \tag{6}$$

The electromagnetic field in a 2-D model separates into two independent modes: the inductive TE mode with the components E_x, H_y, and H_z and the galvanic TM mode with the components E_y, E_z, and H_x. The TE mode gives rise to the longitudinal impedance $Z^{\parallel}$ and the tipper W_{zy}, which reflect the inductive effect of geoelectric structures (inductive anomalies), whereas the TM mode gives rise to the transverse impedance $Z^{\perp}$, reflecting the galvanic effect of geoelectric structures (galvanic anomalies). Thus, we have three independent formulations of the inverse problem, separating inductive and galvanic anomalies of different physical origins.

(1) MT inductive inversion: $\sigma(y, z)$ is found from the longitudinal impedance $\tilde{Z}^{\parallel}$. To determine the operator

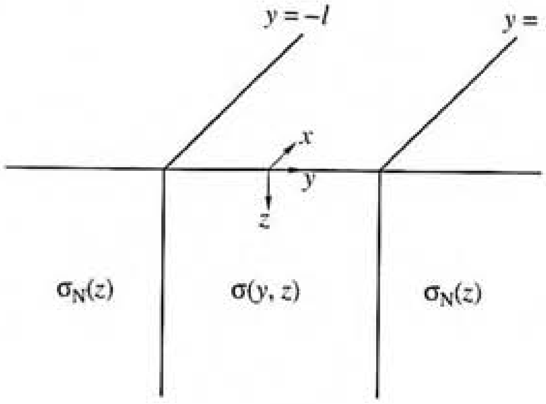

Fig. 3.

$Z^{\parallel}\{y, z = 0, \omega, \sigma(y, z)\}$, the longitudinal impedance is written in the form

$$Z^{\parallel}(y, z = 0, \omega) = \frac{E_x(y, z = 0, \omega)}{H_y(y, z = 0, \omega)} = i\omega\mu_0 \left.\frac{E_x(y, z, \omega)}{\frac{\partial E_x(y, z, \omega)}{\partial z}}\right|_{z=0}, \tag{7}$$

where $E_x(y, z, \omega)$ is obtained from the Helmholtz equations

$$\frac{\partial^2 E_x(y, z, \omega)}{\partial y^2} + \frac{\partial^2 E_x(y, z, \omega)}{\partial z^2} + i\omega\mu_0\sigma_N(z)E_x(y, z, \omega) = 0 \quad |y| > l,$$
$$\frac{\partial^2 E_x(y, z, \omega)}{\partial y^2} + \frac{\partial^2 E_x(y, z, \omega)}{\partial z^2} + i\omega\mu_0\sigma(y, z)E_x(y, z, \omega) = 0 \quad |y| \le l \tag{8}$$

with the conditions at infinity

$$E_x(y, z, \omega)_{|y| \to \infty} \longrightarrow E_x^N(z, \omega), \quad E_x(y, z, \omega)_{z \to \infty} \longrightarrow 0 \tag{9}$$

and the boundary conditions

$$[E_x(y, z, \omega)]_S = 0, \quad \left[\frac{\partial E_x(y, z, \omega)}{\partial n}\right]_S = 0. \tag{10}$$

Here, $E_x^N(z, \omega)$ is the normal electric field in the region surrounding the anomalous zone, and n is the normal to the boundary S between blocks or layers of different conductivities. The square brackets in (10) indicate a discontinuity of a function at the boundary S.

The anomalous electric field in air $E_x^A(y, z) = E_x(y, z) - E_x^N(z)$ satisfies the radiation condition.

According to (7) and (9), we have

$$Z^{\parallel}(y, z, \omega)_{|y| \to \infty} \longrightarrow Z_N(z, \omega) = \frac{E_x^N(z, \omega)}{H_y^N(z, \omega)}, \quad (11)$$

where $Z_N(z, \omega)$ and $H_y^N(z, \omega)$ are the normal (1-D) impedance and the normal magnetic field in the region surrounding the anomalous zone.

(2) MV inductive inversion: $\sigma(y, z)$ is found from the tipper $\tilde{W}_{zy}$. To determine the operator $W_{zy}\{y, z = 0, \omega, \sigma(y, z)\}$, the tipper is represented in the form

$$W_{zy}(y, z = 0, \omega) = \frac{H_z(y, z = 0, \omega)}{H_y(y, z = 0, \omega)} = \left. \frac{\dfrac{\partial E_x(y, z, \omega)}{\partial y}}{\dfrac{\partial E_x(y, z, \omega)}{\partial z}} \right|_{z=0}, \quad (12)$$

where $E_x(y, z, \omega)$ is obtained from Helmholtz equations (8) with conditions at infinity (9) and with boundary conditions (10). According to (9) and (12), we have

$$W_{zy}(y, z, \omega)_{|y| \to \infty} \longrightarrow 0; \quad (13)$$

i.e., the tippers vanish as the distance from the anomalous zone tends to infinity.

(3) MT galvanic inversion: $\sigma(y, z)$ is found from the transverse impedance $\tilde{Z}^{\perp}$. To determine the operator $Z^{\perp}\{y, z = 0, \omega, \sigma(y, z)\}$, the transverse impedance is written as

$$Z^{\perp}(y, z = 0, \omega) = -\frac{E_y(y, z = 0, \omega)}{H_x(y, z = 0, \omega)} = -\left. \frac{1}{\sigma(y, z)} \frac{\dfrac{\partial H_x(y, z, \omega)}{\partial z}}{H_x(y, z = 0, \omega)} \right|_{z=0}, \quad (14)$$

where $H_x(y, z, \omega)$ is obtained from the Helmholtz equations

$$\frac{\partial^2 H_x(y, z, \omega)}{\partial y^2} + \frac{\partial^2 H_x(y, z, \omega)}{\partial z^2} + i\omega\mu_0\sigma_N(z)H_x(y, z, \omega) = 0 \quad |y| > l,$$
$$\frac{\partial^2 H_x(y, z, \omega)}{\partial y^2} + \frac{\partial^2 H_x(y, z, \omega)}{\partial z^2} + i\omega\mu_0\sigma(y, z)H_x(y, z, \omega) = 0 \quad |y| \le l, \quad (15)$$

with the conditions at infinity

$$H_x(y, z, \omega)_{|y| \to \infty} \longrightarrow H_x^N(z, \omega), \quad H_x(y, z, \omega)_{z \to \infty} \longrightarrow 0 \quad (16)$$

and the boundary conditions

$$[H_x(y, z, \omega)]_S = 0, \quad \left[\frac{1}{\sigma(y, z)} \frac{\partial H_x(y, z, \omega)}{\partial n}\right]_S = 0, \quad H_x(y, z = 0, \omega) = \text{const}. \quad (17)$$

Here, $H_x^N(z, \omega)$ is the normal magnetic field in the region surrounding the anomalous zone.

According to (14) and (17), we have

$$Z^{\perp}(y, z, \omega)_{|y| \to \infty} \longrightarrow Z_N(z, \omega) = -\frac{E_y^N(z, \omega)}{H_x^N(z, \omega)}, \quad (18)$$

where $Z_N(z, \omega)$ and $E_y^N(z, \omega)$ are the normal (1-D) impedance and the normal electric field in the region surrounding the anomalous zone.

Inversion in the Class of 3-D Models

We considered MT and MV inversions in the classes of 1- and 2-D models. It is evident that these models are connected with a rather rough mathematical abstraction and the validity of their application to an approximate description of real geoelectric structures always requires analysis and substantiation. Presently, we have a well-developed mathematical apparatus for the solution of 1- and 2-D inverse problems of magnetotellurics, and criteria for assessing the conditions favorable for the successful application of this apparatus are available. Achievements of two-dimensional magnetotellurics are widely known, but one should not forget cases of failure caused by the disregard of 3-D effects. The development of effective computational programs ensuring a sufficiently fast 3-D inversion of impedances and tippers is the main challenge of modern magnetotellurics.

Now, we address MT and MV inversions in the class of 3-D models. Let a homogeneously layered Earth with a normal conductivity $\sigma_N(z)$ depending on depth z contain a closed anomalous region V in which the conductivity $\sigma(x, y, z)$ is an arbitrary piecewise-continuous function of the horizontal coordinates x and y and the depth z (Fig. 4). This model admits two independent formulations of an inverse problem that separate MT and MV inversions with their different sensitivities to near-surface distortions.

(1) MT inversion: $\sigma(x, y, z)$ is found from the impedance tensor $[\tilde{\mathbf{Z}}]$. We determine the operator $[\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]$ with the help of the method of integral equations.

The electromagnetic field in a 3-D model satisfies the integral relations

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^N(\mathbf{r}) + \iiint_V \Delta\sigma(\mathbf{r}_v)[\mathbf{G}^E(\mathbf{r}|\mathbf{r}_v)]\mathbf{E}(\mathbf{r}_v)dV, \tag{19a}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^N(\mathbf{r}) + \iiint_V \Delta\sigma(\mathbf{r}_v)[\mathbf{G}^H(\mathbf{r}|\mathbf{r}_v)]\mathbf{E}(\mathbf{r}_v)dV, \tag{19b}$$

where $\mathbf{E}^N$ and $\mathbf{H}^N$ are the normal electric and magnetic fields, $[\mathbf{G}^E]$ and $[\mathbf{G}^H]$ are the electric and magnetic Green tensors, $\Delta\sigma = \sigma - \sigma_N$ is the excess (anomalous) conductivity, $M(\mathbf{r})$ is an arbitrary point in the Earth or on its surface, and $M_v(\mathbf{r}_v)$ is a point in the anomalous region V.

An integral equation for the electric field inside the anomalous region is readily derived from (19a). If $M(\mathbf{r}) \in V$, we have

$$\mathbf{E}(\mathbf{r}'_v) - \iiint_V \Delta\sigma(\mathbf{r}_v)[\mathbf{G}^E(\mathbf{r}'_v|\mathbf{r}_v)]\mathbf{E}(\mathbf{r}_v)dV = \mathbf{E}^N(\mathbf{r}'_v). \tag{20}$$

Solving integral equation (20) and determining the electric field $\mathbf{E}(\mathbf{r}_v)$ inside V, we substitute $\mathbf{E}(\mathbf{r}_v)$ into (19) and find the electric and magnetic fields on the Earth's surface.

An advantage of this approach consists in the fact that the electric and magnetic Green tensors are calculated only once for a given normal distribution of conductivity $\sigma_N(z)$. The conductivity $\sigma(\mathbf{r}_v)$ then changes in the iterative inversion process, and kernels of integrals are simply obtained through the multiplication of the known Green tensors by the excessive conductivity $\Delta\sigma(\mathbf{r}_v)$. This substantially shortens the computational time because kernels of integrals need not be calculated anew whenever the model of the medium changes.

The electric and magnetic fields are found for two different polarizations of the normal field:

$$\mathbf{E}^{N(1)} = \{E_x^{N(1)}, 0, 0\}, \quad \mathbf{H}^{N(1)} = \{0, H_y^{N(1)}, 0\},$$
$$\mathbf{E}^{N(2)} = \{0, E_y^{N(2)}, 0\}, \quad \mathbf{H}^{N(2)} = \{H_x^{N(2)}, 0, 0\}.$$

The resulting electromagnetic fields on the Earth's surface $\mathbf{E}^{(1)} = \{E_x^{(1)}, E_y^{(1)}, 0\}$, $\mathbf{H}^{(1)} = \{H_x^{(1)}, H_y^{(1)}, H_z^{(1)}\}$ and $\mathbf{E}^{(2)} = \{E_x^{(2)}, E_y^{(2)}, 0\}$, $\mathbf{H}^{(2)} = \{H_x^{(2)}, H_y^{(2)}, H_z^{(2)}\}$ provide the system of linear equations for the determination of impedance tensor components

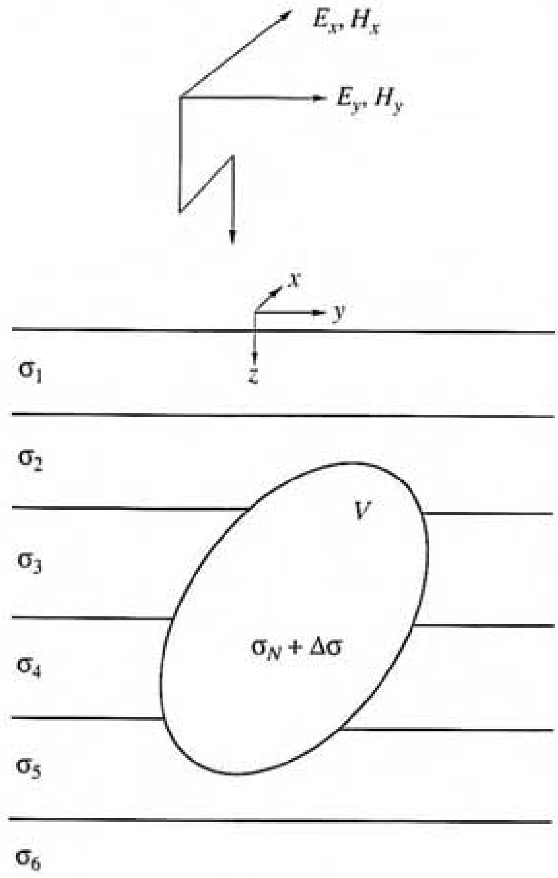

Fig. 4.

$$\begin{cases} Z_{xx}H_x^{(1)} + Z_{xy}H_y^{(1)} = E_x^{(1)}, \\ Z_{xx}H_x^{(2)} + Z_{xy}H_y^{(2)} = E_x^{(2)}, \end{cases}$$
$$\begin{cases} Z_{yx}H_x^{(1)} + Z_{yy}H_y^{(1)} = E_y^{(1)}, \\ Z_{yx}H_x^{(2)} + Z_{yy}H_y^{(2)} = E_y^{(2)}. \end{cases} \tag{21}$$

Hence,

$$Z_{xx} = \frac{E_x^{(1)}H_y^{(2)} - E_x^{(2)}H_y^{(1)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}}, \quad Z_{xy} = \frac{E_x^{(2)}H_x^{(1)} - E_x^{(1)}H_x^{(2)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}},$$
$$Z_{yx} = \frac{E_y^{(1)}H_y^{(2)} - E_y^{(2)}H_y^{(1)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}}, \quad Z_{yy} = \frac{E_y^{(2)}H_x^{(1)} - E_y^{(1)}H_x^{(2)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}}. \tag{22}$$

(2) MV inversion: $\sigma(x, y, z)$ is found from the tipper $\tilde{\mathbf{W}}$. To determine the operator $\mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}$, we use the magnetic fields $\mathbf{H}^{(1)} = \{H_x^{(1)}, H_y^{(1)}, H_z^{(1)}\}$ and $\mathbf{H}^{(2)} = \{H_x^{(2)}, H_y^{(2)}, H_z^{(2)}\}$, obtained on the Earth's

surface for two different polarizations of the normal field, and solve the system of linear equations

$$\begin{cases} W_{zx}H_x^{(1)} + W_{zy}H_y^{(1)} = H_z^{(1)}, \\ W_{zx}H_x^{(2)} + W_{zy}H_y^{(2)} = H_z^{(2)}, \end{cases} \tag{23}$$

which yields the tipper components

$$\begin{aligned} W_{zx} &= \frac{H_z^{(1)}H_y^{(2)} - H_z^{(2)}H_y^{(1)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}}, \\ W_{zy} &= \frac{H_z^{(2)}H_x^{(1)} - H_z^{(1)}H_x^{(2)}}{H_x^{(1)}H_y^{(2)} - H_x^{(2)}H_y^{(1)}}. \end{aligned} \tag{24}$$

3. THREE QUESTIONS OF HADAMARD

Starting to solve an inverse problem, one should answer three questions of Hadamard: Does a solution to this problem exist? Is it unique? Finally, is it stable with respect to small perturbations (errors) in initial data? These questions determine the correctness of the formulation of the inverse problem. If its solution exists and if it is unique and stable, the problem is well-posed. If one of these conditions is violated, the problem is regarded as ill-posed. We show that inverse problems of magnetotellurics are unstable and, therefore, ill-posed.

On the Existence of the Solution to the Inverse Problem of Magnetotellurics

At first glance, the problem of the existence of the solution appears to be simple, because the impedance tensor $[\tilde{\mathbf{Z}}]$ and the tipper $\tilde{\mathbf{W}}$ measured on a set of Earth's surface points should correspond to the really existing distribution of conductivity in the heterogeneous Earth. However, the experimental values of $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$ being in practice more or less inaccurate, conflicts between real and model conditions are possible.

Let $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$ contain measurement and model errors δ_Z and δ_W. It is evident that the real distribution of conductivity in the Earth and the real MT and MV response functions do not belong to the chosen model class on which the inverse problem (3) is defined. Such an inverse problem does not have a rigorous solution. To remove this contradiction, the notion of a quasi-solution is introduced; namely, a distribution of conductivity $\sigma(x, y, z)$ is said to be a quasi-solution of inverse problem (3) if the misfits of the impedance tensor and tipper calculated from this distribution do not exceed the errors in the initial information δ_Z and δ_W. Inverse problem (3) having a set of quasi-solutions, we have to select from this set a quasi-solution that provides the best approximation to the real geoelectric structure. This distribution of conductivity $\bar{\sigma}(x, y, z)$ is called an exact model solution. In solving the inverse problem, our goal is to find the exact model solution.

Using the notion of the exact model solution, we can formalize the definition of measurement and model errors. Let $[\bar{\mathbf{Z}}]$ and $\overline{\mathbf{W}}$ be the impedance tensor and the tipper obtained from a model that belongs to the chosen model class and has the conductivity $\bar{\sigma}(x, y, z)$. Then, measurement errors are determined as

$$\delta_Z^{ms} = \|[\tilde{\mathbf{Z}}] - [\bar{\mathbf{Z}}]\| \quad \delta_W^{ms} = \|\tilde{\mathbf{W}} - \overline{\mathbf{W}}\|, \tag{25}$$

and model errors are determined as

$$\begin{aligned} \delta_Z^{md} &= \|[\bar{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\|, \\ \delta_W^{md} &= \|\overline{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\|. \end{aligned} \tag{26}$$

Setting $\delta_Z = \delta_Z^{ms} + \delta_Z^{md}$ and $\delta_W = \delta_W^{ms} + \delta_W^{md}$ and applying the triangle rule, we reduce (25), (26) to the initial inverse problem (3).

On the Uniqueness of the Solution to the Inverse Problem of Magnetotellurics

Considering inverse problems of magnetotellurics, we proceed from the following heuristic statement. If the impedance tensor and tipper belong to a model class on which the inverse problem is defined and are exactly specified on the Earth's surface in the entire frequency range, the inverse problem has a unique solution. This statement was proven in four partial cases.

Tikhonov [1965] proved the theorem of uniqueness for 1-D MT inversion in the class of piecewise-analytical functions $\sigma(z)$. He considered this theorem as a basis for MT sounding.

We present a simplified proof of the theorem of Tikhonov for the case of a piecewise-constant distribution of the conductivity. Let $\sigma(z)$ be a piecewise-constant function of the depth z:

$$\sigma(z) = \sigma_n \text{ at } z_{n-1} < z < z_n,$$
$$n \in [1, N],\ z_0 = 0,\ z_N = \infty,\ h_n = z_n - z_{n-1},$$

where σ_n and h_n are the conductivity and thickness of the nth layer and z_n is the depth of its lower boundary. The model rests at the depth $z = z_{N-1}$ on an infinite homogeneous basement having the conductivity σ_N = const. The admittance $Y(z, \omega)$ in this homogeneously layered model satisfies the Riccati equation

$$\frac{dY(z, \omega)}{dz} + i\omega\mu_0 Y^2(z, \omega) = -\sigma(z), \qquad z \in [0, z_{N-1}],\ \omega \in [0, \infty] \tag{27}$$

with the boundary conditions

$$[Y(z, \omega)]_S = 0, \quad Y(z_{N-1}, \omega) = (1+i)\sqrt{\frac{\sigma_N}{2\omega\mu_0}}.$$

A recurrent formula expressing $Y_{n-1} = Y(z_{n-1}, \omega)$ through $Y_n = Y(z_n, \omega)$ is easily derived from the Riccati equation:

$$Y_{n-1} = \beta_n \frac{(\beta_n + Y_n) - (\beta_n - Y_n)e^{2ik_nh_n}}{(\beta_n + Y_n) + (\beta_n - Y_n)e^{2ik_nh_n}}, \tag{28}$$

where k_n is the wavenumber of the nth layer,

$$k_n = (1+i)\sqrt{\frac{\omega\mu_0\sigma_n}{2}},$$

and

$$\beta_n = \frac{k_n}{\omega\mu_0} = (1+i)\sqrt{\frac{\sigma_n}{2\omega\mu_0}}.$$

Inverting (28), we obtain a formula determining Y_n through Y_{n-1} (converting the admittance from the upper boundary of the nth layer to its lower boundary):

$$Y_n = \beta_n \frac{(\beta_n + Y_{n-1}) - (\beta_n - Y_{n-1})e^{2ik_nh_n}}{(\beta_n + Y_{n-1}) + (\beta_n - Y_{n-1})e^{2ik_nh_n}}. \tag{29}$$

Let the admittance $Y_0 = Y(0, \omega)$ be known at the Earth's surface. Then, the successive application of (29) provides the admittance $Y_n = Y(z_n, \omega)$ at any depth z_n if the distribution $\sigma(z)$ is known in the interval $0 < z < z_n$.

Now, we prove the theorem of uniqueness, which is formulated as follows. If $Y^{(1)}(z, \omega)$ and $Y^{(2)}(z, \omega)$ are the solutions of problem (27) for $\sigma^{(1)}(z)$ and $\sigma^{(2)}(z)$, then $Y_0^{(1)}(\omega) \equiv Y_0^{(2)}(\omega)$ implies that $\sigma^{(1)}(z) \equiv \sigma^{(2)}(z)$. This theorem is easily proven ad absurdum. Assume that

$$Y_0^{(1)}(\omega) \equiv Y_0^{(2)}(\omega), \tag{30a}$$

$$\sigma^{(1)}(z) \equiv \sigma^{(2)}(z) \text{ at } 0 < z < z_{n-1}, \tag{30b}$$

$$\sigma^{(1)}(z) \neq \sigma^{(2)}(z) \text{ at } z > z_{n-1}. \tag{30c}$$

Then, applying (29) to (30a) and (30b) and extending $Y_0^{(1)}$ and $Y_0^{(2)}$ to the depth z_{n-1}, we obtain $Y_{n-1}^{(1)}(\omega) \equiv Y_{n-1}^{(2)}(\omega)$. The high-frequency asymptotics of $Y_{n-1}^{(1)}(\omega)$ and $Y_{n-1}^{(2)}(\omega)$ are described, according to (28), by the formula

$$Y_{n-1}(\omega) \underset{\omega\to\infty}{\sim} \beta_n = (1+i)\sqrt{\frac{\sigma_n}{2\omega\mu_0}}.$$

Thus, the assumption $Y_{n-1}^{(1)}(\omega) \equiv Y_{n-1}^{(2)}(\omega)$ leads to $\sigma_n^{(1)} = \sigma_n^{(2)}$, which contradicts assumption (30c). Successively increasing n, we reach the model basement and obtain $\sigma^{(1)}(z) \equiv \sigma^{(2)}(z)$. The theorem of uniqueness is proven.

The next step was made by Weidelt [1978], who proved the theorem of uniqueness for a 2-D model excited by an E-polarized field. The electrical conductivity in this model is described by an analytical function $\sigma(y, z)$. The conductivity $\sigma(y, z)$ was proven to be uniquely determined by simultaneous observations of horizontal components of electric and magnetic fields in the entire frequency range $0 < \omega < \infty$ along a y-profile of a finite length.

The theorem of Weidelt was generalized by Gusarov [1981], who considered a 2-D E-polarized model with a piecewise-analytical distribution of conductivity $\sigma(y, z)$. The theorem proven by Gusarov states that the piecewise-analytical function $\sigma(y, z)$ is uniquely determined by MT inversion of the longitudinal impedance $Z^{\parallel} = Z_{xy}$ specified in the entire frequency range $0 < \omega < \infty$ on an infinite y-profile $-\infty < y < \infty$.

The idea underlying all these proofs is based on the properties of the skin effect. The latter implies the existence of a high frequency such that the field or impedance can be approximated by a high-frequency asymptotics depending on a local value of σ. Comparison of high-frequency asymptotics for various geoelectric structures suggests that different distributions of conductivity σ correspond to different fields and different impedances. Unfortunately, the realization of this simple idea encounters significant mathematical difficulties due to the complexity of the determination of high-frequency asymptotics of the field in heterogeneous media.

Resorting to intuition, the above proofs of uniqueness can be extended to the general case of MT inversions. Intuition suggests that the ω dependence of the impedance tensor (the skin effect) ensures the determination of vertical variations in the conductivity, whereas the horizontal variations in the conductivity can be determined from the dependence of the impedance tensor on x and y. Thus, it appears evident that measurements of the MT impedance made in a wide frequency range on sufficiently long profiles or in a sufficiently large area can provide information adequate for the reconstruction of the geoelectric structure of the region studied.

The problem of uniqueness of the solution of the MV inverse problem requires a special consideration. At first glance, it seems that the tipper characterizes horizontal heterogeneity of the medium and cannot provide information on its normal structure $\sigma_N(z)$ because we have $W_{zx} = W_{zy} = 0$ in a horizontally homogeneous model. However, if the medium is horizontally inhomogeneous, MV sounding can be considered as ordinary frequency sounding utilizing the magnetic field of a local embedded source. The latter can be represented by any inhomogeneity $\Delta\sigma(x, y, z)$ in which an excessive electric current is induced that spreads into the host medium. It is evident that the distribution of this current and its magnetic field depend not only on the structure of the inhomogeneity $\Delta\sigma(x, y, z)$ but also

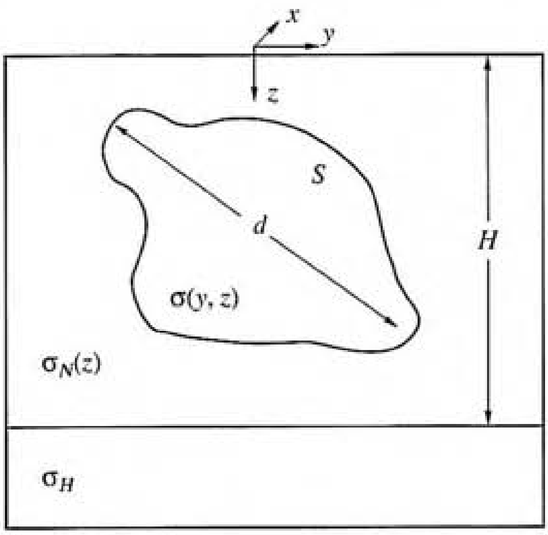

Fig. 5.

on the normal structure $\sigma_N(z)$. Thus, the solution of the MV inverse problem $\sigma(x, y, z) = \sigma_N(z) + \Delta\sigma(x, y, z)$ exists and we should elucidate whether it is unique.

The theorem of uniqueness for the MV inversion was proven by Dmitriev [Berdichevsky *et al.*, 1997]. The model shown in Fig. 5 is considered. In this model, a homogeneously layered Earth with the normal conductivity

$$\sigma_N(z) = \begin{cases} \sigma(z), & 0 < z < H \\ \sigma_H, & H < z \end{cases}$$

contains a 2-D inhomogeneous region S of an excess conductivity $\Delta\sigma(y, z) = \sigma(y, z) - \sigma_N(z)$. The inhomogeneity is elongated along the x axis, and the maximum size of its cross section is d. The functions $\sigma_N(z)$ and $\Delta\sigma(y, z)$ are piecewise-analytical. An infinite homogeneous basement with a conductivity σ_H = const occurs at depths greater than H.

Dmitriev's theorem states that the piecewise-analytical distribution of conductivity

$$\sigma(M) = \begin{cases} \sigma_N(z) & M \notin S \\ \sigma_N(z) + \Delta\sigma(y, z) & M \in S \end{cases}$$

is uniquely determined by exact values of the tipper

$$W_{zy}(y) = \frac{H_z(y, z = 0)}{H_y(y, z = 0)}, \quad -\infty < y < \infty, \quad 0 \le \omega \le \infty,$$

given at the Earth's surface $z = 0$ at all points of the y axis from $-\infty$ to ∞ in the entire range of frequencies from 0 to ∞.

This theorem of uniqueness is proven in two stages. The asymptotics of the tipper $W_{zy}(y)$ at a great distance from the inhomogeneity S is first derived, and the frequency dependence of this asymptotics is shown to uniquely determine the distribution of the normal conductivity $\sigma_N(z)$. Then, it is proven that, with the known conductivity $\sigma_N(z)$, the tipper uniquely determines the impedance of the inhomogeneous medium.

The anomalous magnetic field at the Earth's surface can be represented as the field produced in a horizontally homogeneous layered medium by excess currents of density j_x induced in the region S:

$$\begin{aligned} \overline{H}_y^A(y) &= \frac{H_y^A(y, z = 0)}{H_y^N(z = 0)} = \int_S j_x(M_0) h_y(y, M_0) dS, \\ \overline{H}_z(y) &= \frac{H_z^A(y, z = 0)}{H_y^N(z = 0)} = \int_S j_x(M_0) h_z(y, M_0) dS, \end{aligned} \tag{31}$$

where $h_y(y, M_0)$ and $h_z(y, M_0)$ are magnetic fields produced at the surface of a horizontally homogeneous medium by an infinitely long linear current of unit density flowing at the point $M_0(y_0, z_0) \in S$. The functions $h_y(y, M_0)$ and $h_z(y, M_0)$ have the form [Dmitriev, 1969; Berdichevsky and Zhdanov, 1984]

$$\begin{aligned} h_y(y, M_0) = \frac{i}{\omega\mu_0} \lim_{z \to 0} \int_0^\infty \cos\lambda(y - y_0) \\ \times e^{\lambda z} U(\lambda, z = 0, z_0) \lambda d\lambda, \end{aligned} \tag{32}$$

$$\begin{aligned} h_z(y, M_0) = -\frac{i}{\omega\mu_0} \lim_{z \to 0} \int_0^\infty \sin\lambda(y - y_0) \\ \times e^{\lambda z} U(\lambda, z = 0, z_0) \lambda d\lambda, \end{aligned} \tag{33}$$

where the factor $e^{\lambda z}$ relates to the upper half-space $z \le 0$ and the function $U(\lambda, z, z_0)$ is the solution of the boundary problem

$$\begin{aligned} &\frac{d^2 U(\lambda, z, z_0)}{dz^2} - \eta^2(\lambda, z) U(\lambda, z, z_0) = -\delta(z - z_0), \\ &z, z_0 \in [0, H]; \\ &\eta(\lambda, z) = \sqrt{\lambda^2 - i\omega\mu_0\sigma_N(z)}, \quad \mathrm{Re}\,\eta > 0; \\ &\frac{dU(\lambda, z, z_0)}{dz} + \lambda U(\lambda, z, z_0) = 0 \text{ at } z = 0; \\ &\frac{dU(\lambda, z, z_0)}{dz} - \eta_H(\lambda) U(\lambda, z, z_0) = 0 \text{ at } z = H; \\ &\eta_H(\lambda) = \sqrt{\lambda^2 - i\omega\mu_0\sigma_H}, \quad \mathrm{Re}\,\eta_H > 0. \end{aligned} \tag{34}$$

Note that the components $\overline{H}_y^A$ and $\overline{H}_z$ of the anomalous magnetic field can be found from values of the tipper W_{zy}, known at all points of the y axis. To determine $\overline{H}_y^A$, we solve the integral equation

$$W_{zy}(y)\overline{H}_y^A(y) + \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\overline{H}_y^A(y_0)}{y_0 - y}dy_0 = -W_{zy}(y), \quad (35)$$

after which $\overline{H}_z$ is found from the known value of $\overline{H}_y^A$:

$$\overline{H}_z = W_{zy}(1 + \overline{H}_y^A). \quad (36)$$

Now, we find the asymptotics of the functions $h_y(y, M_0)$ (32) and $h_z(y, M_0)$ (33) at $|y - y_0| \longrightarrow \infty$. Harmonics of low spatial frequencies λ make the major contribution to (33) at large $|y - y_0|$. Expanding $U(\lambda, z = 0, z_0)$ in powers of small λ, we have

$$U(\lambda, z = 0, z_0) = U(\lambda = 0, z = 0, z_0) + \lambda\frac{dU(\lambda, z = 0, z_0)}{d\lambda}\bigg|_{\lambda = 0} + \ldots;$$

hence, upon the substitution into (33) and integration, we obtain

$$h_y(y, M_0) = \frac{i}{\omega\mu_0}\frac{U(\lambda = 0, z = 0, z_0)}{(y - y_0)^2} + O\left(\frac{1}{(y - y_0)^4}\right). \quad (37)$$

Similarly, we obtain from (33)

$$h_z(y, M_0) = \frac{2i}{\omega\mu_0}\frac{1}{(y - y_0)^3}\frac{dU(\lambda, z = 0, z_0)}{d\lambda}\bigg|_{\lambda = 0} + O\left(\frac{1}{(y - y_0)^5}\right). \quad (38)$$

In order to write the relations between $\overline{H}_y^A$ and $\overline{H}_z$ in the form containing the MT impedance, we introduce the functions

$$V_y(z) = U(\lambda = 0, z, z_0),$$
$$V_z(z) = \frac{dU(\lambda, z, z_0)}{d\lambda}\bigg|_{\lambda = 0}. \quad (39)$$

The function $V_y(z)$ is the solution of problem (34) at $\lambda = 0$. The problem for the function $V_z(z)$ is obtained by differentiating (34) with respect to λ and setting $\lambda = 0$. Then,

$$\frac{d^2V_z(z)}{dz^2} + i\omega\mu_0\sigma(z)V_z(z) = 0 \quad z \in [0, H],$$

$$\frac{dV_z(z)}{dz}\bigg|_{z = +0} = -V_y(0), \quad (40)$$

$$\frac{dV_z(z)}{dz}\bigg|_{z = H} - \sqrt{-i\omega\mu_0\sigma_H}V_z(H) = 0.$$

In this notation,

$$h_y(y, M_0) = \frac{i}{\omega\mu_0}\frac{V_y(0)}{(y - y_0)^2} + O\left(\frac{1}{(y - y_0)^4}\right),$$
$$h_z(y, M_0) = \frac{2i}{\omega\mu_0}\frac{V_z(0)}{(y - y_0)^3} + O\left(\frac{1}{(y - y_0)^5}\right). \quad (41)$$

Returning to (31), we determine the asymptotics of the anomalous magnetic field. At $|y - y_0| \longrightarrow \infty$, we have

$$\overline{H}_y^A(y) = \frac{i}{\omega\mu_0}V_y(0)\int\limits_S\frac{j_x(M_0)}{(y - y_0)^2}dS$$
$$= \frac{i}{\omega\mu_0}\frac{V_y(0)}{(y - y_S)^2}\int\limits_S j_x(M_0)dS = \frac{i}{\omega\mu_0}\frac{V_y(0)}{(y - y_S)^2}J_x,$$
$$\overline{H}_z(y) = \frac{2i}{\omega\mu_0}V_z(0)\int\limits_S\frac{j_x(M_0)}{(y - y_0)^3}dS \quad (42)$$
$$= \frac{2i}{\omega\mu_0}\frac{V_z(0)}{(y - y_S)^3}\int\limits_S j_x(M_0)dS = \frac{2i}{\omega\mu_0}\frac{V_z(0)}{(y - y_S)^3}J_x,$$

where

$$J_x = \int\limits_S j_x(M_0)dS$$

is the total excess current in the inhomogeneity and y_S is the coordinate of the central point of its cross section S. Thus, with regard for (38), we have

$$\frac{\overline{H}_z(y)}{\overline{H}_y^A(y)} = \frac{2}{(y - y_S)}\frac{V_z(0)}{V_y(0)} = -\frac{2}{(y - y_S)}\frac{V_z(0)}{\left.\frac{dV_z(z)}{dz}\right|_{z = 0}} \quad (43)$$

in a region sufficiently far from the inhomogeneity S ($|(y - y_S| \gg d$, where d is the maximum size across the inhomogeneity).

In order to show that the ratio $\overline{H}_z/\overline{H}_y^A$ can be expressed through the normal impedance of the Earth, we introduce the function

$$Z(z) = i\omega\mu_0\frac{V_z(z)}{\frac{dV_z(z)}{dz}}. \quad (44)$$

According to (40), this function satisfies the Riccati equation

$$\frac{dZ(z)}{dz} - \sigma_N(z)Z^2(z) = i\omega\mu_0 \quad (45)$$

with the boundary condition

$$Z(H) = \sqrt{\frac{-i\omega\mu_0}{\sigma_H}}. \quad (46)$$

We obtained the known problem (4) for the impedance of a 1-D medium having the conductivity $\sigma_N(z)$. The function $Z(z)$ in the model under consideration evidently represents the normal impedance $Z_N(z)$. Setting $Z(z) = Z_N(z)$ and taking into account (43)–(46), we find the far-field asymptotics

$$Z_N(0) = -\frac{i\omega\mu_0(y-y_S)}{2}\frac{\overline{H}_z(y)}{\overline{H}^A_y(y)}\bigg|_{|y-y_S|\gg d}, \quad (47)$$

coinciding with the known expression for the field of a remote infinitely long linear current [Vanyan, 1965]. The normal impedance Z_N is connected with the ratio of the anomalous magnetic field components $\overline{H}_z$ and $\overline{H}^A_y$, which can be determined, according to (35) and (36), from values of the tipper W_{zy} known at all points of the y axis from $-\infty$ to ∞. Knowing the tipper W_{zy} all along the y axis, we synthesize the anomalous magnetic field ($\overline{H}_z$, $\overline{H}^A_y$) and calculate the normal impedance Z_N from the far-field asymptotics. Knowing the anomalous magnetic field ($\overline{H}_z$, $\overline{H}^A_y$) and the normal impedance Z_N, we integrate the second of the Maxwell equations (the Faraday law) and continue the longitudinal impedance $Z^{\parallel}$ to the entire y axis:

$$Z^{\parallel}(y) = \frac{E_x(y)}{H_y(y)} = \frac{1}{1+\overline{H}^A_y}\left\{Z_N - i\omega\mu_0\int_{-\infty}^{y}\overline{H}_z(y)dy\right\}. \quad (48)$$

Thus, we find $Z^{\parallel}$ from values of W_{zy}. A one-to-one correspondence exists between W_{zy} and $Z^{\parallel}$. Therefore, we can apply the theorem of Gusarov [1981], stating that inversion of $Z^{\parallel}$ has a unique solution, and extend this result to inversion of W_{zy}. The theorem of uniqueness for 2-D MV inversion reduces to that for 2-D MT inversion (the TE mode). Both methods, MV sounding and MT sounding, have a common mathematical basis. The 2-D conductivity distribution is uniquely determined from exact values of impedances or tippers given at all points of the Earth's surface in the entire frequency range.

On the Instability of Inverse Problems of Magnetotellurics

Inverse problems of magnetotellurics are unstable. The set Σ_δ, characterized by small misfits of the impedance tensor and tipper, can contain equivalent solutions that arbitrarily differ from one another and from the exact model solution.

We consider this property of inverse problems of magnetotellurics using, as an example, the 1-D inverse problem. The analysis is based on the theorem of stability of the S-distribution proven by Dmitriev in [Berdichevsky and Dmitriev, 1991, 2002].

Recall that the S-distribution is the function

$$S(z) = \int_0^z \sigma(z)dz, \quad (49)$$

determining the conductance of the Earth on the interval $[0, z]$. The conductivity σ is connected with the conductance S through the differential relation $\sigma(z) = dS(z)/dz$.

The theorem of stability of the S-distribution consists of two statements.

(1) The admittance $Y(\omega) = Y(z = 0, \omega)$ measured at the Earth's surface depends continuously on $S(z)$. Thus, the condition

$$\|S^{(1)}(\omega) - S^{(2)}(\omega)\|_C \le \varepsilon \quad (50)$$

implies that

$$\|Y^{(1)}(\omega) - Y^{(2)}(\omega)\|_{L_2} \le \delta(\varepsilon), \quad (51)$$

where $\delta \longrightarrow 0$ at $\varepsilon \longrightarrow 0$.

(2) The conductance $S(z)$ is stably determined from the admittance $Y(\omega) = Y(z = 0, \omega)$ specified at the Earth's surface. Thus,

$$\|S^{(1)}(\omega) - S^{(2)}(\omega)\|_C \longrightarrow 0 \quad (52)$$

if

$$\|Y^{(1)}(\omega) - Y^{(2)}(\omega)\|_{L_2} \longrightarrow 0. \quad (53)$$

The set of conductivity distributions obtained from the inversion of 1-D admittance is

$$\sigma_\delta \in \Sigma_\delta = \{\sigma(z): \|\tilde{Y}(\omega) - Y[\omega, \sigma(z)]\|_{L_2} < \delta_Y\}, \quad (54)$$

where $\tilde{Y}(\omega)$ is the measured admittance, $Y[\omega, \sigma(z)]$ is the operator calculating the admittance from a given distribution $\sigma(z)$, and δ_Y is the error in the admittance values. The theorem of stability of the S-distribution implies that, for any two distributions $\sigma^{(1)}_\delta(z)$ and $\sigma^{(2)}_\delta(z)$ from the set Σ_δ, the following condition is valid:

$$\left\|\int_0^z \sigma^{(1)}_\delta(z)dz - \int_0^z \sigma^{(2)}_\delta(z)dz\right\|_C \le \varepsilon(\delta_Y), \quad (55)$$

where $\varepsilon \longrightarrow 0$ at $\delta_Y \longrightarrow 0$. If the distributions $\sigma^{(1)}_\delta(z)$ and $\sigma^{(2)}_\delta(z)$ meet condition (55), they are equivalent; i.e., they are characterized by close S-distributions and cannot be resolved by MT observations performed with the uncertainty δ_μ Such distributions of the conductivity are called S-equivalent distributions, so that Σ_δ is the set of S-equivalent distributions of the conductivity. In the case of 1-D magnetotellurics, we can formulate the following generalized principle of S-equivalence: the conductance $S(z)$ characterizes the whole set Σ_δ of equivalent solutions of the 1-D inverse MT problem. To specify the entire set Σ_δ, it is sufficient to know its S-distribution.

Differentiating the conductance $S(z)$, one intends to find the conductivity $\sigma(z)$. However, the immediate numerical differentiation of $S(z)$ is an unstable operation generating a scatter in the distributions $\sigma(z)$. The determination of $\sigma(z)$ from $Y(\omega)$ is evidently an unstable problem. As can easily be shown, there exist essentially different distributions $\sigma^{(1)}(z)$ and $\sigma^{(2)}(z)$ corresponding to close distributions $S^{(1)}(z)$ and $S^{(2)}(z)$, and thereby to close distributions $Y^{(1)}(\omega)$ and $Y^{(2)}(\omega)$.

Now, we return to the 1-D model with an infinite homogeneous basement at a depth h. Let, for example,

$$\sigma^{(1)}(z)-\sigma^{(2)}(z) = \begin{cases} 0 \text{ at } z \notin [z', z'+\Delta h] \\ c/\sqrt{\Delta h} \text{ at } z \in [z', z'+\Delta h], \end{cases}$$

where $z' + \Delta h < h$. Then,

$$S^{(1)}(z)-S^{(2)}(z) = \int_0^z [\sigma^{(1)}(z)-\sigma^{(2)}(z)]dz$$

$$= \begin{cases} 0 \text{ at } 0 \le z \le z' \\ c(z-z')/\sqrt{\Delta h} \text{ at } z' \le z \le z'+\Delta h \\ c\sqrt{\Delta h} \text{ at } z'+\Delta h \le z \le h. \end{cases}$$

The norm of deviations is determined as

$$N_\sigma = \|\sigma^{(1)}(z)-\sigma^{(2)}(z)\|_{L_2}$$

$$= \left\{\int_0^h [\sigma^{(1)}(z)-\sigma^{(2)}(z)]^2 dz\right\}^{1/2} = c,$$

$$N_S = \|S^{(1)}(z)-S^{(2)}(z)\|_{L_2}$$

$$= \left\{\int_0^h [S^{(1)}(z)-S^{(2)}(z)]^2 dz\right\}^{1/2} = c\sqrt{\Delta h(h-z'-2\Delta h/3)}.$$

Choosing large c and small Δh, the deviation N_σ can always be made arbitrarily large, and the deviation N_S, arbitrarily small. Consequently, arbitrarily close distributions of the conductance and arbitrarily close admittances can correspond to arbitrarily differing distributions of the conductivity. The inferred estimates have a simple physical interpretation. Let the medium studied contain a thin layer whose conductance S_0 is much smaller than the conductance S of the overlying layers. The conductivity of the layer can vary within wide limits constrained by the condition $S_0 \ll S$, but these variations affect only insignificantly the admittance measured at the Earth's surface.

The 1-D inverse problem is unstable. Evidently, we have every reason to extend this conclusion to 2- and 3-D inverse problems. Compare, for example, a model with a smoothly varying boundary between two deep layers and a model in which this boundary rapidly fluctuates around an average smooth variation. Their MT and MV response functions observed at the Earth's surface will virtually coincide, although these models are largely different.

Inverse MT problems are unstable. An arbitrarily small error in initial MT and MV data can lead to an arbitrarily large error in the results of inversion of these data, i.e., in the conductivity distribution. Using the terminology of Hadamard, we state that inverse problems of magnetotellurics are ill-posed. A direct approach to the solution of ill-posed (unstable) problems is geophysically useless, because it can yield results far from reality.

4. INVERSE PROBLEMS OF MAGNETOTELLURICS IN LIGHT OF TIKHONOV'S THEORY OF REGULARIZATION OF ILL-POSED PROBLEMS

The cornerstone of MT and MV data interpretation is the theory of regularization of ill-posed problems. Its basic principles were formulated by Tikhonov [1963]. Presently, methods of this theory have been developed rather comprehensively and are widely used in practice [Tikhonov and Arsenin, 1977; Tikhonov and Goncharsky, 1987; Zhdanov, 2002]. The Russian mathematical school headed by A.N. Tikhonov gave rise to a new science of interpretation of observations encompassing various fields of science and technology.

In accordance with [Berdichevsky and Dmitriev, 1991, 2002; Zhdanov, 2002], we consider inverse problems of magnetotellurics in light of Tikhonov's theory, which provides a basis for developing the strategy of MT and MV inversions.

Conditionally Well-Posed Formulation of Inverse Problems of Magnetotellurics

The interpretation of an unstable MT or MV inverse problem is meaningful if a priori geological–geophysical information on the region under consideration is used and certain restraints are imposed on its geoelectric structure. This is a way to transform an unstable problem into a stable one. In the absence of a priori information restricting the scope of the search, we can obtain only one of the equivalent models or, at best, a model with a significantly smoothed distribution of conductivity leveling out contrasts of sought-for structures.

Thus, the transformation of an unstable problem into a stable one is attained by imposing an additional condition restricting the scope of the search. A compact subset Σ_δ^C containing the exact model solution and consisting of solutions sufficiently close to the exact model solution is chosen in the set Σ_δ of equivalent solutions. (Recall that a functional set is compact if any sequence of functions of this set contains a subsequence converging to a function also belonging to this set and that the necessary condition of compactness of a set is its boundedness.) The theory of regularization is based on the theorem of Tikhonov on the stability of an inverse problem defined on a compact subset [Tikhonov and Arsenin, 1977; Berdichevsky and Dmitriev, 2002]. This theorem is formulated as follows: if the error δ of the initial information tends to zero, the solution of the inverse problem on a compact subset Σ_δ^C converges to the exact model solution. An ill-posed inverse problem that has a unique solution and is stable on the compact subset Σ_δ^C is called conditionally well-posed (or well-posed after Tikhonov), and the subset Σ_δ^C is called a correctness set. Thus, the inverse problem ill-posed after Hadamard becomes well-posed after Tikhonov.

The compact subset Σ_δ^C (the correctness set) is chosen with the help of the generalization of a priori geological–geophysical information (experimental evidence, reasonable hypotheses, and general ideas of the origin and configuration of geoelectric structures). In essence, this means that a new geoelectric model is constructed on the basis of previous geological and geophysical models. The solution of an MT or MV inverse problem is effective if magnetotellurics provides new information as compared to what was known before MT and MV observations. Naturally, the solution of an inverse problem should be preceded by analysis of a priori information (in conjunction with the visualization of MT and MV characteristics, which facilitates the identification and localization of geoelectric structures).

In constructing the correctness set (the compact subset Σ_δ^C), i.e., in imposing restraints on the geoelectric structure of the medium, one should keep in mind that the condition $\delta \longrightarrow 0$ is unrealizable in practice, because initial information, which is obtained by processing of field measurements, is never free from uncertainties. Therefore, we speak of the practical stability of a conditionally well-posed problem. The problem is regarded as practically stable if the width of the correctness set is such that, given real errors δ, it consists of solutions that are sufficiently close to the exact model solution.

The correctness set, in which the solution to the inverse problem is sought, forms an interpretation model. The latter should incorporate modern ideas (hypotheses) on the stratification of the medium and on heterogeneities disturbing this stratification. The interpretation models of magnetotellurics separate into two classes: (1) layered models and (2) locally inhomogeneous models.

A layered model consists of a finite number of infinite or wedging-out layers. This model class includes horizontally layered models, consisting of homogeneous layers with horizontal boundaries, and quasi-horizontally layered models, in which the electrical conductivities of layers and their boundaries slowly vary in horizontal directions. A very important feature of the quasi-horizontally layered models is the presence (or absence) of high-resistivity layers playing the role of galvanic screens. This property, characterized by the galvanic constant of the model, determines the extent of near-surface galvanic anomalies and the sensitivity of the model to deep conductive structures.

The locally inhomogeneous models are layered models with breaks in their layers, sharp variations in their conductivity and boundaries, and heterogeneous inclusions of an arbitrary shape (for example, conductive bodies or conductive channels).

The construction of an interpretation model is based on a priori geological and geophysical information and qualitative constraints obtained from the analysis of MT and MV characteristics determined directly from field measurements. This analysis narrows the choice of the correctness set and allows one to construct an interpretation model described by a small number of parameters.

The interpretation model should meet the following two requirements:

it should be informative, i.e., describe main properties of the geoelectric medium, including objective layers and structures; and

it should be simple, being determined by a small number of free parameters that ensure the practical stability of the inverse problem.

It is evident that these conditions are opposite: the more informative the model, the more complicated it is. Thus, we have to choose an optimal model that will be sufficiently informative and sufficiently simple. This is a crucial point of the interpretation, predetermining both the strategy of the inversion and, to an extent, its results. It is at this stage of interpretation that the intuition of a researcher and his or her professional skill and academic qualification, understanding of the real geological situation and goals of the MT survey, and adherence to traditions, as well as willingness to deviate from them, become significant. Although the choice of an interpretation model is subjective, it is nevertheless limited by a priori information, results of qualitative analysis of field measurements, and reasonable hypotheses. It is in this sense that we state that the interpretation of MT and MV data is effective under the condition of sufficiently complete a priori information constraining the search. Although the statement that the better the geoelectric structure of the medium is known the better it can be determined seems paradoxical, it actually means that, solving the inverse problem, we improve and widen our knowledge of the structure of the medium, and therefore, the better this structure is known, the more meaningful and detailed the new results will be.

The amount of a priori information required to construct an optimal interpretation model depends on the complexity of the medium studied and on the goals of the interpretation. Whereas detailed a priori information on the tectonics and geodynamics of a region is required in rift or subduction zones, even very general ideas on the stratification of the study medium are sufficient in the case of stable platforms. Moreover, we can reject altogether a priori information at the preliminary stage of interpretation and perform the smoothed Occam inversion, which is, in essence, a formal transformation of experimental data. Such a transformation provides a gross geoelectric regionalization helpful for the identification of zones of interest for further interpretation.

Tikhonov's theory of ill-posed problems offers two basic approaches to the interpretation of MT and MV data: (1) optimization method and (2) regularization method [Berdichevsky and Dmitriev, 1991, 2002]. We briefly describe these approaches.

Optimization Method

This approach is only effective in studying simple media, described by a small number of parameters. Let us return to inverse problem (3) and assume that we have a priori information constraining a compact set M of admissible solutions of this problem including the exact model solution. If approximate values of the impedance tensor $[\tilde{\mathbf{Z}}]$ and the tipper $\tilde{\mathbf{W}}$ are known from observations, the conductivity distributions $\tilde{\sigma}^Z(x, y, z)$ and $\tilde{\sigma}^W(x, y, z)$ minimizing the misfit functionals

$$\begin{aligned}\Phi^Z\{\tilde{\sigma}^Z\} &= \|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \tilde{\sigma}^Z(x, y, z)\}]\| \\ &= \inf_{\sigma \in M}\|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\|, \\ \Phi^W\{\tilde{\sigma}^W\} &= \|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \tilde{\sigma}^W(x, y, z)\}\| \\ &= \inf_{\sigma \in M}\|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\|\end{aligned} \tag{56}$$

are approximate solutions of problem (3). As a rule, the misfit minimization procedure is iterative. A starting model is constructed through the parametrization of the interpretation model. The forward problem is further solved and the misfits between model and experimental values of the impedance tensor or the tipper are calculated. A new model, decreasing the misfits, is then chosen. The iterations are performed until the misfits approach the level of errors in the initial values of $[\tilde{\mathbf{Z}}]$ or $\tilde{\mathbf{W}}$. If the misfits cannot be decreased to the level of errors in the initial data, this implies that the compact set M was chosen to be overly narrow. In this case, we test successively widening compacta (e.g., we increase the density of subdivision of the model). A compactum on which the equation misfit is equal to the uncertainty of initial data is regarded as an optimal compact set. However, an overly wide compactum makes the problem unstable and can yield a solution that differs strongly from the exact model solution. This limits the actual applicability of the optimization method.

It is evident that separate inversions of the impedance and tipper are effective if the solutions $\tilde{\sigma}^Z(x, y, z)$ and $\tilde{\sigma}^W(x, y, z)$ are close to each other. Otherwise, magnetotelluric and magnetovariational inversions should be performed self-consistently. For example, we can minimize the functional of the total misfit

$$\begin{aligned}&\Phi\{\sigma(x, y, z)\} \\ &= \|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\|^2 \\ &\quad + \|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\|^2\end{aligned} \tag{57}$$

and control the contributions of MT and MV inversions by assigning them specific weights. Alternatively, we can accomplish successive interrelated partial inversions: starting with the MV inversion, which is free from distorting effects of local near-surface inhomogeneities, we proceed to the MT inversion with a starting model constructed from the results of the MV inversion. This approach is advantageous in that it eliminates near-surface distortions, forming geoelectric noise.

Regularization Method

Regularization of solutions substantially widens the possibilities of interpretation. Given a sufficient amount of a priori information, this approach provides maximum geoelectric information consistent with the accuracy of field observations. The main feature specific of the regularization method is that the criterion for choosing an approximate solution is included directly in the inversion algorithm. In solving the inverse problem, the compactum M narrows around the exact model solution. The regularization method admits the input of any type of a priori information, controls its influence on the solution of an inverse problem, and focuses the inversion on objective layers and structures.

This approach is based on the regularization principle: the criterion for the selection of a solution should be such that the inferred approximate solution should tend to the exact model solution of the inverse problem, when the errors in the initial information tend to zero.

The regularization principle for MT (3a) and MV (3b) inversions has the form

$$\lim_{\delta_Z \to 0} \tilde{\sigma}^Z(x, y, z) = \bar{\sigma}^Z(x, y, z) \quad \text{MT inversion,}$$
$$\lim_{\delta_W \to 0} \tilde{\sigma}^W(x, y, z) = \bar{\sigma}^W(x, y, z) \quad \text{MV inversion,} \tag{58}$$

where $\tilde{\sigma}^Z$, $\tilde{\sigma}^W$ and $\bar{\sigma}^Z$, $\bar{\sigma}^W$ are approximate and exact model solutions of MT and MV problems and δ_Z, δ_W are errors in the initial information.

The regularization principle is implemented with the help of a regularizing operator. A set of analytical and numerical operations that allows one to obtain an approximate solution satisfying the regularization principle is called the regularizing operator R of the inverse problem. In inverse problems of geophysics, it is advantageous to use a regularizing operator R_α depending on a numerical parameter $\alpha > 0$, which is called the regularization parameter. As the level of errors in the initial information δ tends to zero, the regularization parameter α should also tend to zero:

$$\lim_{\delta_Z \to 0} \alpha \longrightarrow 0 \quad \text{(MT inversion),}$$
$$\lim_{\delta_W \to 0} \alpha \longrightarrow 0 \quad \text{(MV inversion),} \tag{59}$$

and the regularizing operator R_α, when applied to the approximate response function, should yield the exact model solution of the problem:

$$\lim_{\delta_Z \to 0} R_\alpha[\tilde{\mathbf{Z}}] = \bar{\sigma}^Z(x, y, z) \quad \text{(MT inversion,),}$$
$$\lim_{\delta_W \to 0} R_\alpha \tilde{\mathbf{W}} = \bar{\sigma}^W(x, y, z) \quad \text{(MV inversion).} \tag{60}$$

The determination of the approximate solution of an inverse problem stable with respect to uncertainties in the initial information reduces to the construction of the regularizing operator R_α and the determination of the regularization parameter consistent with the accuracy of observations. The approximate solution obtained in this way is called a regularized solution.

Variational methods of constructing the regularizing operator are most widespread in geophysics. A stabilizing functional (a stabilizer), providing a criterion for the selection of admissible solutions, plays a key role in this approach. The stabilizer is usually written in the form

$$\Omega\{\sigma(x, y, z)\} \le C, \tag{61}$$

where C is a positive constant. The functional $\Omega\{\sigma(x, y, z)\}$ determines a compact set of functions $\sigma(x, y, z) \in \Sigma_C$. The smaller the value of C, the narrower the set Σ_C. Introducing (61), inverse problem (3) is formulated as a variational problem for a conditional extremum:

$$\begin{cases} \inf \Omega\{\sigma(x, y, z)\} \\ \|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\| \le \delta_Z & \text{(62a)} \\ \|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\| \le \delta_W; & \text{(62b)} \end{cases}$$

i.e., we find a minimum compactum Σ_C consisting of functions $\sigma(x, y, z)$ that satisfy conditions (62a) and (62b). The set Σ of approximate solutions of such an inverse problem is the intersection of the compactum Σ_C with the sets Σ_{δ_Z} and Σ_{δ_W} of equivalent solutions of MT and MV inverse problems:

$$\Sigma = \Sigma_C \cap \Sigma_{\delta_Z} \cap \Sigma_{\delta_W}.$$

It is convenient to replace the conditional extremum problem by an unconditional extremum problem:

$$\inf \Phi_\alpha\{\sigma(x, y, z)\}, \tag{63}$$

where Φ_α is Tikhonov's regularizing functional,

$$\Phi_\alpha\{\sigma(x, y, z)\} = M\{\sigma(x, y, z)\} + \alpha\Omega\{\sigma(x, y, z)\} \tag{64}$$

consisting of the functional of the total misfit

$$M\{\sigma(x, y, z)\}$$
$$= \|[\tilde{\mathbf{Z}}] - [\mathbf{Z}\{x, y, z = 0, \omega, \sigma(x, y, z)\}]\|^2$$
$$+ \|\tilde{\mathbf{W}} - \mathbf{W}\{x, y, z = 0, \omega, \sigma(x, y, z)\}\|^2$$

and the stabilizing functional $\Omega(\sigma)$. The solution of this inverse problem reduces to the minimization of $\Phi_\alpha(\sigma)$, i.e., to the minimization of $M(\sigma)$ and $\Omega(\sigma)$. Whereas the initial problem (3) is unstable, the solution obtained by minimizing the functional Φ_α is stable with respect to small variations in $[\tilde{\mathbf{Z}}]$ and $\tilde{\mathbf{W}}$. This is due to the fact that the functional $\Omega(\sigma)$ narrows the class of possible solutions and stabilizes the problem. Such a functional is called a stabilizer.

The structure of the functional $\Omega(\sigma)$ depends on a priori restraints imposed on the solution of an inverse problem. This can be, for example, the requirement of smoothness of $\sigma(x, y, z)$, satisfied by minimizing the functional

$$\Omega(\sigma) = \int_V \left\{ \left(\frac{\partial\sigma}{\partial x}\right)^2 + \left(\frac{\partial\sigma}{\partial y}\right)^2 + \left(\frac{\partial\sigma}{\partial z}\right)^2 \right\} dxdydz, \tag{65}$$

or the requirement of closeness of $\sigma(x, y, z)$ to a hypothetical model $\sigma_0(x, y, z)$, satisfied by minimizing the functional

$$\Omega(\sigma) = \int_V \{\sigma(x, y, z) - \sigma_0(x, y, z)\}^2 dxdydz. \tag{66}$$

The weight of the stabilizing functional, i.e., the amount of its effect on the solution of an inverse problem, is controlled by the regularization parameter α (Fig. 6). At large α, the minimization of $\Phi_\alpha(\sigma)$ leads to the predominant minimization of $\Omega(\sigma)$, i.e., smoothes the solution too much or retains it near the a priori hypothetical model, ignoring results of observations. At small α, the minimization of $\Phi_\alpha(\sigma)$ leads to the predominant minimization of $M(\sigma)$: the stabilizing effect of $\Omega(\sigma)$ is suppressed and an unstable incorrect solution is obtained. An optimum value of α providing a sufficiently small misfit and ensuring sufficiently strong stabilization of the solution should evidently be found.

The regularization parameter should be consistent with the uncertainty of the initial information δ. The optimum value of α can be chosen by testing a monotonically decreasing sequence $\alpha_1 > \alpha_2 > \ldots > \alpha_n$. For each α, variational problem (63) is solved and the iterative sequence of solutions characterized by their misfit values is determined. The parameter $\alpha = \alpha_{opt}$ at which the misfit attains the uncertainty of the initial information δ is regarded as an optimum parameter. The optimum parameter of regularization provides a conductivity distribution fitting best the exact model solution.

This simple technique is applicable if the uncertainty δ is well known. However, we commonly have a more or less gross estimate:

$$\delta_{min} \le \delta \le \delta_{max}. \tag{67}$$

In this case, solutions consistent with various values of δ from interval (67) are tested. Close solutions selected from the resulting set are averaged, providing an approximation of the exact model solution.

If nothing is known of measurement and model errors, the choice of the parameter α_{opt} cannot be based on solution misfits. In this case, a quasi-optimal value of the regularization parameter is determined. For example, α_{opt} can be determined as a value α at which the solution of the problem significantly deviates from requirements of the stabilizer (smoothness or closeness to the hypothetical model) but remains sufficiently stable. A similar method for the determination of α_{opt} was proposed by Hansen [1998]. This heuristic method is based on the so-called L-representation. A monotonically decreasing sequence of regularization parameters $\alpha_1 > \alpha_2 > \ldots > \alpha_n$ is tested and the misfit $M(\alpha)$ and the stabilizer $\Omega(\alpha)$ are determined for various α and a fixed minimum of Tikhonov's functional $\Phi(\alpha)$. Figure 7 presents, on a log–log scale, the $\Omega(\alpha)$ versus $M(\alpha)$ plot. This curve has typically an L-shaped form, with a fairly distinct bend separating a nearly horizontal branch with large $M(\alpha)$ and small $\Omega(\alpha)$ from a nearly vertical branch with small $M(\alpha)$ and large $\Omega(\alpha)$. The exact model solution is best approximated by assuming that the central point of the bend, characterized by the largest curvature, defines the quasi-optimal parameter of regularization α_{opt}.

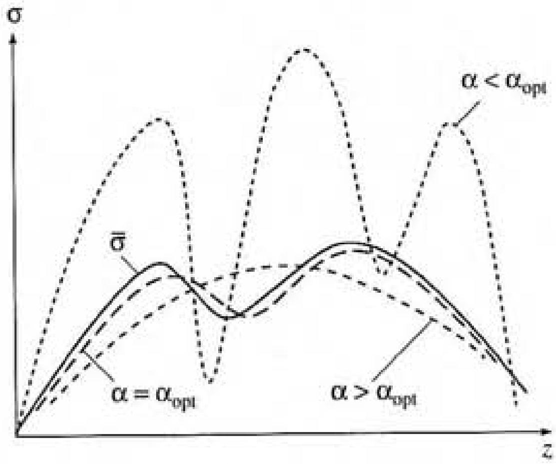

Fig. 6.

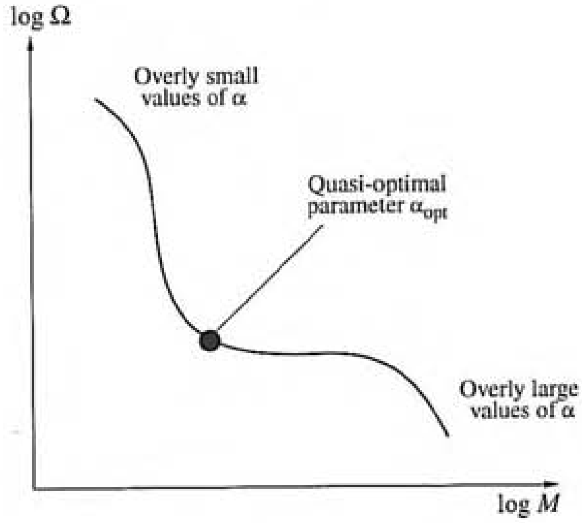

Fig. 7.

We considered the regularization of the inverse problem of magnetotellurics (3), based on the minimization of the functional $M\{\sigma(x, y, z)\}$, summarizing the misfits of MT and MV inversions. An alternative approach consists in a sequence of interrelated partial inversions in which the MT inversion stabilizer is constructed on the basis of the solution obtained from MV inversion. Note that such an approach gives more reliable results because it eliminates near-surface distortions forming geoelectric noise.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project nos. 02-05-64079 and 03-05-64167.

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