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CELESTIAL MECHANICS OF PLANET SHELLS

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The motion of a planet consisting of an external shell (mantle) and a core (rigid body), which are connected by a visco-elastic layer and mutually gravitationally interact with each other and with an external celestial body (considered as a material point), is studied (Barkin, 1999, 2002a,b; Vilke, 2004). Relative motions of the core and mantle are studied on the assumption that the centres of mass of the planet and external body move on unperturbed Keplerian orbits around the general centre of mass of the system. The core and mantle of the planet have axial symmetry and have different principal moments of inertia. The differential action of the external body on the core and mantle cause the periodic relative displacements of their centres of mass and their relative turns. An approximate solution of the problem was obtained on the basis of the linearization, averaging and small-parameter methods. The obtained analytical results are applied to the study of the possible relative displacements of the core and mantle of the Earth under the gravitational action of the Moon. For the suggested two-body Earth model and in the simple case of a circular (model) lunar orbit the new phenomenon of periodic translatory-rotary oscillations of the core with a fortnightly period the mantle was observed. The more remarkable phenomenon is the cyclic rotation with the same period (13.7 days) of the core relative to the mantle with a ‘large’ amplitude of 152 m (at the core surface).

The results obtained confirm the general concept described by Barkin (1999, 2002a,b) that induced relative shell oscillations can control and dictate the cyclic and secular processes of energization of the planets and satellites in definite rhythms and on different time scales.

The results obtained mean that giant moments and forces produce energy which causes in particular deformations of the viscoelastic layer between planet shells. This process is realized with different intensities on different time scales. Here we have almost some machine of transformation of mechanical energy of translatory–rotary motions of the shells to elastic energy of deformation of the intermediate layer. Owing to the inelastic (dissipative) properties of this layer, part of elastic energy will become warm energy. This fundamental process has a cyclic character so the variations in the mechanical energy of translational and rotational motions of the shells are cyclic. The rhythms and types of relative wobble of the shells define periodic variations and transformations of mechanical, elastic and warm energies on different time scales. These fundamental positions maintain a constant value in the particular problem considered about the dynamics of the Earth’s shell and core–mantle dynamics of resonant objects: the Moon and Mercury.

The cyclic accumulation of elastic energy and warm energy of intermediate layer (between the core and mantle) in realized owing to the action of the inner moments and forces between shells. A considerable part of this energy transforms to the energy of numerous dynamic and physical processes on the planet. It is the mechanism of energization of the planet that defines its endogenous activity (Barkin, 2002a,b).

In a number of studies (see for example Barkin (1999, 2002a,b) and Ferrandiz and Barkin (2003)), celestial bodies are studied as objects with a complex structure (elastic, liquid or gaseous core, with shells). The dynamics of such objects in a gravitational field are described by a system of integrodifferential equations in ordinary and partial derivatives (Vilke, 1997a,b) for which research is difficult. At the same time, the complex structure of planets can appear as one of the factors determining the course of dynamic processes (the rotation of a planet around the centre of mass, tidal phenomena, orbit evolution, and tectonic processes as a consequence of relative displacement of parts of a planet) (Barkin, 2002b).

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In this article the two-layer model of a planet as a system consisting of a core and a mantle, which are considered as gravitationally interacting rigid bodies and moving in a gravitational field of an external celestial body, is investigated. The problem is studied in a restricted way, namely an the assumption that the motions of the centres of mass of a planet and external body (considered as a material point) occur on unperturbed Keplerian orbits, and relative displacements of the core and mantle are subject to determination. The obtained results are illustrated with the example of the motion of the Earth–Moon system.

1 CELESTIAL MECHANICS OF PLANET SHELLS

1.1 General concept

Gravitational interaction of the Moon and Sun with non-spherical inhomogeneous shells of the Earth generates very large additional mechanical forces (and moments) of the interaction of the neighbouring shells (the rigid core, liquid core, mantle and its layers, lithosphere and separate plates, and others). The action of these forces on different time scales on the corresponding shells generates cyclic perturbations of the tensional state of the shells, their deformations, small relative translational displacements and slow rotation of the shells, the formation of the planetary crack system, the redistribution of the plastic and fluid masses, the planetary redistribution of the ice–water masses, and others. In the geological period of time this leads to a fundamental tectonic reconstruction of the Earth (Barkin, 1999; 2002a,b).

In accordance with our concept of the global processes of the planets and satellites, time evolution and cyclicity have a celestial–mechanical nature and primarily are caused by translational displacements of their shells. In this paper we analyse the force interaction between neighbouring non-spherical shells cause by different gravitational influences of the external celestial bodies. These additional forces of a cyclic celestial–mechanical nature produce deformations of all the layers of the body and organize and control almost all natural processes. The analytical expressions of the components of these forces in the inertial system and in the body reference system were obtained and their structure was studied.

1.2 Mechanisms of the cyclic external perturbations of the shells of the given celestial body (Earth)

These mechanisms are as follows:

(i) small relative rotation (nutations) of the shells on different time scales;
(ii) small relative translational motions of the shells (displacements of their centres of mass);
(iii) relative displacements and rotations of the shells due to eccentricity of their centre-of-mass positions.

All these mechanisms arise because of the differential gravitational influence of the external celestial bodies (for the Earth it is the Moon, the Sun and planets). Another possible mechanism of translational displacements of the core in the layer D'' can be caused by relative core–mantle librations.

1.3 Shell interactions

Owing to the different oblatenesses of the shells a external body produces different accelerations of their centres of mass. Giant tension appears as a result between neighbouring shells and they deform all layers of the shells. This effect occurs even when the centres of mass of the shells coincide. For planets and satellites, similar forces appear between the core, mantle, external shell and their inhomogeneous layers. For the Earth, for example, giant forces appear between the core and the mantle, between the shells with a boundary at 670 km, between the lithosphere and separate plates, and others. So the additional cyclic force of mutual interaction of the liquid core and mantle (caused by the Moon's attraction) is about 10^{-6} from the full value of the force of the gravitational attraction between the Earth and the Moon. This force in a few orders larger than the classical tidal force in the Earth–Moon system. Of course this force is cyclic and characterized by a very wide basis of frequencies typical for orbital motions (of the Sun, Moon, planets and Solar System in the Galaxy), for rotational motion of the Earth, Moon and Sun and for many observed natural processes.

To understand the role of the inner interaction let us consider the simplest model of a celestial body, consisting of two axisymmetric shells (two-shell body). The inner shell S_1 and external shell S_2 can execute small relative translatory–rotary displacements due to the thin elastic layer between shells. In unperturbed motion of the celestial body the axes of its shells coincide. The shells are subjected to different gravitational actions from every external celestial body and, as a result, small relative displacements of the shells must be observed.

1.3.1 Force between shells. Let us consider the gravitational interaction of the shells S_1 and S_2 with the external celestial body P, which moves on an unperturbed (or perturbed) Keplerian elliptical orbit. The body P (satellite) we shall consider as a material point. In unperturbed motion the two-shell body rotates as a rigid body relative to the axis of symmetry, $O\zeta$. r is the modulus of the radius vector $\boldsymbol{r}$ of the satellite S with respect to the centre of mass of the planet (two-shell body). γ is the angle between the radius vector $\boldsymbol{r}$ and the polar axis of inertia of the satellite, $O\zeta$.

From differential equations of the relative translatory–rotary motion of the shells it follows that the additional force acting on the external shell from the side of the inner shell can be characterized by the formula

$$F(r,\gamma) = k\frac{(1-2\gamma^2+5\gamma^4)^{1/2}}{r^4},$$

where k is a constant parameter depending on the mass m_i of the shells and on their moments of inertia (C_i is the polar moment of inertia and A_i is the equatorial moment of inertia):

$$k = \frac{3}{2}f(m_1(C_2 - A_2) - m_2(C_1 - A_1)),$$

where f is a gravitational constant.

Obviously, depending on the character of the satellite orbit the force $F(r,\gamma)$ is a cyclic function of time. Even in the case of a circular orbit this force is subjected to large variations. In reality the direction cosine changes in diapason $\gamma \in (-\sin\rho, \sin\rho)$, where ρ is the angle between the axis of rotation of the two-shell body and the normal to the plane of the satellite orbit. The orientation of the force in the two-shell body reference system (latitude φ_F) depends only on γ:

$$\sin\varphi_F = \frac{\gamma(3-5\gamma^2)}{(1-2\gamma^2+5\gamma^4)^{1/2}}.$$

The presented force characteristics of the shell interaction are very important. They define the direction and intensity of the action of the inner shell on the external shell and the control processes of variations in the tension state of interacting shells and dictate variations in many natural processes of the shells and the surface of the two-shell body.

1.3.2 Shell interactions and inclinations of the axes of the planets. In the case of a circular orbit of the planet the minimal possible value of the force $F(r,\gamma)$ is achieved with $\gamma = \pm 1/5^{1/2}$. The corresponding inclination of the axis of planet rotation will be $26^{\circ}.6$. With a smaller inclination the force (1) will slightly increase and with $\gamma = 0$ ($\rho = 0$) it will be 112% from the minimal value. Practically this means that for all diapason of the values $\rho \in 0^{\circ}$–39° the force of interaction of the planet shells will be minimal: $F(r,\gamma) \leqslant F_0 = k/r^4$.

It is very interesting to note that for planets of the Solar System the conditions of minimal interaction of their shells are fulfilled. These conditions are satisified not only for Uranus and for the double external planet Pluto–Charon. The result obtained can be considered as confirmation of the important dynamic evolutionary role of the discussed mechanism of shell dynamics for the planets and satellites. Probably the shells use special inclined orientations for minimization of the mutual interaction. In other words the observed inclinations of the planet axes of rotation correspond to a quieter joint life of the shells. It is natural to assume that similar situations occur in other planetary systems in the Galaxy. This means that, in future, typical inclinations of the planetary axes of rotation of about 25°–29° will be observed. However, for complete conclusions on the considered problems a new treatment of the mechanical problems with respect to their relative oscillations of the planet shells and their evolution (Barkin, 2002a), must be applied.

1.3.3 Evaluations of the inner forces. In Tables 1 and 2 the values of the differential force F_S of the shell interaction are given for the simplest two-layer models of the planets and satellites. Here this force is illustrated as a defined characteristic of the tension state and deformation of the elastic shell (or some layer between shells).

Table 1. Evaluations of the inner forces between the shells of satellites.

N	*Satellite*	m_S *(kg)*	J_{2S}/I	F_S *(dyme)*
1	Io	8.94×10^{22}	5.96×10^{-3}	0.136×10^{23}
2	Ganymede	1.48×10^{23}	5.74×10^{-4}	0.109×10^{21}
3	Tethys	7.55×10^{20}	1.38×10^{-2}	0.281×10^{20}
4	Ariel	1.27×10^{21}	6.09×10^{-3}	0.227×10^{20}
5	Europa	4.80×10^{22}	1.52×10^{-3}	0.901×10^{19}
6	Pluto	1.24×10^{22}	0.40×10^{-3}	0.842×10^{19}
7	Dione	1.05×10^{21}	5.44×10^{-3}	0.641×10^{19}
8	Triton	2.14×10^{22}	1.83×10^{-4}	0.598×10^{19}
9	Mimas	3.80×10^{19}	5.98×10^{-2}	0.551×10^{19}
10	Titan	1.35×10^{23}	1.09×10^{-4}	0.318×10^{19}
11	Miranda	6.33×10^{19}	2.26×10^{-3}	0.300×10^{19}
12	Enceladus	8.40×10^{19}	2.42×10^{-2}	0.275×10^{19}
13	Rhea	2.49×10^{21}	1.92×10^{-3}	0.263×10^{19}
14	Amalthea	7.17×10^{18}	7.43×10^{-2}	0.227×10^{19}
15	Umbriel	1.27×10^{21}	1.09×10^{-4}	0.213×10^{19}
16	Callisto	1.08×10^{23}	1.54×10^{-4}	0.188×10^{19}
17	Charon	0.19×10^{22}	0.50×10^{-3}	0.426×10^{18}
18	Moon	7.35×10^{22}	5.13×10^{-4}	0.397×10^{18}
19	Titania	3.49×10^{21}	5.13×10^{-4}	0.350×10^{18}
20	Oberon	3.03×10^{21}	2.14×10^{-4}	0.396×10^{17}
21	Phobos	5.74×10^{15}	0.196	0.175×10^{16}
22	Hyperion	1.77×10^{19}	0.295	0.735×10^{15}
23	Deimos	1.05×10^{15}	0.176	0.231×10^{13}
24	Nereid	4.27×10^{19}	1.00×10^{-3}	0.175×10^{11}
25	Iapetus	1.88×10^{21}	4.70×10^{-6}	0.748×10^{9}
26	Phoebe	4.00×10^{18}	0.0647	0.173×10^{9}

Table 2. Evaluations of the inner forces between the shells of planets.

Planet	J_2	m_σ ($\times 10^{24}$ kg)	F_σ
Mercury	0.600×10^{-4}	0.3302	8.920×10^{13}
Venus	0.597×10^{-5}	4.8685	6.616×10^{13}
Earth	1.083×10^{-4}	5.9736	4.469×10^{15}
Mars	1.959×10^{-4}	0.6419	4.568×10^{13}
Jupiter	1.470×10^{-2}	1898.6	3.309×10^{18}
Saturn	1.633×10^{-2}	568.46	6.926×10^{16}
Uranus	3.513×10^{-3}	86.831	2.519×10^{13}
Neptune	3.539×10^{-3}	102.43	4.608×10^{12}
Pluto	1.300×10^{-4}	0.0124	1.508×10^{4}

Of course this parameter can be considered as that which characterizes the tectonic energy (the elastic energy of the deformations, the mechanical energy of the destruction of the Earth's layers, the liquid redistribution and the warm energy). Here we present a simplified expression for this parameter (Barkin, 2002 b,c)

$$F_S = m_S 10.12 \frac{C_S - (A_S + B_S)/2}{C_S} \frac{R_S}{a_S} R_S \frac{1}{T_S^2},$$

where m_S is the mass the satellite, $\alpha_S = [C_S - (A_S + B_S)/2]/C_S$ is the dynamic oblateness of the satellite, A_S, B_S and C_S are the principal moments of inertia, $J_{2S} = \alpha_S I_S$, $I_S = C_S/m_S R_S^2$ is a non-dimensional moment of inertia; R_S is the mean radius of the satellite radius, a_S is the major semiaxis of the satellite orbit and T_S is the orbital period.

Similar evaluations of the parameter of endogenous activity of the satellites caused by gravitational attraction of the mother planet have been obtained.

1.3.4 Nature of the layer D$''$ *of the Earth.* For the Earth's system (core–mantle) for example the role of the intermediate layer is played by the well-known layer D$''$. Owing to the elastic and plastic properties of this layer a transformation of mechanical energy of the relative displacements of the shells to elastic and warm energy must be observed. As a result, definite variations (cyclic in time) in the tension of the mantle, warm flows, and fluid redistributions of Earth mass and many other natural processes must be realized. In this sense, of course, layer D$''$ in reality is a 'kitchen of tectonic processes' as stated by Zonenshine and Kuzmin (1993). We can suggest that the nature and origin of the layer D$''$ are directly caused by relative displacements and wobble of the core and mantle of the Earth. However, the mechanism of forced shell interactions works in the cases of other planets and satellites of the Solar System. It is natural to suggest that similar intermediate layers must be discovered for Mercury, Venus and Mars and for many other bodies of the Solar System (Barkin, 2002c).

1.4 Mechanism of the endogenous activity of the Earth

Having carried out a preliminary analysis of the shell interactions, let us suggest that the relative translatory–rotary oscillations of the Earth's shells due to the differential gravitational influences of the external celestial bodies (the Moon, the Sun and others) is the main mechanism of their endogenous activity and in particular is the main mechanism for the ordering of the positions of the centres of geological formations (Barkin, 2002b,c).

The endogenous energy of the Earth can be evaluated from the following formulae (Barkin, 2002a,b,c);

$$U_\epsilon = \chi U, \quad U = \frac{3}{2} \cdot \frac{f m_0}{r^3}(C - A)\gamma^2, \quad W_\epsilon = \frac{dU_\epsilon}{dt}.$$

Here m_0 is the mass of perturbing body (the Moon or the Sun). C, A, C_1, A_1 and C_2, A_2 are the polar and equatorial moments of inertia of the full Earth and its core and mantle. γ is an angle between radius-vector of perturbing body and polar axis of inertia of satellite, r is a module of this radius-vector,

$$\chi = \left(\frac{C_2 - A_2}{C_2} - \frac{C_1 - A_1}{C_1}\right) \Big/ \frac{C - A}{C} = 0.507.$$

The values for the moments of inertia of the full Earth and its main shells are as follows (1 unit $= 10^{36}$ kg m^2):

Earth's core: $C_1 = 9.1387$, $A_1 = 9.1206$;
Earth's mantle: $C_2 = 71.2623$, $A_2 = 70.9914$;
Earth: $C = 80.4010$, $A = 80.1120$;
core: $(C_1 - A_1)/C_1 = 1.9806 \times 10^{-3}$;
mantle: $(C_2 - A_2)/C_2 = 3.8015 \times 10^{-3}$;
Earth: $(C - A)/C = 3.5945 \times 10^{-3}$.

The following evaluations of energetic characteristics of the Earth endogenous activity for the main planetary geodynamical processes have been obtained (Barkin, 2002b,c):

energy of influence of the Moon–Sun: $U = 10^{29}$ erg;
energy of the inner interaction of the shells: $\chi U = 0.5 \times 10^{29}$ erg;
endogenous energy (elastic, inelastic and warm): $U_{\text{endogenic}} \approx 10^{26}$–$10^{28}$ erg;
power of the inner energy transformation: $W = 10^{29}$ erg;
power of endogenous processes: $W_{\text{endogenic}} = 10^{29}$ erg;
power of volcanic processes: $W_{\text{volcanos}} = 10^{29}$ erg;
power of seismic processes: $W_{\text{earthqukes}} = 10^{29}$ erg.

Here m_0 is the mass of perturbing body (the Moon or the Sun). C, A, C_1, A_1 and C_2, A_2 are the polar and equatorial moments of inertia of the full Earth and its core and mantle.

1.5 Geodynamic consequences

Force variations are reflected in the variations in the tensional states of the body shells and, moreover, on different time scales from hours to hundreds of millions of years. The above-mentioned shell interactions lead to considerable variations in the tensional state of the planet, to its planetary geological and geophysical reconstructions and to processes characterized by the properties of cyclicity, polarity, asymmetry, inversion and others. These fundamental properties of the planet and satellite life can be explained as consequences of the shell oscillations in definite rhythms in geological periods of time. The cycles of the Galaxy play the main role here. They are caused by the perturbed orbital motion of the Solar System in the Galaxy and by the Galaxy's gravitational attraction. The radial character of the shell displacements and their non-permanent drift are the dynamic reasons for the phenomena of asymmetry and inversion of the geodynamic states. On the basis of the developed approach a mechanical interpretation can be given for following observed phenomena and processes:

(i) cyclicity of the natural processes on the planets and satellites;
(ii) cycles of geoevolution of the Galaxy;

(iii) origin and existence of the latitudinal and longitudinal ordering of the planet and satellite formations (grid phenomenon);
(iv) correlation's of the positions of the largest formations of the planets (satellites) and orientation of their axes of rotation with peculiarities of the Solar System's trajectory in the Galaxy's gravitational field.

All these phenomena are reflections of the small and slow relative displacements of the shells in geological periods of time.

1.6 Natural cycles

Relative oscillations of the two shells of the planet under the action of the gravitating external bodies were studied on the basis of some simple model problems. We have assumed that the shells are rigid axisymmetric bodies separated by a thin elastic or inelastic layer. Because of this layer the shells can execute small relative translational and rotational motions when subjected to a definite additional reaction from this layer. Firstly, we take into account the elastic forces acting from the layer on the shells.

It was shown that perturbations in the orbital motion of the external celestial bodies (the Moon, the Sun and planets) are reflected directly in the relative displacements of the shells. The hierarchical role of the orbital perturbations of the Moon, the Sun and planets in the shell dynamics was established. Relative oscillations of the shells are characterized by short periodic perturbations (P) with periods comparative with the orbital periods of the planets of tens and hundreds of years, by secular perturbations (S) with periods of the secular orbital motions of the Solar System of tens and hundreds of thousands of years and by galactic perturbations (G) with periods of tens and hundreds of millions of years. All these orbital perturbations give similar classes of relative oscillations and displacements of the Earth's shells. Different combinations of these perturbations generate new classes of shell oscillations with intermediate periods. The amplitude of the perturbations increase in the sequence P–S–G. Studies of the shell oscillations show an increase in the geodynamic role of the above-mentioned orbital perturbations, which is confirmed by the results of the observations and geological data.

The relative drift and displacements of the Earth's shells in geological periods of time are more siginificant and important for life on the Earth. They are dictated by the orbital perturbations of the Moon, the Sun and planets caused by the Galaxy's attraction and Galactic motion of the Solar System. On the other hand the direct perturbing influence of the Galaxy's attraction to the Earth's shells also takes place. Both mechanisms of the Earth's evolution are very important and need mechanical studies.

The influence of the Galaxy on the shells leads to more marked variations in their tensional states and to planetary tectonic, geological and geography reconstructions of the Earth which are characterized by the properties of cyclicity, polarity, asymmetry, inversion and others (Barkin, 2002c).

The main properties of the considered shell displacements are as follows:

(i) ordering, cyclicity and directricity of their motions (the main cycles of the shell's oscillations are characterized by periods of the perturbed galactic motion of the Sun);
(ii) radial character of displacements leading to antisymmetry in the change in the geodynamic states (in opposite hemispheres of the Earth);
(iii) non-uniform displacements of the shells, leading to the irregular variations in the activity of the geodynamic and geophysical processes.

2 ABOUT THE FORCED MOTION OF THE TWO-SHELL PLANET IN THE GRAVITATIONAL FIELD OF A PERTURBING BODY

2.1 Treatment of the problem

Let the planet consist of a core and an external shell (mantle) between which there is a thin spherical layer of a homogeneous viscoelastic environment. The core and mantle are considered as rigid bodies. The coordinate system $C_1x_1x_2x_3$ is connected to a core occupying domain $V_1 = \{\boldsymbol{r}_1^2 < a_1^2, \boldsymbol{r}_1 = (x_1, x_2, x_3)\}$, the point C_1 is the centre of mass of a core, and $\mathcal{J}_1 = \operatorname{diag}(A_1, A_1, C_1)$ is a tensor of inertia of a core relative to the coordinate system $C_1x_1x_2x_3$.

Let us assume that the main central moment of inertia $C_1 > A_1$ (it is connected to sufficiently fast rotation of a planet around of the axis C_1x_3). Concerning the external shell also we shall assume that it occupies the domain V_2 for which the internal surface is set by the equation $\boldsymbol{r}_2^2 = (a_1 + l)^2, \boldsymbol{r}_2 = (y_1, y_2, y_3)$ in the coordinate system $C_2y_1y_2y_3$. Here C_2 is the centre of mass of the shell, l is the thickness of the viscoelastic layer between the core and the external shell. The tensor of inertia of the environment coordinate system $C_2y_1y_2y_3$ is equal to $\mathcal{J}_2 = \operatorname{diag}(A_2, A_2, C_2)$, $C_2 > A_2$. If the coordinate system $C_1x_1x_2x_3$ and $C_2y_1y_2y_3$ coincide, the viscoelastic layer is in a non-tensional state.

The coordinate system $CX_1X_2X_3$ is connected to the centre of mass of a planet, and its axes are parallel to the axes of the inertial coordinate system $OX_1X_2X_3$ with origin at the centre of weights of a planet and a material point of mass m. Vector equalities in the coordinate system $CX_1X_2X_3$ are fulfilled:

$$
\begin{aligned}
\boldsymbol{P}_1 &= -\frac{m_{\mathrm{r}}}{m_1}\boldsymbol{Q} + \boldsymbol{\Gamma}_1\boldsymbol{r}_1, \boldsymbol{r}_1 \in V_1, \quad m_{\mathrm{r}} = \frac{m_1 m_2}{m_1 + m_2}, \\
\boldsymbol{P}_2 &= \frac{m_{\mathrm{r}}}{m_2}\boldsymbol{Q} + \boldsymbol{\Gamma}_2\boldsymbol{r}_2, \quad \boldsymbol{r}_2 \in V_2,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are the vectors of the points of the core and mantle, $\boldsymbol{Q}$ is the vector C_1C_2, and $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are orthogonal operators of transition from coordinate systems $C_1x_1x_2x_3$ and $C_2y_1y_2y_3$ to the coordinate system $CX_1X_2X_3$. If $\boldsymbol{Q} = 0$ and $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2$, the core of a planet is inside the external shell and their principal axes of inertia coincide. On motion of a planet, small displacements and rotations of the mantle of the planet relative to its core take place. In the coordinate system $C_1x_1x_2x_3$ this displacement is equal to $\boldsymbol{q} = \boldsymbol{\Gamma}_1^{-1}\boldsymbol{Q}$, and the relative operator of turn is $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_1^{-1}\boldsymbol{\Gamma}_2$. For the points lying on the surface of the core and the internal surface of the mantle, the following vector equality is valid:

$$
\boldsymbol{r}_2 = \boldsymbol{q} + \left(1 + \frac{l}{a_1}\right)\boldsymbol{\Gamma}\boldsymbol{r}_1, \quad \boldsymbol{r}_1 = (x_1, x_2, x_3), \quad \boldsymbol{r}_2 = (y_1, y_2, y_3), \quad \boldsymbol{q} = (q_1, q_2, q_3), \tag{2}
$$

where $\boldsymbol{q}$ is a displacement of the centre of mass of the mantle relative to the centre of a core, $\boldsymbol{\Gamma}$ is the matrix of the rotation of the mantle relative to the coordinate system $C_1x_1x_2x_3$ connected to a core, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors of points on the surface of the core and the internal surface of the mantle respectively. The displacement vector $\boldsymbol{q}$ is represented as small in the sense that the ratio $|\boldsymbol{q}|/l$ is small, and the operator $\boldsymbol{\Gamma}$ is close to unity so that

$$
\boldsymbol{\Gamma} = \mathrm{O}_1(\gamma_1)\mathrm{O}_2(\gamma_2)\mathrm{O}_3(\gamma_3), \tag{3}
$$

with

$$
\mathrm{O}_1(\gamma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma_1 & -\sin\gamma_1 \\ 0 & \sin\gamma_1 & \cos\gamma_1 \end{pmatrix},
$$

$$\mathrm{O}_2(\gamma_2) = \begin{pmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 1 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 \end{pmatrix},$$

$$\mathrm{O}_3(\gamma_3) = \begin{pmatrix} \cos\gamma_3 & -\sin\gamma_3 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where γ_1, γ_2 and γ_3 are small angles of turn of the external shell relative to the respective axes. If the operators $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are set with the help of Euler angles and taking into account equation (3) the following equalities hold:

$$\begin{aligned}
\mathbf{\Gamma}_k &= \mathrm{O}_3(\psi + \psi_k)\mathrm{O}_1(\theta + \theta_k)\mathrm{O}_3(\varphi + \varphi_k), \quad k = 1, 2, \\
\mathbf{\Gamma} &\approx \begin{pmatrix} 1 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 1 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 1 \end{pmatrix}, \\
\Delta\psi &= \psi_2 - \psi_1, \quad \Delta\theta = \theta_2 - \theta_1, \quad \Delta\varphi = \varphi_2 - \varphi_1, \\
\gamma_1 &= \Delta\psi \sin\theta \sin\varphi + \Delta\theta \cos\varphi, \\
\gamma_2 &= \Delta\psi \sin\theta \cos\varphi - \Delta\theta \sin\varphi, \\
\gamma_3 &= \Delta\psi \cos\theta + \Delta\varphi.
\end{aligned} \tag{4}$$

The Euler angles ψ, θ and φ describe the rotation of the planet when its core and mantle make a single whole, and the perturbations ψ_k, θ_k and φ_k, $k = 1, 2$, are equal to zero. In equations (4), small terms of second order and higher order relative to $\Delta\psi$, $\Delta\theta$ and $\Delta\varphi$ are omitted.

2.1.1 Elastic energy and dissipation function. We shall consider the interaction of the core and external shell to be modelled by elastic forces with the potential (Vilke, 1997a,b)

$$E[\boldsymbol{u}] = \iiint\limits_{V_0} \left(\frac{\lambda}{2}(\operatorname{div}\boldsymbol{u})^2 + \mu \sum_{i,j=1}^{3} e_{ij}^2 \right) \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3, \tag{5}$$

$$e_{ij} = \tfrac{1}{2}(u_{ij} + u_{ji}), \quad u_{ij} = \frac{\partial u_i}{\partial x_j}.$$

Here λ and μ are Lamé coefficients of the homogeneous elastic layer between the core and mantle, $V_0 = \{a_1 < |\boldsymbol{r}| < a_1 + l\}$ is the domain occupied by this layer, $\boldsymbol{u}(\boldsymbol{r}, t)$ is the vector of displacements of the points of the elastic medium relative to the coordinate system connected to the core. To estimate the modulus of elastic potential (5) we shall take the field of elastic displacements in accordance with equation (2) to be

$$\boldsymbol{u} = \frac{r - a_1}{l}[\boldsymbol{q} + (\mathbf{\Gamma} - E)\boldsymbol{r}], \tag{6}$$

with

$$\boldsymbol{r} = (x_1, x_2, x_3), \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

The displacement field (6) is zero on the surface of the core and corresponds to displacements of points on the internal surface of the mantle. It is possible to see that equation (6) is an approximation of the displacement field due to the first term of its Taylor series along a beam coincident with one of the radiis of the core. Using equality (6), we find the elastic force potential (5) as

$$E[\boldsymbol{u}] = \tfrac{1}{2}[N_1\boldsymbol{q}^2 + N_2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)], \tag{7}$$

with

$$N_1 = \frac{4\pi E(1-\nu)[(a_1+l)^3 - a_1^3]}{9l^2(1+\nu)(1-2\nu)}, \quad N_2 = \frac{8\pi E(1-\nu)[(a_1+l)^5 - a_1^5]}{75l^2(1+\nu)}.$$

Here E and ν are the elasticity modulus and Poisson's ratio respectively of the material of the viscoelastic layer. We shall assume further that mutual displacements of the mantle and core are accompanied by the dissipation of energy described by the Rayleigh function

$$D = \frac{\chi_1}{2}[N_1(\dot q_1^2 + \dot q_2^2 + \dot q_3^2) + N_2(\dot\gamma_1^2 + \dot\gamma_2^2 + \dot\gamma_3^2)], \tag{8}$$

where χ_1 is a factor determining the dissipation of energy, and the superscript dot means the derivative with respect to time.

2.1.2 Kinetic energy of the shells. The coordinate system $OX_1X_2X_3$ with the origin as the centre of mass is a inertial system, and $\boldsymbol{R}$ is a radius vector connecting the centre of mass of the planet and the material point. The kinetic energy of the system according to the Konig theorem in view of equation (1) is given by

$$T = \frac{M_\mathrm{r}}{2}\dot{\boldsymbol{R}}^2 + \frac{m_\mathrm{r}}{2}\dot{\boldsymbol{q}}^2 + \tfrac{1}{2}(J_1\omega_1, \omega_1) + \tfrac{1}{2}(J_2\omega_2, \omega_2), \tag{9}$$

with

$$M_\mathrm{r} = \frac{m(m_1+m_2)}{m+m_1+m_2}, \quad m_\mathrm{r} = \frac{m_1 m_2}{m_1+m_2},$$

where ω_1 and ω_2 are the angular velocities of the core and the mantle in the coordinate systems connected to each of them. The following relations are valid:

$$\begin{aligned}\tfrac{1}{2}(J_k\omega_k, \omega_k) &= \tfrac{1}{2}A_k[(\dot\psi + \dot\psi_k)^2 \sin^2(\theta+\theta_k) + (\dot\theta + \dot\theta_k)^2] \\ &\quad + \tfrac{1}{2}C_k[\dot\varphi + \dot\varphi_k + (\dot\psi + \dot\psi_k)\cos(\theta+\theta_k)]^2.\end{aligned}$$

2.1.3 Potential energy. Let us find expressions for the potential energy of gravitational interaction of the three components of the system, namely, the core, mantle and material point m. We have

$$\begin{aligned}
&\Pi_{12} = -\gamma \iint\limits_{V_1\cup V_2} \frac{\rho_1\rho_2\,\mathrm{d}v_1\,\mathrm{d}v_2}{f_{12}(z)}, \\
&f_{12}(z) = [(\mathbf{q} + \Gamma \boldsymbol{r}_2 - \boldsymbol{r}_1)^2]^{1/2}, \quad \boldsymbol{r}_i \in V_i, \quad i = 1, 2, \\
&\Pi_{13} = -\gamma m \int_{V_1} \frac{\rho_1\,\mathrm{d}v_1}{f_{13}(z)}, \quad f_{13}(z) = \left[\left(\boldsymbol{R} + \boldsymbol{\Gamma}_1\boldsymbol{q}\frac{m_\mathrm{r}}{m_1} - \boldsymbol{\Gamma}_1\boldsymbol{r}_1\right)^2\right]^{1/2}, \\
&\Pi_{23} = -\gamma m \int_{V_2} \frac{\rho_2\,\mathrm{d}v_2}{f_{23}(z)}, \quad f_{23}(z) = \left[\left(\boldsymbol{R} - \boldsymbol{\Gamma}_1\boldsymbol{q}\frac{m_\mathrm{r}}{m_2} - \boldsymbol{\Gamma}_2\boldsymbol{r}_2\right)^2\right]^{1/2}, \\
&\boldsymbol{z} = (z_1, \ldots, z_6), \quad z_i = q_i, \quad z_{i+3} = \gamma_i, \quad i = 1, 2, 3.
\end{aligned} \tag{10}$$

Here γ is a universal gravitational constant, V_1, V_2, ρ_1 and ρ_2 are domains occupied by the core, the mantle and their corresponding densities respectively, subjected accordingly to the even functions on the argument x_3 or y_3 and dependent also on the sum of squares of other arguments. The function Π_{12} depends on a component of the vector $\boldsymbol{z}$ and is represented as the following Taylor series:

$$\Pi_{12}(\boldsymbol{z}) = \Pi_{12}(0) + \sum_{i=1}^{6} \frac{\partial \Pi_{12}(0)}{\partial z_i} z_i + \frac{1}{2} \sum_{i,j=1}^{6} \frac{\partial^2 \Pi_{12}(0)}{\partial z_i \partial z_j} z_i z_j + \cdots . \tag{11}$$

As the components of the vector $\boldsymbol{z}$ are small, in equation (11) we are limited to square-law members. Further this agrees with equation (10) and so we shall find that

$$\frac{\partial f_{12}^{-1}(0)}{\partial q_i} = \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{e}_i}{f_{12}^3(0)}, \qquad \frac{\partial f_{12}^{-1}(0)}{\partial \gamma_i} = \frac{(\boldsymbol{r}_2 \times \boldsymbol{r}_1) \cdot \boldsymbol{e}_i}{f_{12}^3(0)}, \qquad i = 1, 2, 3,$$

where $\boldsymbol{e}_i$ is a unit vector along the axis Cx_i. Taking into account the symmetry of the domains V_1 and V_2, we find that $\partial \Pi_{12}(0)/\partial z_i = 0$, $i = 1, \ldots, 6$. For the second partial derivatives the following relations hold:

$$\begin{aligned}
\frac{\partial^2 f_{12}^{-1}(0)}{\partial q_i\, \partial q_j} &= -f_{12}^{-3}(0)\delta_{ij} + 3(\boldsymbol{r}_2 - \boldsymbol{r}_1, \boldsymbol{e}_i)(\boldsymbol{r}_2 - \boldsymbol{r}_1, \boldsymbol{e}_j) f_{12}^{-5}(0),\\
\frac{\partial^2 f_{12}^{-1}(0)}{\partial q_i\, \partial \gamma_j} &= f_{12}^{-3}(0)(\boldsymbol{e}_j \times \boldsymbol{e}_i, \boldsymbol{r}_2) - 3(\boldsymbol{r}_2 - \boldsymbol{r}_1, \boldsymbol{e}_i)(\boldsymbol{r}_2 \times \boldsymbol{r}_1, \boldsymbol{e}_j) f_{12}^{-5}(0),\\
\frac{\partial^2 f_{12}^{-1}(0)}{\partial \gamma_i\, \partial \gamma_j} &= -f_{12}^{-3}(0)(\boldsymbol{e}_i \times \boldsymbol{r}_2, \boldsymbol{e}_j \times \boldsymbol{r}_1) + 3(\boldsymbol{r}_2 \times \boldsymbol{r}_1, \boldsymbol{e}_i)(\boldsymbol{r}_2 \times \boldsymbol{r}_1, \boldsymbol{e}_j) f_{12}^{-5}(0), \quad i \geq j.
\end{aligned}$$

Thus, the potential is represented by only the square-law part of the variables z_i as

$$\begin{aligned}
\Pi_{12} &= \tfrac{1}{2}[g_1(q_1^2 + q_2^2) + g_3 q_3^2 + h_1(\gamma_1^2 + \gamma_2^2) + h_3\gamma_3^2],\\
g_k &= -\iint\limits_{V_1 \cup V_2} \gamma \rho_1 \rho_2 \frac{\partial^2 f_{12}^{-1}}{\partial q_k^2}\, \mathrm{d}v_1\, \mathrm{d}v_2, \quad k = 1, 3,\\
h_k &= -\iint\limits_{V_1 \cup V_2} \gamma \rho_1 \rho_2 \frac{\partial^2 f_{12}^{-1}}{\partial \gamma_k^2}\, \mathrm{d}v_1\, \mathrm{d}v_2, \quad k = 1, 3.
\end{aligned} \tag{12}$$

The coefficient $h_3 = 0$ as both rigid bodies have symmetry relative to the axes C_1x_3 and C_2y_3 and an calculation of the appropriate integrals in equation (12) the rotation of the mantle or core about these axes does not change the relative positionings of points on these two rigid bodies. We shall estimate the value of the coefficient h_1 by assuming that the core and mantle have approximately spherical symmetry, and on their surfaces, at the points with coordinates $(0, 0, a_1)$ and $(0, 0, a_2)$, the negative masses $\Delta m_k = (A_k - C_k)a_k^{-2}$, $k = 1, 2$, the modulus of which coincides with values of superfluous masses in the equatorial areas of the core and external shell, are situated. The integral in equation (12) is replaced by a summation of these points and as a result it appears that

$$h_1 = \frac{\gamma(C_1 - A_1)(C_2 - A_2)}{a_1 a_2 (a_2 - a_1)^3}. \tag{13}$$

The coefficients g_1 and g_2 can be estimated approximately, by having assumed that the superfluous masses $4^{-1}(C_2 - A_2)a_2^{-2}$ are located at points with the coordinates $(\pm a_2, \pm a_2, 0)$.

In this case these coefficients are equal:

$$g_3 = -\tfrac{1}{2}g_1 = \frac{\gamma m_1(C_2 - A_2)}{a_2^5}. \tag{14}$$

We shall find the approximate values of the functions Π_{13} and Π_{23}. It should be noted that $\mathbf{\Gamma}_2 = \mathbf{\Gamma}_1\mathbf{\Gamma}$, and we shall consider initially a situation where $\boldsymbol{q} = \mathbf{0}$, $\mathbf{\Gamma} = E$. In this case the core and mantle make a single whole (an axisymmetrical rigid body), and the potential energy of gravitational forces will be given by Beletskii (1975)

$$\Pi_{13}(0) + \Pi_{23}(0) \approx -\frac{\mu}{R} + \frac{\mu}{2R^3}(A - C)(1 - 3\alpha_3^2), \tag{15}$$
$$A = A_1 + A_2, \quad C = C_1 + C_2, \quad \mu = \gamma(m_1 + m_2),$$
$$\alpha_3 = \boldsymbol{e}_3 \cdot \boldsymbol{R}^0, \quad \boldsymbol{R}^0 = \frac{\boldsymbol{R}}{|\boldsymbol{R}|},$$

where $\boldsymbol{e}_3$ is a vector along axis Cx_3. The potential energy of the core and mantle at their interaction with a material point is represented according to equation (15) in the following form:

$$\begin{aligned} \Pi_{13}(z) &\approx -\gamma m\left(\frac{m_1}{|\boldsymbol{R}_1|} - \frac{A_1 - C_1}{2|\boldsymbol{R}_1|^3}(1 - 3\alpha_{31}^2)\right), \quad \boldsymbol{R}_1 = \boldsymbol{R} + \mathbf{\Gamma}_1\boldsymbol{q}\frac{m_{\mathrm{r}}}{m_1}, \\ \Pi_{23}(z) &\approx -\gamma m\left(\frac{m_2}{|\boldsymbol{R}_2|} - \frac{A_2 - C_2}{2|\boldsymbol{R}_2|^3}(1 - 3\alpha_{32}^2)\right), \quad \boldsymbol{R}_2 = \boldsymbol{R} - \mathbf{\Gamma}_1\boldsymbol{q}\frac{m_{\mathrm{r}}}{m_2}, \end{aligned} \tag{16}$$

with

$$\alpha_{3k} = \mathbf{\Gamma}_k\boldsymbol{e}_3 \cdot \boldsymbol{R}_k^0, \quad \boldsymbol{R}_k^0 = \frac{\boldsymbol{R}_k}{|\boldsymbol{R}_k|}, \quad k = 1, 2.$$

In equations (16) there are terms of the order of smallness which is determined by the degree of attitude of $\max(a_1, a_2)/R$, and terms of the order of smallness $|\boldsymbol{q}|/R$, γ_1, γ_2, γ_3 and higher.

Taking into account these circumstances, we shall keep in equation (16) terms of the lowest order of smallness and we shall use the following expression:

$$\begin{aligned} \Pi_{13} + \Pi_{23} \approx{}& -\gamma\frac{m(m_1 + m_2)}{R} + \gamma\frac{mm_{\mathrm{r}}}{2R^3}[\boldsymbol{q}^2 - 3(\boldsymbol{R}^0, \Gamma_1\mathbf{q})^2] + \gamma\frac{m(A_1 - C_1)}{2R^3} \\ & \times\left\{1 - 3(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{e}_3)^2 - 3\frac{m_{\mathrm{r}}}{Rm_1}[(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{q})\right. \\ & \left. +2(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{e}_3)q_3 - 5(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{e}_3)(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{q})]\right\} \\ & + \gamma\frac{m(A_2 - C_2)}{2R^3}\left\{1 - 3(\boldsymbol{R}^0, \mathbf{\Gamma}_2\boldsymbol{e}_3)^2 - 3\frac{m_{\mathrm{r}}}{Rm_2}\right. \\ & \left. \times[(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{q}) + 2(\boldsymbol{R}^0, \mathbf{\Gamma}_2\boldsymbol{e}_3)q_3 - 5(\mathbf{R}^0, \mathbf{\Gamma}_2\boldsymbol{e}_3)(\boldsymbol{R}^0, \mathbf{\Gamma}_1\boldsymbol{q})]\right\}. \end{aligned} \tag{17}$$

As $\mathbf{\Gamma} = \mathbf{\Gamma}_1^{-1}\mathbf{\Gamma}_2$, according to equation (4) taking into account the smallness of the angles γ_k, $k = 1, 2, 3$, the following relations hold:

$$\gamma_1^2 = (\mathbf{\Gamma}_2\boldsymbol{e}_2, \mathbf{\Gamma}_1\boldsymbol{e}_3)^2, \quad \gamma_2^2 = (\mathbf{\Gamma}_2\boldsymbol{e}_3, \mathbf{\Gamma}_1\boldsymbol{e}_1)^2, \quad \gamma_3^2 = (\mathbf{\Gamma}_2\boldsymbol{e}_1, \mathbf{\Gamma}_1\boldsymbol{e}_2)^2, \tag{18}$$

which will be used in equations (7), (8) and (12).

To describe the motion of the system we shall take advantage of the Lagrange equations of the second sort within the framework of the restricted treatment of the problem when the centre of mass of a planet moves on an unperturbed Keplerian orbit, and the core and mantle of a planet make a single whole, that is a rigid body that rotates with a constant angular velocity around an axis of symmetry. Thus the unperturbed motion of a system is described by the well-known formulae

$$R = \frac{p}{1 + e\cos\vartheta}, \quad \boldsymbol{R}^0 = (\cos\vartheta, \sin\vartheta, 0),$$
$$\psi = \psi_0, \quad \theta = \theta_0, \quad \dot{\varphi} = \dot{\varphi}_0, \quad \psi_k = 0, \quad \theta_k = 0, \quad \dot{\varphi}_k = 0 \quad (k = 1, 2).$$

Here p is a parameter, e is the eccentricity and ϑ is the true anomaly of an orbit of the centre of mass of a planet. All variables with an index a zero are constant.

2.1.4 Lagrange function The Lagrange variables, subject to determination, in the considered restricted treatment of a problem are the components of the relative displacement vector of the core and mantle of a planet and their relative rotations. The Lagrange function in view of equations (9), (17) and (18) is represented in the form

$$\begin{aligned}
&-\tfrac{1}{2}(\mathbf{K}\boldsymbol{q}, \boldsymbol{q}) - \gamma\frac{mm_{\mathrm{r}}}{2R^3}[\boldsymbol{q}^2 - 3(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{q})^2] - \gamma\frac{m(A_1 - C_1)}{2R^3}\big(1 - 3(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)^2 \\
&\quad - 3\frac{m_{\mathrm{r}}}{Rm_1}[(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{q}) + 2(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)q_3 - 5(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{q})]\big) \\
&\quad - \gamma\frac{m(A_2 - C_2)}{2R^3}\Bigg(1 - 3(\boldsymbol{R}^0, \boldsymbol{\Gamma}_2\boldsymbol{e}_3)^2 - 3\frac{m_{\mathrm{r}}}{Rm_2} \\
&\quad \times[(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{q}) + 2(\boldsymbol{R}^0, \boldsymbol{\Gamma}_2\boldsymbol{e}_3)q_3 - 5(\boldsymbol{R}^0, \boldsymbol{\Gamma}_2\boldsymbol{e}_3)(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{q})]\Bigg) \\
&\quad - \tfrac{1}{2}n_1[(\boldsymbol{\Gamma}_2\boldsymbol{e}_2, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)^2 + (\boldsymbol{\Gamma}_2\boldsymbol{e}_3, \boldsymbol{\Gamma}_1\boldsymbol{e}_1)^2] - \tfrac{1}{2}n_1(\boldsymbol{\Gamma}_2\boldsymbol{e}_1, \boldsymbol{\Gamma}_1\boldsymbol{e}_2)^2, \qquad (19)
\end{aligned}$$

where

$$\mathbf{K} = \mathrm{diag}(k_1, k_2, k_3), \quad k_i = N_1 + g_i, \quad n_i = N_2 + h_i \quad (i = 1, 3).$$

We use the Lagrange function (19) to obtain the equations of motion of the mechanical system.

2.2 Equations of motion and their approximate solution

The variables $\boldsymbol{q}$ in the Lagrange equations have the following form:

$$\begin{aligned}
&m\ddot{\boldsymbol{q}} + \mathbf{K}q + \frac{\gamma mm_{\mathrm{r}}}{R^3}[\boldsymbol{q} - 3(\boldsymbol{R}^0, \boldsymbol{\Gamma}_0\boldsymbol{q})\Gamma_0^{-1}\boldsymbol{R}^0] \\
&\quad - \frac{3\gamma mm_{\mathrm{r}}(A_1 - C_1)}{2m_1R^4}[\boldsymbol{\Gamma}_1^{-1}\boldsymbol{R}^0 + 2(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)\boldsymbol{e}_3 - 5(\boldsymbol{R}^0, \boldsymbol{\Gamma}_1\boldsymbol{e}_3)\boldsymbol{\Gamma}_1^{-1}\boldsymbol{R}^0] \\
&\quad + \frac{3\gamma mm_{\mathrm{r}}(A_2 - C_2)}{2m_2R^4}[\boldsymbol{\Gamma}_1^{-1}\boldsymbol{R}^0 + 2(\boldsymbol{R}^0, \boldsymbol{\Gamma}_2\boldsymbol{e}_3)\boldsymbol{e}_3 - 5(\boldsymbol{R}^0, \boldsymbol{\Gamma}_2\boldsymbol{e}_3)\boldsymbol{\Gamma}_1^{-1}\boldsymbol{R}^0] = \mathbf{0}. \qquad (20)
\end{aligned}$$

In the equations (20) the linear variables: $\boldsymbol{q}$, $\psi_k, \theta_k, \varphi_k, k = 1, 2$, are kept, and the operator $\boldsymbol{\Gamma}_0$ corresponds to the unperturbed motion of the planet. The dissipative forces are not taken into account. Their presence will result in the damping of the oscillations and evolution of the slow variables. We shall assume that the frequencies of free oscillations of the core–mantle system, because large elastic forces, are higher than the frequencies of the perturbing forces $\dot{\varphi}_0 : \omega = (\gamma\, m\, R^{-3})^{1/2}$. Also the frequency of orbital motion is $\omega \ll \dot{\varphi}_0$. In equation (20) we shall omit the inertial terms and we shall average all coefficients for the variables $\boldsymbol{q}$, $\psi_k, \theta_k, \varphi_k$, $k = 1, 2$, the variables φ and ϑ, and the perturbing forces only the variable φ. As the orbital eccentricities of planets of the Solar System are small, this means that we can consider initially circular orbits when $e = 0$, $p = R$. As a result, the equation (20) becomes

$$
\begin{aligned}
&\mathbf{K}_0 q + \frac{12P_0}{R}\left\{\left[\left(\frac{A_1 - C_1}{m_1} - \frac{A_2 - C_2}{m_2}\right)[5\sin^2\theta\{[1-\cos(2g)] + 6\sin\theta\sin g\}\right.\right. \\
&\quad \left.\left. + \; 5\left(\frac{A_1 - C_1}{m_1}\theta_1 - \frac{A_2 - C_2}{m_2}\frac{\theta_1 + \theta_2}{2}\right)\sin(2\theta)\right]\right\}\boldsymbol{e}_3 = \mathbf{0}, \\
&P_0 = \frac{m_{\mathrm{r}}\omega^2}{4}, \\
&\boldsymbol{K}_0 = \boldsymbol{K} + P_0\,\mathrm{diag}(1 - 3\cos^2\theta, 1 - 3\cos^2\theta, -2 + 6\cos^2\theta), \quad g = \vartheta - \psi.
\end{aligned}
\tag{21}
$$

In equation (21) and the following, the subscript zero on the variables θ and ψ is omitted. From equation (21) it follows that the components of the displacement vector of the centre of the mantle relative to the core, namely q_1 and q_2, are equal to zero. We shall linearize the Lagrange equations corresponding to the angular generalized coordinates $\psi_k, \theta_k, \varphi_k, k = 1, 2$, with respect to these variables and we shall average with respect to the fast variable φ of unperturbed motion. As a result we shall obtain the following reduced equations:

$$
\begin{aligned}
&A_k\ddot{\psi}_k\sin^2\theta - H_k\dot{\theta}_k\sin\theta + C_k(\ddot{\varphi}_k + \ddot{\psi}_k\cos\theta)\cos\theta \\
&\quad + (-1)^k[(\psi_2 - \psi_1)(n_1\sin^2\theta + n_3\cos^2\theta) + n_3(\varphi_2 - \varphi_1)\cos\theta] \\
&\quad = -\tfrac{3}{2}\omega^2\sin^2\theta\sin(2g)\sum_{j=1}^{2}(A_j - C_j)\delta_{jk}, \quad H_k = C_k\dot{\varphi}_0, \\
&A_k\ddot{\theta}_k + H_k\dot{\psi}_k\sin\theta - \tfrac{3}{4}\omega^2\sum_{j=1}^{2}(A_j - C_j)\{\sin(2\theta)[1 - \cos(2g)] \\
&\quad + 2\theta_j\cos(2\theta)\}\delta_{jk} + (-1)^k n_1(\theta_2 - \theta_1) = 0, \\
&C_k(\ddot{\varphi}_k + \ddot{\psi}_k\cos\theta) + (-1)^k n_3[(\psi_2 - \psi_1)\cos\theta + \varphi_2 - \varphi_1] = 0, \quad k = 1, 2.
\end{aligned}
\tag{22}
$$

Here δ_{jk} is the Kronecker symbol. In equations (22), terms of the order of R^{-3} are saved and terms of the order of R^{-4} are rejected. Averaging is made with respect to the variables φ and g in expressions for the coefficients of the variables ψ_k, θ_k, and φ_k and with respect to the variable φ in perturbations. As a result the system of equations (22) describing the rotations of the mantle relative to the core is separated from equations (21) which determine the displacement of a core relative to the external shell and are represented as

$$
\begin{aligned}
&q_1 = q_2 = 0, \quad q_3 = S_0 + S_1\theta_1 + S_2\theta_2, \\
&S_0 = \frac{12P_0}{k_{30}R}\left(\frac{C_1 - A_1}{m_1} - \frac{C_2 - A_2}{m_2}\right)\{5\sin^2\theta[1 - \cos(2g)] + 6\sin\theta\sin g\},
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \frac{60P_0}{k_{30}R}\left[\frac{C_1 - A_1}{m_1} - \frac{C_2 - A_2}{2m_2}\right]\sin(2\theta), \qquad (23)\\
S_2 &= -\frac{30P_0}{k_{30}R}\frac{C_2 - A_2}{m_2}\sin(2\theta),\\
k_{30} &= k_3 + 2P_0(3\cos^2\theta - 1).
\end{aligned}
$$

In accordance with equations (7), (14), (19) and (21), for coefficient k_{30} we obtain the following expression:

$$
k_3 = Ea_1\frac{4\pi(1-\nu)[(a_1+l)^3 - a_1^3]}{9l^2a_1(1+\nu)(1-2\nu)} + m_2\frac{\gamma m_1}{a_2^3}\frac{C_2 - A_2}{m_2a_2^2} + m_{\mathrm{r}}\omega^2\tfrac{1}{2}(3\cos^2\theta - 1).
$$

The established displacements of the mantle relative to the core of a planet are described by equation (22) in which we shall reject inertial members that exclude from consideration fast nutation oscillations of a core relative to the mantle with a small amplitude, and their motion is considered within the framework of the approximate theory of gyroscopes. As a result the following system of equations is obtained:

$$
\begin{gathered}
H_1\dot{\theta}_1 + n_1\Delta\psi\sin\theta = s_1,\\
H_2\dot{\theta}_2 - n_1\Delta\psi\sin\theta = s_2,\\
s_k = 3\omega^2\sin\theta(C_k - A_k)\mathrm{Re}[\mathrm{i}\exp(2\mathrm{i}g)],\\
H_1\dot{\psi}_1 + L_{11}\theta_1 + L_{21}\theta_2 = L_{01},\\
H_2\dot{\psi}_2 + L_{12}\theta_1 + L_{22}\theta_2 = L_{02}, \qquad (24)\\
\Delta\psi\cos\theta + \Delta\varphi = 0,\\
L_{kk} = \frac{3\omega^2(C_k - A_k)\cos(2\theta) + 2n_1}{2\sin\theta}, \quad L_{21} = L_{12} = -\frac{n_1}{\sin\theta},\\
L_{0k} = \frac{3\omega^2(A_k - C_k)}{2}\cos\theta\{1 - \mathrm{Re}[\exp(2\mathrm{i}g)]\}, \quad k = 1, 2.
\end{gathered}
$$

In system (24) the fifth equation is separated from the first four equations, and from the third and fourth equations of this system the equation follows

$$
\begin{aligned}
&\Delta\dot{\psi} - M_1\theta_1 + M_2\theta_2 = M_0\{1 - \mathrm{Re}[\exp(2\mathrm{i}g)]\},\\
&M_0 = \frac{3\omega^2}{2\dot{\varphi}_0}\left(\frac{A_2}{C_2} - \frac{A_1}{C_1}\right)\cos(2\theta), \qquad (25)\\
&M_k = \frac{n_1}{\sin\theta}\left(\frac{1}{H_1} + \frac{1}{H_2}\right) + \frac{3\omega^2\cos(2\theta)(C_k - A_k)}{2H_k\sin\theta}, \quad k = 1, 2.
\end{aligned}
$$

Solving the linear system of the differential equations consisting of the first two equations of system (24) and equation (25), we shall obtain its general solution in the form

$$
\begin{aligned}
\theta_1 &= \frac{D_0 - M_0}{2M_1} - H_1^{-1}[D_1\cos(\Omega t) - D_2\sin(\Omega t)] + G_1\cos(2g),\\
\theta_2 &= \frac{D_0 + M_0}{2M_2} + H_2^{-1}[D_1\cos(\Omega t) - D_2\sin(\Omega t)] + G_2\cos(2g),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\psi &= -\frac{\Omega}{n_1 \sin\theta}[D_1 \sin(\Omega t) + D_2(\cos\Omega t)], \qquad (26)\\
\Omega &= \left[n_1^2(H_1^{-1} + H_2^{-1})^2 + \frac{3}{2}n_1\omega^2\left(\frac{C_1 - A_1}{H_1^2} + \frac{C_2 - A_2}{H_2^2}\right)\cos(2\theta)\right]^{1/2},\\
G_k &= G_0^{-1}[12\omega^3 H_1 H_2 H_k^{-1}(C_k - A_k) + 2(-1)^k n_1 H_1 H_2 H_k^{-1} M_0\\
&\quad - 3\omega n_1 M_1 M_2 M_k^{-1} \sin\theta(C_1 - A_1 + C_2 - A_2)]\sin\theta, \quad k = 1, 2,\\
G_0 &= 8H_1 H_2\omega^2 - 2n_1 \sin\theta(H_1 M_2 + H_2 M_1).
\end{aligned}
$$

Here D_0, D_1 and D_2 are arbitrary constants. Because the core of the planet and its mantle as started above take the form of axysimmetric flattened ellipsoids under the influence of the centrifugal forces of inertia, this means that $C_k - A_k > 0, k = 1,\ 2$. Then, from equation (26), it follows that the size Ω^2 is positive. Dissipative forces which are not taken into account above cause the oscillations to be damped. The dissipative forces correspond to insertion in equation (24) of a coefficient $n_1(1 + \chi p)$ containing a multiplier n_1 and where p is the operator of differentiation with respect to time, and $\chi = \chi_1 N_1 n_1^{-1}$. As a result the characteristic equation of the system will become

$$
H_1 H_2 p^3 + pn_1 \sin\theta(\chi p + 1)[(\chi Mp + M_2)H_1 + (\chi Mp + M_1)H_2] = 0, \qquad (27)
$$

with

$$
M = n_1 \sin^{-1}\theta(H_1^{-1} + H_2^{-1}).
$$

Equation (27) has as before a zero root and two complex conjugate roots with a negative real part:

$$
p_{1,2} = \frac{-\chi Z_1 \pm (\chi^2 Z_1^2 - 4Z_0 Z_2)^{1/2}}{2Z_0}, \qquad (28)
$$

with

$$
\begin{aligned}
Z_0 &= n_1^2\frac{(C_1 + C_2)^2}{C_1 C_2} + \frac{3\omega^2 n_1 \cos(2\theta)}{2}\frac{[(C_1 - A_1)C_1^2 + (C_2 - A_2)C_2^2]}{C_1 C_2} > 0,\\
Z_1 &= \frac{2n_1(C_1 + C_2)^2}{C_1 C_2 \sin\theta} + \frac{3\omega^2 \cos(2\theta)}{2\sin\theta}\frac{(C_1 - A_1)C_2^2 + (C_2 - A_2)C_1^2}{C_1 C_2} > 0,\\
Z_2 &= H_1 H_2 + \chi^2 n_1^2\frac{(C_1 + C_2)^2}{C_1 C_2} > 0.
\end{aligned}
$$

After damping of the oscillations the solution of the system will be given by equations (26) in which it is necessary to put $D_1 = D_2 = 0$, and the forced oscillations at frequency 2ω to be given by the relation $[G_k + O(\chi)]\cos(2g) + O(\chi)\sin(2g)$, $k = 1, 2$, where $O(\chi)$ means the small terms of order χ and higher in the assumption about the smallness of the product $\chi\Omega$. Neglecting these terms, we shall obtain as a first approximation the established solution in the following form:

$$
\begin{aligned}
\theta_1 &= \frac{D_0 - M_0}{2M_1} + G_1 \cos(2g),\\
\theta_2 &= \frac{D_0 + M_0}{2M_2} + G_2 \cos(2g), \qquad (29)\\
\Delta\psi &= 0.
\end{aligned}
$$

The angular velocity of precession in the established regime of motion is determined from equations (24) taking into account equations (29) and is

$$\begin{aligned}\dot{\psi}_1 &= \dot{\psi}_2 \\ &= \frac{3\omega^2 \cos(2\theta)}{2(H_1+H_2)\sin\theta}\left[\left(\frac{A_1-C_1}{M_1}+\frac{A_2-C_2}{M_2}\right)D_0+\left(\frac{A_1-C_1}{M_1}-\frac{A_2-C_2}{M_2}\right)M_0\right] \\ &\quad+\frac{3\omega^2\cos\theta}{2(H_1+H_2)}(A_1-C_1+A_2-C_2)[1-\cos(2g)]. \end{aligned}\tag{30}$$

The value $\Delta\varphi$ in the established state is equal to zero. If the core and mantle of the planet are dynamically similar each other, this means equality of the ratios, that is, $A_1C_1^{-1} = A_2C_2^{-1}$; then $M_0 = 0$, $M_1 = M_2$ and $G_1 = G_2$ and, according to equation (29), the angles θ_1 and θ_2 have the identical constant components and identical periodic components. The constant component can be accepted as equal to zero, in an appropriate way having corrected the unperturbed value of the angle θ. Similar reasoning can be used in the general case, having assumed an arbitrary constant $D_0 = M_0(M_2 - M_1)(M_1 + M_2)^{-1}$ and having presented equations (29) and (30) as

$$\begin{aligned}\theta_1 &= -\frac{M_0}{M_1+M_2}+G_1\cos(2g), \\ \theta_2 &= \frac{M_0}{M_1+M_2}+G_2\cos(2g), \\ \Delta\psi &= 0, \\ \dot{\psi}_1 &= \dot{\psi}_2 = \frac{3\omega^2 M_0\cos(2\theta)}{(H_1+H_2)(M_1+M_2)\sin\theta}\left(\frac{(A_1-C_1)M_2}{M_1}-\frac{(A_2-C_2)M_1}{M_2}\right) \\ &\quad+\frac{3\omega^2\cos\theta}{2(H_1+H_2)}(A_1-C_1+A_2-C_2)[1-\cos(2g)]. \end{aligned}\tag{31}$$

The fulfilled analysis of the equations of motion was based on the approximate equations obtained as a result of averaging the fast variables, and by a set of assumptions concerning terms containing accelerations, which naturally leads to the corresponding approximate results.

2.3 Mutual motions of the core and mantle in the Earth–Moon system

As a numerical example we shall consider the Earth –Moon system, having taken advantage of the known data on a structure of the Earth (Zharkov, 1983; Barkin, 2002c; Sidorenkov, 2002). As basic units of dimensional sizes we use kilograms, metres and seconds, and all the following quantities are given is this system of units except where the units are otherwise stated. For the Earth's core we shall take the numerical values of its characteristics as

$$a_1 = 3.47\times10^6,\quad m_1 = 1.94\times10^{24},\quad C_1 = 9.1387\times10^{36},\quad A_1 = 9.1206\times10^{36},$$

and for the Earth's mantle

$$a_2 = 6.37\times10^6,\quad m_2 = 4.03\times10^{24},\quad C_2 = 7.1266\times10^{37},\quad A_2 = 7.0994\times10^{37}.$$

Using the density of the mantle at the depth of 2900 km as equal to 5200 kg m^{-3}, the velocity of longitudinal waves as 14 000 m s^{-1}, and the velocity of transverse waves as 6000 m s^{-1}, we obtain the value of Young's modulus $E = 4.53\times10^{11}$ and Poisson's ratio $\nu = 0.21$. According to Sidorenkov (2002) the thickness of the intermediate layer separating the core and the mantle is $l = 3 \times 10^4$. The Moon is modelled by a material point and its motion relative to the Earth occurs on a circular orbit of radius $R = 3.84 \times 10^8$ with a period of 27.3 days that corresponds to the mean orbital motion with $\omega = 2.66 \times 10^{-6}$. According to equations (7), (13) and (14), we obtain

$$N_1 = 8.64 \times 10^{20}, \quad N_2 = 2.485 \times 10^{33}, \quad g_3 = 3.36 \times 10^{14}, \quad h_1 = 6.09 \times 10^{25}.$$

Then we calculate that

$$H_1 = 6.66 \times 10^{32}, \quad H_2 = 51.95 \times 10^{32};$$

so the angular velocity of the Earth's diurnal rotation is $\dot{\varphi}_0 = 7.29 \times 10^{-5}$. On the assumption that the inclination of the axis of rotation of the Earth to the plane of the orbit of the Moon is equal to $\theta = 23°$ (generally speaking, this angle varies as the normal to the plane of the orbit of the Moon precesses relative to the normal to the ecliptic plane and describes a cone with an angle of solution of about 10° for a period of 18.6 years), we shall calculate according to equation (25) the following values:

$$M_0 = -1.848 \times 10^{-10}, \quad M_1 = 105.83, \quad M_2 = 10.589.$$

The coefficients G_1 and G_2 are determined from equation (26) and are approximately equal:

$$G_1 = 0.077\,069, \quad G_2 = 0.077\,025.$$

The difference between the angles of nutation according to equations (31) appears to be

$$\theta_2 - \theta_1 = \frac{2M_0}{M_1 + M_2} + (G_1 - G_2)\cos(2g) = -1.746 \times 10^{-11} - 4.4 \times 10^{-5}\cos(2g)$$

or, in arcsececonds,

$$\theta_2 - \theta_1 = -0''36 \times 10^{-5} - 9''076\cos(2g). \tag{32}$$

From equation (32) it follows that the axes of symmetry of core and mantle have insignificant stationary splitting (on the surface of the core of the Earth with a radius of 3470 km the points where they crossing are situated from each other at a distance of 0.06 mm), and the periodic part of a the splitting (its period is equal to a half the Moon's orbital period) has the maximum value of 152 m. This means that the core 'is periodically rotated' relative to the mantle at the above-mentioned distance. According to measurements of the angular velocity of rotation of the Earth, harmonics with the fortnightly and lunar month periods are observed in its spectrum (Sidorenkov, 2002, p. 28).

Displacements of the mantle of the planet relative to the core are determined on the basis of equations (23). The numerical values of the corresponding characteristics in the considered numerical example are small and given by

$$\begin{aligned} S_0 &= -3.71 \times 10^{-6}[1 - \cos(2g)] - 11.42 \times 10^{-6}\sin g, \\ S_1\theta_1 &= -1.17 \times 10^{-17} - 5.66 \times 10^{-7}\cos(2g), \\ S_2\theta_2 &= 1.69 \times 10^{-17} - 7.85 \times 10^{-7}\cos(2g). \end{aligned} \tag{33}$$

The dynamic effects described above, together with tidal effects have a global planetary character. They can influence seismic activity alongside tidal effects in the deformable mantle of the Earth. Of course, to obtain more reliable results about amplitudes and the full spectra of relative oscillations of the core-mantle system, models of the Earth must be studied more accurately and realistically.

3 SOME EFFECTS IN GEOCENTRE MOTION

3.1 About geocentre oscillation with fortnightly period

The observed effect of the relative displacements of the core and mantle of the Earth described above allow us to evaluate geocentre displacements with the same period and to compare with the data obtained from Global Positioning System (GPS) and Doppler Orbitography and Radiopositioning Integrated by Satellite (DORIS) observations.

The fortnightly oscillation of the geocenter was observed in GPS and DORIS data of observations in the paper (Tatevian *et al.*, 2004). Time series of geocentre variations for the period 1993–2003 have been analysed. A constant term and trend have been estimated in order to express the time series in a common reference frame. The amplitudes and phases of the annual and semiannual variations in the geocentre components x_C, y_C, and z_C have been evaluated for different sets of data (GPS daily data for the 10 year period January 1993–October 2003 and DORIS weekly solutions for the period January 1993–October 2003 and March 1999–August 2002).

The periodicity analysis technique based on the least-squares iterations was used for mathematical modelling. As a result the trend components (linear components) and the set of harmonics were observed. The computations were made in two steps: firstly, estimation and removal of the trend; secondly, finding one after another and removal of the periodic components (harmonics) with amplitudes twice exceeding their standard errors (see Table 3 for GPS data). The final model is a polyharmonic function that has the best fit to the initial time series. An analysis of the obtained series of the geocentre variations shows that the precision of the x_C and y_C components is generally significantly better than for the z_C component, and the amplitudes of the z_C component are twice to three times those for the x_C and y_C components for both GPS and DORIS solutions. Geocentre motions with a definite spectrum of frequencies as determined using DORIS and GPS data are of the order of 5–18 mm for each coordinate, and the secular trend (for the considered period) is about 2–4 mm year^{-1}.

The relative displacement of the centres of mass $C_{\mathrm{m}}^{(0)}$ and C_{c} of the undeformable mantle and core in $\boldsymbol{r}_C = (x_C, y_C, z_C)$. Corresponding displacements of these centres relative to the general centre of mass $C^{(0)}$ are r_{c} and $r_{\mathrm{m}}^{(0)}$. They satisfy the simple relations

$$r_{\mathrm{c}} + r_{\mathrm{m}}^{(0)} = r_C = |\boldsymbol{r}_C| , \quad r_{\mathrm{c}} m_{\mathrm{c}} = r_{\mathrm{m}} m_{\mathrm{m}}^{(0)} .$$

The centre of mass of the mantle is displaced in the opposite direction to direction of the center of the core displacement. So

$$O_{\mathrm{m}}C = r_{\mathrm{m}} = r_{\mathrm{m}}^{(0)} - \delta r_{\mathrm{m}} = \rho \left(-k_{-2} + \frac{m_{\mathrm{c}}}{m_{\mathrm{c}} + m_{\mathrm{m}}} \right) .$$

We have considered a restricted treatment of the problem except for consideration of the fortnightly variation in relative positions of the core and mantle. From our model it follows that for the described oscillation of the shells for the Earth, oscillation of the centre of mass of the Earth must be observed for a fortnightly period. This oscillation mainly occurs along the polar axis of the Earth with an amplitude of about 2.32 mm and with a period of 13.78 days. This result is sufficiently close to the observed oscillation of the geocentre of 2.17 ± 0.74 mm for GPS data (period, 13.41 days) and 3.88 ± 2.19 mm for DORIS data (period, 13.73 days). For this evaluation we have introduced a superfluous mass of the core which is equal to $\Delta m_C = 0.1932 m_\oplus$ ($m_\oplus$ is the mass of the Earth) (Barkin, 1999).

In a more exact treatment of problem (taking into account all the spectrum of orbital perturbations in the motion of Earth–Moon system) we plan to evaluate and explain again the observed oscillations in the motion of the geocentre. Data about amplitudes, the phases and

Table 3. Variations in the geocentre positions on the data of GPS observations in last 10 years.

A (mm)	φ *(years)*	T_i *(years)*$_{observed}$	T_i *(days)*$_{observed}$	T_i *(days)*$_{predicted}$
21.33 ± 0.87	998.682 ± 0.013	1.0065 ± 0.0021	367.6 ± 0.77	365.3
12.45 ± 0.85	998.084 ± 0.040	1.8265 ± 0.0115	667.1 ± 4.2	668.4
10.82 ± 0.86	997.613 ± 0.182	8.0053 ± 0.2443	2924.0 ± 89	2922
8.55 ± 0.81	997.698 ± 0.051	1.6071 ± 0.0129	587.0 ± 4.7	591.7
6.95 ± 0.81	996.727 ± 0.134	3.6245 ± 0.0788	1324.0 ± 29	1293
6.32 ± 0.80	998.903 ± 0.013	0.3147 ± 0.0006	114.9 ± 0.22	113
6.35 ± 0.80	998.858 ± 0.020	0.5026 ± 0.0016	183.6 ± 0.58	183
6.64 ± 0.79	998.460 ± 0.036	0.9423 ± 0.0054	344.2 ± 1.97	346
5.49 ± 0.78	998.784 ± 0.021	0.4601 ± 0.0015	168.1 ± 0.55	172
5.17 ± 0.78	998.633 ± 0.042	0.8678 ± 0.0059	317.0 ± 2.16	326
5.01 ± 0.77	998.829 ± 0.010	0.1923 ± 0.0003	70.2 ± 0.11	69.8
4.47 ± 0.77	998.999 ± 0.013	0.2396 ± 0.0005	87.5 ± 0.18	90.7
4.49 ± 0.77	998.641 ± 0.033	0.5945 ± 0.0031	217.1 ± 1.13	221
4.04 ± 0.76	996.222 ± 0.329	5.5339 ± 0.3112	2021.0 ± 114	2181
3.76 ± 0.76	998.697 ± 0.137	2.1036 ± 0.0466	768.3 ± 17	770.7
3.98 ± 0.76	998.866 ± 0.018	0.2881 ± 0.0008	105.23 ± 0.29	113; 100
3.43 ± 0.76	998.901 ± 0.011	0.1608 ± 0.0003	58.73 ± 0.11	60.1
3.42 ± 0.76	998.776 ± 0.016	0.2287 ± 0.0006	83.53 ± 0.22	80.4
3.29 ± 0.75	998.946 ± 0.020	0.2652 ± 0.0008	96.86 ± 0.29	100
3.09 ± 0.75	998.682 ± 0.026	0.3334 ± 0.0014	121.8 ± 0.51	122
2.81 ± 0.75	998.639 ± 0.038	0.4442 ± 0.0027	162.2 ± 0.99	161
2.77 ± 0.75	998.956 ± 0.005	0.0577 ± 0.0001	21.1 ± 0.04	19.4
2.74 ± 0.75	996.765 ± 0.239	2.6419 ± 0.0979	965.0 ± 35.8	1023
2.71 ± 0.75	998.901 ± 0.047	0.5335 ± 0.0040	194.9 ± 1.46	192
2.40 ± 0.75	998.786 ± 0.022	0.2175 ± 0.0008	79.4 ± 0.29	80.4
2.36 ± 0.75	998.948 ± 0.012	0.1153 ± 0.0002	42.1 ± 0.07	40.4
2.33 ± 0.75	998.935 ± 0.009	0.0872 ± 0.0001	31.9 ± 0.04	30.5
2.27 ± 0.74	998.961 ± 0.005	0.0444 ± 0.0001	16.2 ± 0.04	15.9
2.24 ± 0.74	998.942 ± 0.008	0.0710 ± 0.0001	25.9 ± 0.04	27.4
2.24 ± 0.74	998.944 ± 0.014	0.1278 ± 0.0003	46.7 ± 0.11	50.4
2.17 ± 0.74	998.966 ± 0.006	0.0557 ± 0.0001	20.3 ± 0.04	19.4
2.18 ± 0.74	998.999 ± 0.004	0.0367 ± 0.0001	13.4 ± 0.04	13.8
2.15 ± 0.74	998.860 ± 0.019	0.1669 ± 0.0005	61.0 ± 0.18	60.1
2.28 ± 0.74	998.961 ± 0.034	0.3211 ± 0.0017	117.3 ± 0.62	113
2.10 ± 0.74	998.962 ± 0.013	0.1105 ± 0.0002	40.4 ± 0.07	40.4

periods of these oscillations are given in the Table 3. In the last column the theoretical values of the periods of expected oscillations of the geocentre are presented. These values have been predicted (Barkin, 2002c) on the basis of the hypothesis that all planetary processes have the same frequency basis. This hypothesis is confirmed by analytical solution obtained in this paper and by the observed data for other natural processes.

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