Van der Waerden's theorem and games

The well-known **Van der Waerden's theorem** states: *If the natural numbers are colored in finite number of colors then one can find a finite arbitrarily large arithmetic progression, all of the same color.*

The theorem can be reformulated as follows: *Let k and l be natural numbers. Then there exists a number $N(k,l)$ such that for any natural $N > N(k,l)$ and any partitioning of the set $1,\dots,N$ into k subsets, one of the subsets contains an arithmetic progression of length l.*

Upper bounds for the numbers $N(k,l)$ are huge. For example, complexity of the upper bound for $N(k,3)$ is an exponent of the exponent etc. k times taken. For $N(k,4)$, instead of the exponent, the function $N(k,3)$ is iterated, and so on. (The best estimates were obtained quite recently and by non-elementary methods). Lower estimates are only exponential. In particular, the project at the Summer Conference of the Tournament of Towns in 1995 was devoted to this theorem. **Its proof is not our goal**. Related questions on which is suggested to think for participants are:

1. Is it possible to color the set of natural numbers so that there would be no infinite monochromatic arithmetic progression?
2. Prove an exponential lower bound: $N(s,l) > s^{l/2}$. Implement an explicit construction.
3. Is it possible to color black more than half of the vertices of some regular n-gon so that the union of the original coloring and any ten its rotations will not make all vertices black? Try to get the best estimates for n as well as explicit constructions.

Szemeredi's theorem (also known as **Erdös-Turan conjecture**, 1936) states that *if A is an arbitrary set of natural numbers and the sequence $a_n = \frac{A\cap\{1,\dots,N\}}{N}$ has a subsequence with a nonzero limit point then for any natural $k > 3$ the set A contains an arithmetic progression of length k. This is one of the main results of Szemeredi (obtained in 1975), for which he received the Abel Prize. For progressions of length 3 this result was obtained by Klaus Roth in 1953.*

https://en.wikipedia.org/wiki/Szemer%C3%A9di%27s_theorem.

One of the main results of Terence Tao was the proof that *among prime numbers there is an arbitrarily long arithmetic progression.* The result of I.D.Shkhedov refers to finding of a black corner in the square $N \times N$ when a certain fraction of cells is colored black.

(http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=im&paperid=912&option_lang=rus).

Our main goal is to look at this topic from the game point of view. Perhaps this alternative view will enable us to better understand the issues related to the theorems of Van der Waerden and Szemerédi and to obtain new results, in particular, the estimates in the Van der Waerden's theorem.

Series A

1. Two players play the following game. On a both-ways endless checkered strip, the first player puts one cross into a free cell, and the second player puts 1000 zeroes in free cells. The goal of the first player is to create an arithmetic progression of crosses of length 3, and the goal of the second player is to prevent him.

   A) Prove that the first player has a winning strategy.

   B) Prove that there is a number n such that the first player safely can finish the game in n moves.

   C)Try to estimate the minimum number of moves necessary for the first to surely win under the assumptions above.

   D)Solve previous task for **one-way** strip.

2. Two players are playing the foloowing game on an infinite checkered plane. The first one puts a cross in a free cell, the second one answers by putting 1000 zeros. The goal of the first player is to create an isosceles rectangular triangle of crosses whose sides are parallel to the coordinate axes.

   A) Prove that the first player has a winning strategy

   B) Find the minimum number of moves necessary for the guaranteed win of the first.

3. Can the first player create an arithmetic progression of length 4 under the assumptions of the problem 1?

4. Under the assumptions of problem 2, can the first player construct a square?

5. Suppose that in the game on the strip the first player puts one cross, and the second one answers by putting n zeros. Can the first one create an arithmetic progression of length k?

6. Estimate the minimum number of moves necessary for the first player to win in the previous problem.

7. Now two players play a game on the plane. The first one puts one red dot, the second one answers by putting k blue dots. Prove that for any set F that consists of n points, the first player can create a figure F' similar to F (that is, obtained from F by a composition of homothety and parallel translation), all of whose vertices are red.

Series B

Generalized van der Waerden theorem: *Let rational points of the plane be painted in a finite number of colors. Then for any figure M consisting of a finite number of rational points, one can find a one-color figure M' similar to M (that is, obtained from M by a composition of homothety and parallel transport) consisting of rational points.*

1. Prove that in the formulation of the generalized Van der Waerden's theorem, the concept of *similar set* can not be replaced by the concept of *congruent* (or *equal* set, according to the terminology of the modern school program). In other words, a one-color set congruent to the set M not becessarily exists.

   And what if we consider the group of transformations preserving the area, or *hyperbolic rotations* (namely, transformations that shrink the X-axis and dilate Y-axis with the same factor?

2. The rational points of the plane are colored in several colors. Prove that there is a right triangle with sides parallel to the coordinate axes and monochrome vertices. (For solving this problem, Van der Waerden's theorem may be used.)

3. (** The solution of this problem is unknown.) Under the assumptions of the previous problem, prove that there is a monochrome rectangle of area 1 with monochrome vertices.

   We call a simplex in the n-dimensional space *standard* if it has n edges parallel to the coordinate axes.

4. Rational points of the n-dimensional space are colored in several colors. Prove that there is a standard simplex with monochrome vertices.

   (For solving this problem, Van der Waerden's theorem may be used.)

5. Formulate and prove the game form of the previous items.

   The second problem of this series reduced to the game form is solvable. In this connection, the question arises:

6. Formulate and solve the game form of the problem about the image of a given finite set under the action of hyperbolic rotations.

Series C

1. There is an unlimited number of boards 8×8. The first player puts 2 chips on any of them, the second one answers setting one black chip on any of the boards.

   a) Can the first player pave one of the boards for sure?

   b) What is the smallest number of moves sufficient for the first player to do this?

2. Two play tic-tac-toe on a one-way endless checkered strip. The first one puts two crosses at each move, the second one replies with a zero. Can the first player obtain 100 consecutive crosses in a row?

3. Under the assumptions of the first, can the first player obtain the above result in 2^{45} moves?

4. And in 2^{90} turnes?

Series D

Two painters play the following game. The first one marks a point on a plane and connects it with some points that were marked before. Previously drawn arcs cannot be crossed. The second painter colors the point just marked by the first painter in such a manner that no two points of the same color are connected by an arc.

1. Can the first painter make the second one use more than n colors?

2. Can the first painter achieve that in polinomial (in n) number of moves?

3. A tree is drawn. Two players paint its vertices in turn so that no two neighbouring vertices are of the same color. Can the first player make the second one use more than four colors?

4. Under assumptions of the previous problem, can the first player make the second one use more than three colors?

5. Let G be a bipartite graph. Two players in turn color its vertices so that any two neighbouring vertices are of different color. The second player tries to use the minimum possible number of colors, and the first one wants to make him use the maximum possible number of colors as possible. Is it true that for any natural N this bipartite graph G can be chosen so that N is not a sufficient number of colors for the second player?

Series E

1. A complete graph with n vertices is given. Two players paint its edges in turn. Theirst player colors one edge red, the other one colors 100 edges blue, and this is repeated until all edges are painted. Can the first player obtain for sure for sufficiently large n a complete subgraph with 100 vertices all whose edges are red?

2. 99% of a complete graph's edges are colored red. Given sufficiently large n, does there exist a complete subgraph on 1000 vertices with red edges?

3. Every subset with k elements in the set $\{1, \dots, n\}$ is colored into one of s colors (for example, for $k = 2$ we get a coloring of edges of a complete graph with n vertices). Prove that for sufficiently large n there exists a subset $U \subset \{1, \dots, n\}$ such that all its subsets having k elements are of the same color, and if x is the minimal element of U then the number of elements in U is at least $\exp(x) + s$.