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On Classes of Groups of Finite Metabelian Rank

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Abstract—We introduce the notion of the metabelian rank of a group and study non-Abelian groups of finite metabelian rank. We prove the following result: If G is the extension of a locally finite group by a locally nilpotent-by-finite group and the metabelian rank of G is finite then the special rank of G is finite too.

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1. INTRODUCTION

One of important directions in group theory is studying classes of groups with restrictions on ranks for systems of their subgroups. The notion of the rank of a group appeared in the theory of Abelian groups as an analog of the notion of the dimension of a vector space. Mal'tsev extended this notion to the case of arbitrary groups; namely, he introduced the notions of the general and special ranks [26]. By definition, the general rank of a group is finite and equal to r if r is the least number such that each finitely generated subgroup of this group is included in a subgroup with at most r generators. The special rank of a group is finite and equal to r if r is the least number such that each finitely generated subgroup of this group admits a system of generators with at most r elements. The notion of a group of finite special rank became widespread. Now the special rank is usually called simply "the rank."

Numerous articles of Soviet and foreign authors were devoted to studying groups of finite rank. Mal'tsev's article [27] played a fundamental role. In that article, linear solvable groups were studied and, on their basis, a program was formulated (and mostly implemented) for studying various classes of solvable groups of finite rank. In particular, the famous theorem was proven on the almost triangulability of a solvable matrix group over an algebraically closed field (later called the Kolchin—Mal'tsev theorem) and a series of other well-known results was obtained. Solvable groups of finite rank and their automorphism groups were studied in [17, 23, 28, 3—7, 35—37]. Kargapolov [23] proved his deep theorem saying that the rank of a solvable group is finite if the rank of each Abelian subgroup is finite. Later this result was generalized by Baer and Heineken to radical groups [1]. Gorchakov showed that a similar result is valid for periodic locally solvable groups [18]. Shunkov proved the corresponding theorem for locally finite groups [34]. On the other hand, Merzlyakov constructed an example of a locally solvable group of infinite rank such that the rank of each Abelian subgroup is finite (moreover, these subgroups are finitely generated) [29].

Notice that the class of solvable groups of finite rank includes the classes of polycyclic groups, solvable minimax groups (i.e., groups admitting a subnormal series whose factors are Abelian and satisfy either the minimum condition or the maximum condition), groups admitting a finite rational series, etc. These types of groups are closely connected with groups of finite rank. For example, Robinson proved that a finitely generated solvable group of finite rank is a minimax group [30].

For a more complete survey of results on groups of finite rank, the reader is referred to [24, Sec. 25; 32]. Requirement that the rank of a group be finite is now widely used in various group theoretical problems.

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Comparing the requirement that the rank be finite with other finiteness conditions (for example, the minimum and maximum conditions on subgroups), it is not difficult to observe that the latter were considered for subgroups of various types (for example, Abelian, normal, non-Abelian, primary, etc.). In this connection, Zaĭtsev introduced the notion of the $\mathfrak{F}$-rank of a group, see [9].

Definitinon 1. Let G be a group and let $\mathfrak{F}$ be a nonempty system of its finitely generated subgroups. The *$\mathfrak{F}$-rank of the group G* is the least number r such that each subgroup belonging to $\mathfrak{F}$ admits a generating set with at most r elements. If such a number r does not exist then the $\mathfrak{F}$-rank of the group G is infinite.

In [9, 10], non-Abelian groups of finite $\mathfrak{F}$-rank were studied, where

(1) the system $\mathfrak{F}$ consists of all non-Abelian finitely generated subgroups of a non-Abelian group (i.e., groups of finite non-Abelian rank);

(2) the system $\mathfrak{F}$ consists of all non-Abelian non-periodic finitely generated subgroups of a non-periodic non-Abelian group (i.e., groups of finite non-Abelian 0-rank).

In [11—13], generalizations of groups of finite $\mathfrak{F}$-rank were studied. Notice that rank restrictions for systems of subgroups were widely used in studying linear groups of infinite dimension [14—16].

In the present article, we study non-Abelian groups of finite $\mathfrak{F}$-rank, where $\mathfrak{F}$ is the system of all metabelian (i.e., non-Abelian 2-step solvable) finitely generated subgroups of a group (i.e., groups of finite metabelian rank). Investigation of this class was initiated in [8].

The main results of the present article are Theorems 1—3 and Corollary 2.

2. PRELIMINARIES

Throughout the article, we consider non-Abelian groups only. By a *metabelian group* we always mean a non-Abelian 2-step solvable group. We denote the metabelian rank of a group G by $r_{ma}(G)$. We call the special rank of a group G the *rank of* G and denote, as usual, by $r(G)$.

Recall that the 0-rank of an Abelian group is the rank of its quotient group modulo the periodic part. The 0-rank of a solvable group is the sum of the 0-ranks of the factors of its subnormal series with Abelian factors. The 0-rank of a solvable group G is denoted by $r_0(G)$ and is an invariant of G. By the 0-rank of an almost solvable group we mean the 0-rank of its normal solvable subgroup of finite index.

Solvable groups of finite 0-rank are called solvable A_1-groups in Mal'tsev's classification [27]. As is known, the quotient group $G/t(G)$ of a solvable group G of finite 0-rank modulo its periodic radical $t(G)$ (i.e., the greatest normal periodic subgroup) is a solvable A_4-group, i.e., a group of finite rank admitting a finite subnormal series with Abelian factors whose periodic parts are finite [27]. This condition holds if and only if $G/t(G)$ is an almost torsion-free group; moreover, in this case, there is a subgroup $H/t(G)$ of finite index admitting a finite rational series, i.e., a subnormal series whose factors are isomorphic to suitable subgroups of the additive group of rational numbers.

In our study of groups of finite metabelian rank, we will need a series of lemmas.

Lemma 1. *Let G be a metabelian group and let G' denote its commutant. Then the inequality $r(G/G') \leq r_{ma}(G)$ holds.*

Proof. If H/G' is a finitely generated subgroup of G/G' then H is included in a suitable subgroup $H_1 = KG'$, where K is a metabelian finitely generated subgroup. Since K is generated by at most $r_{ma}(G)$ elements, so are the Abelian group H_1/G' and its subgroup H/G'. □

Lemma 2. *Let K be a non-Abelian normal subgroup of a metabelian group G. Then the inequality $r(G/K) \leq r_{ma}(G)$ holds.*

Proof. Let H/K be a finitely generated subgroup of G/K. We represent H as the product $H = SK$, where S is a metabelian finitely generated subgroup. By the definition of the metabelian rank of G, the subgroup S admits a generating set with at most $r_{ma}(G)$ elements. Since $H = SK$, the quotient group H/K is generated by at most $r_{ma}(G)$ elements. This finishes the proof of the inequality $r(G/K) \leq r_{ma}(G)$. □

Corollary 1. *If a metabelian group G can be represented as the product of a central subgroup Z and a metabelian subgroup K then we have*

$$r(Z) \leq r(Z \cap K) + r_{ma}(G).$$

Proof. Since $G/K \simeq Z/Z \cap K$, we have $r(Z) \leq r(Z \cap K)+r(G/K)$. Taking into account the relation $r(G/K) \leq r_{ma}(G)$ from Lemma 2, we obtain the required inequality. ☐

Lemma 3. *Let G be an almost solvable group. Then the inequality $r\big(Z(G)\big) \leq 3 + r_{ma}(G)$ holds.*

Proof. We begin with the case in which G is a finite group. We choose a minimal non-Abelian subgroup F of G (a Miller–Moreno subgroup). Taking into account the description of finite minimal non-Abelian groups (see [22, pp. 285 and 309]), we conclude that F is a metabelian subgroup and the rank of its center is at most 3. Applying Corollary 1 to the subgroup $Z(G)F$, we obtain

$$r\big(Z(G)\big) \leq r\big(Z(G) \cap F\big) + r_{ma}(G) \leq 3 + r_{ma}(G).$$

We turn to the case in which G is a solvable group. Assume that $r\big(Z(G)\big) > 3 + r_{ma}(G)$. We consider a finitely generated subgroup Z of the center $Z(G)$ of G satisfying the condition

$$r(Z) > 3 + r_{ma}(G). \tag{1}$$

We choose a finitely generated metabelian subgroup H of G. The product $H_1 = ZH$ is a finitely generated metabelian subgroup. Let p be a prime number such that

$$r(Z) = r(Z/Z^p). \tag{2}$$

From results of [20] it follows that a finitely generated group is residually finite if its commutant is Abelian. Therefore, there exists a normal subgroup M of H_1 of finite index such that the quotient group H_1/M is not Abelian and there exists a normal subgroup N/Z^p of H_1/Z^p of finite index with $Z \cap N = Z^p$. The quotient group $H_1/M \cap N$ is finite and metabelian. Since $Z \cap N = Z^p$, there exists a subgroup $Z(M \cap N)/(M \cap N)$ of its center that is isomorphic to the quotient group $Z/(Z^p \cap M)$. For the metabelian subgroup H_1, the inequality $r_{ma}(H_1/M \cap N) \leq r_{ma}(G)$ holds. Taking into account the assertion for finite groups proven above, we conclude that the rank of $Z/(Z^p \cap M)$ is at most $3 + r_{ma}(G)$; hence, we have $r(Z/Z^p) \leq 3 + r_{ma}(G)$. According to (2), this contradicts assumption (1) about the rank of Z.

We turn to the case in which G is an almost solvable group. We show that there exists a non-Abelian solvable subgroup K of G. We choose a maximal normal solvable subgroup A of G. If A is a non-Abelian subgroup then we put $K = A$. If A is an Abelian subgroup, we consider the non-Abelian quotient group G/A and choose a minimal non-Abelian subgroup K/A. By [22, pp. 285 and 309], the subgroup K is solvable; hence, the subgroup $Z(G)K$ is non-Abelian and solvable. By the above arguments for solvable groups, we have $r\big(Z(G)\big) \leq 3 + r_{ma}(G)$. ☐

Lemma 4. *Let G be a non-Abelian group of finite metabelian rank and let A be an Abelian normal subgroup of G. Then the rank of A is finite.*

Proof. If A is a central subgroup of G then, by Lemma 3, we have $r(A) \leq 3 + r_{ma}(G)$. If the subgroup A is not central then there exists an element $g \in G$ such that the subgroup $A\langle g\rangle$ is not Abelian. Since the non-Abelian rank of $A\langle g\rangle$ is at most $r_{ma}(G)$, the rank of A is finite in view of results of [9]. ☐

Lemma 5. *Assume that a group G is represented as the product $G = A\langle g\rangle$, where A is an Abelian normal subgroup of G and $g^s \in C_G(A)$ for a suitable natural s. If G is a non-Abelian group then we have*

$$r(A) \leq r_{ma}(G)s + 1.$$

Proof. We consider an arbitrary finitely generated subgroup B of A that is a normal subgroup of G. Since every finitely generated subgroup of A is included in such a subgroup B, it suffices to prove the required inequality for B instead of A.

If $B\langle g\rangle$ is an Abelian subgroup then $B \leq Z(G)$. In view of Lemma 3, we have $r(B) \leq 3 + r_{ma}(G)$; hence, $r(B) \leq r_{ma}(G)s + 3$. If $B\langle g\rangle$ is a non-Abelian subgroup then it is generated by r elements $x_1, x_2, \ldots, x_r$, where $r \leq r_{ma}(G)$. We represent each element x_i in the form $x_i = a_i g_i$, where $a_i \in B$ and $g_i \in \langle g\rangle$. We put $B_0 = \langle a_1^G\rangle \cdots \langle a_r^G\rangle$, where $\langle a_i^G\rangle$ denotes the normal closure of a_i. Since $g^s \in C_G(A)$, there exist at most s elements of G that are conjugate to a_i. We conclude that

$$r\big(\langle a_i^G\rangle\big) \leq s, \quad r(B_0) \leq rs \leq r_{ma}(G)s, \quad r(B) \leq rs \leq r_{ma}(G)s + 1. \qquad \square$$

3. COINCIDENCE OF THE SUBCLASSES OF GROUPS OF FINITE METABELIAN RANK AND OF FINITE SPECIAL RANK FOR SOME CLASSES OF GROUPS

Lemma 6. *If the commutant of a group G is finite then there exists either an Abelian or metabelian subgroup of G of finite index.*

Proof. We denote by C the centralizer of the commutant G' in the group G and by K the commutant of the subgroup C. Since G' is finite, the index $|G : C|$ is finite too. Since $K \leq C$ and $K \leq G'$, we have $K \leq C \cap G'$; hence, $[K, C] = 1$. We conclude that the commutant of the subgroup C is Abelian. Since the index $|G : C|$ is finite, we find that there exists a subgroup of G of finite index that is either Abelian or metabelian. $\square$

Theorem 1. *If G is a locally finite group of finite metabelian rank then the rank of G is finite too.*

Proof. We begin with the case in which G is a periodic locally solvable group. Assume that the rank of G is infinite. By [21, 40], for every finite non-Abelian subgroup K of G, there exists a subgroup of G of the form AK, where A is an Abelian subgroup of infinite rank that is normalized by K. We apply Lemma 4 to the group AK and take into account the fact that the metabelian rank $r_{ma}(G)$ is finite. We find that the rank of the subgroup A is finite, which is a contradiction. Therefore, the rank of the group G is finite.

We turn to the case in which G is a locally finite group (and the metabelian rank $r_{ma}(G)$ is finite). We show that each Sylow p-subgroup of G, where p is prime, is a Chernikov subgroup. Assume the contrary, i.e., let there exist a prime p and a Sylow p-subgroup of G that is not a Chernikov subgroup. Then there exists a countable infinite non-Abelian subgroup K of G with a Sylow p-subgroup that is not a Chernikov subgroup. We represent K as the union of an increasing sequence of finite groups, i.e., let $K_1 < K_2 < \cdots$ and let $\bigcup_{i=1}^{\infty} K_i = K$. We construct a projection Sylow p-subgroup P of K, see [33]. The subgroup P is the union of the finite subgroups $P_i = P \cap K_i$. Since there exists a Sylow p-subgroup of K that is not a Chernikov subgroup, we conclude that P cannot be a Chernikov subgroup. Notice that P is an Abelian subgroup in view of the above arguments for the case of periodic locally solvable groups. Since the rank $r(P)$ is infinite, the ranks of the finite subgroups P_i increase without bound. We may assume that

$$r(P_i) > r_{ma}(G) + 3. \tag{3}$$

If there exists a number i such that the Abelian subgroup P_i is not central in the normalizer $N_{K_i}(P_i)$ then there exists an element $h \in N_{K_i}(P_i)$ such that the subgroup $P_i\langle h\rangle$ is not Abelian. We choose a number j such that $j > i$ and

$$r(P_j) > r_{ma}(G)n + 1, \tag{4}$$

where n is the order of h. If $P_j \neq P_j^h$ then the subgroup $\langle P_j, P_j^h\rangle$ generated by two Sylow p-subgroups of K_j is not Abelian. Since the subgroups P_j and P_j^h are Abelian and the subgroup $P_i = P_i^h$ is included in their intersection, we conclude that P_i is central in $\langle P_j, P_j^h\rangle$. By Lemma 3, we have

$$r(P_i) \leq r_{ma}(G) + 3,$$

which contradicts (3). We conclude that $P_j = P_j^h$ and the element h belongs to the normalizer $N_{K_j}(P_j)$. By Lemma 5, we have

$$r(P_j) \leq r_{ma}(G)n + 1,$$

which contradicts (4).

We now assume that, for every i, the subgroup P_i is included in the center of the normalizer $N_{K_i}(P_i)$. By a theorem of Burnside [19, Theorem 14.3.1], there exists a normal subgroup M_i of K_i such that $K_i = M_i \leftthreetimes P_i$. Since the quotient group K_{i+1}/M_{i+1} is a p-group, its nontrivial subgroup K_iM_{i+1}/M_{i+1} is a p-group too. Since $K_iM_{i+1}/M_{i+1} \simeq K_i/K_i \cap M_{i+1}$ and M_i is the least normal subgroup defining a quotient p-group in K_i, we find that $M_i \leq K_i \cap M_{i+1}$; hence, $M_i \leq M_{i+1}$. We denote by M the union $\bigcup_{i=1}^{\infty} M_i$. Then M forms a normal subgroup of K; moreover, we have $MP = K$ and $M \cap P = 1$. We conclude that $K = M \leftthreetimes P$, the quotient group K/M is an Abelian p-group, and the subgroup M lacks elements of order p.

Thus, if there exist a prime p and a Sylow p-subgroup of K that is not a Chernikov subgroup then $p \notin \pi(K')$. From [2] it follows that every Sylow q-subgroup of K', where q is prime, is a Chernikov subgroup. Hence, the group K' is almost locally solvable. We denote by S the locally solvable radical of the group K'. Then S is a characteristic subgroup of K' and its index in K' is finite. We conclude that the quotient group K/S is the extension of a finite group by an Abelian group and, consequently, the commutant of K/S is finite. By Lemma 6, the group K/S is either Abelian-by-finite or metabelian-by-finite; hence, the group K is almost locally solvable. By the above arguments for the case of periodic locally solvable groups and Lemma 4, the rank of K is finite.

We have proven that, for every prime p, every Sylow p-subgroup of the locally finite group G is a Chernikov subgroup. By [2], the group G is almost locally solvable; hence, the rank of G is finite. □

Theorem 2. *If G is a locally nilpotent group and its metabelian rank $r_{ma}(G)$ is finite then the rank of G is finite too. In particular, if the group G is torsion-free then its rank is bounded by some number $\eta\big(r_{ma}(G)\big)$ that depends on $r_{ma}(G)$ only.*

Proof. We begin with the case in which G is a metabelian nilpotent torsion-free group. Let H be a finitely generated metabelian subgroup of G. Notice that every finitely generated nilpotent group is polycyclic; hence, it is a minimax group. By [39, Lemma 5], there exist two characteristic subgroups K and L of H of finite index such that K/L is an elementary Abelian group of rank r, where r is the length of a rational series of H. Moreover, by [39, Theorem 1], the number r is equal to the rank of H, i.e., we have $r = r(H)$.

If L is an Abelian subgroup then the nilpotent torsion-free group H is Abelian-by-finite. Taking into account [24, Theorem 16.2.8], we find that H is an Abelian group, which is a contradiction. We conclude that the subgroup L is not Abelian. We apply Lemma 2 to the metabelian group H and its subgroup L. We obtain the inequality $r(H/L) \leq r_{ma}(H)$. Since $r(K/L) = r(H)$, we conclude that $r(H) \leq r_{ma}(H)$. On the other hand, the reverse inequality $r_{ma}(H) \leq r(H)$ holds for every metabelian group H. The inequalities $r(H) \leq r_{ma}(H)$ and $r_{ma}(H) \leq r(H)$ imply the equality

$$r(H) = r_{ma}(H). \tag{5}$$

In particular, there exists a metabelian finitely generated subgroup of G generated by exactly $r_{ma}(G)$ elements. Since the subgroup H is arbitrary, from equality (5) it follows that

$$r(G) = r_{ma}(G). \tag{6}$$

We turn to the case in which G is a locally nilpotent torsion-free group. Let A be a finitely generated Abelian subgroup of G and let H be the first non-Abelian upper central factor of a finitely generated non-Abelian subgroup that includes A. Since the nilpotent torsion-free subgroup AH is metabelian, from equality (6) and the obvious inequality $r_{ma}(AH) \leq r_{ma}(G)$ it follows that $r(AH) \leq r_{ma}(G)$. We conclude that $r(A) \leq r_{ma}(G)$. By [28], from the inequality $r(A) \leq r_{ma}(G)$ and the fact that A is an arbitrary Abelian finitely generated subgroup it follows that the rank of G is finite and is bounded by a number $\eta\big(r_{ma}(G)\big)$ that depends on $r_{ma}(G)$ only, i.e., we have

$$r(G) \leq \eta\big(r_{ma}(G)\big) = \frac{1}{2} r_{ma}(G)\big(r_{ma}(G) + 1\big).$$

We turn to the case in which G is a locally nilpotent group (and the metabelian rank $r_{ma}(G)$ is finite). We denote by T the periodic part of G, i.e., the subgroup consisting of all elements of finite order. If T is a non-Abelian subgroup then, by Theorem 1, the rank $r(T)$ is finite. If T is an Abelian subgroup then, by Lemma 4, we have $r(T) < \infty$.

Let A be an Abelian finitely generated torsion-free subgroup of G, let H be the first non-Abelian upper central factor of a finitely generated non-Abelian subgroup that includes A, and let P denote the periodic part of the subgroup AH. If $P = 1$ then from the inequalities $r(AH) \le \eta\bigl(r_{ma}(H)\bigr)$ and $r_{ma}(H) \le r_{ma}(G)$ it follows that $r(AH) \le \eta\bigl(r_{ma}(G)\bigr)$. If P is a nontrivial subgroup then we consider the quotient group AH/P. If this quotient group is Abelian then, by Lemma 1, we have $r(AH/P) \le r_{ma}(G)$. Since $AP/P \simeq A$, we find that $r(A) \le r_{ma}(G)$. If the quotient group AH/P is not Abelian then from (6) and the relations $r_{ma}(AH/P) \le r_{ma}(G)$ and $AP/P \simeq A$ it follows that $r(A) \le r_{ma}(G)$.

Thus, the rank of a finitely generated Abelian torsion-free subgroup A is at most $r_{ma}(G)$. We conclude that, for every Abelian subgroup B of G, the inequality $r(B) \le r_{ma}(G) + r(T)$ is valid. According to [28], the rank of the group G is finite. □

Lemma 7. *Let S be a finitely generated group. Assume that there exists a normal locally finite subgroup T of S of finite rank such that the quotient group S/T is nilpotent-by-finite. Then the group S is nilpotent-by-finite too.*

Proof. By [2, Corollary 13], the subgroup T is almost locally solvable. By [5, Theorem 1], every periodic locally solvable group of finite rank admits an increasing characteristic series with finite factors. By [31, Theorem 3.18], the group S locally satisfies the maximum condition. We conclude that T is a finite subgroup. Since the quotient group S/T is nilpotent-by-finite, the group S is nilpotent-by-finite too. □

Theorem 3. *If G is a locally nilpotent-by-finite group of finite metabelian rank then the rank of G is finite too.*

Proof. Let G be a locally nilpotent-by-finite group of finite metabelian rank and let T denote the periodic radical of G. If T is a non-Abelian subgroup then, by Theorem 1, the rank of T is finite. If T is an Abelian subgroup then, by Lemma 4, we have $r(T) < \infty$.

We first prove the following assertion: The set of the 0-ranks of finitely generated subgroups of G is bounded.

Let K be a finitely generated subgroup of G. We begin with the case in which there exists a finitely generated subgroup H of G that is not Abelian-by-finite. Then the subgroup $\langle K, H\rangle = M$ is not Abelian-by-finite either. Since the subgroup M is nilpotent-by-finite and finitely generated, there exists a normal nilpotent torsion-free subgroup M_1 of M of finite index (in M). Since the subgroup M cannot be represented as the finite extension of an Abelian group and the index $|M : M_1|$ is finite, we conclude that the subgroup M_1 is not Abelian. By Theorem 2, there exists a number $\eta\bigl(r_{ma}(G)\bigr)$ such that $r(M_1) \le \eta\bigl(r_{ma}(G)\bigr)$. Hence, the 0-rank of the subgroup M satisfies the inequality $r_0(M) \le \eta\bigl(r_{ma}(G)\bigr)$. Thus, we have $r_0(K) \le \eta\bigl(r_{ma}(G)\bigr)$.

We turn to the case in which each finitely generated subgroup of G is Abelian-by-finite, i.e., admits an Abelian subgroup of finite index. Let H be a finitely generated non-Abelian subgroup of G satisfying the condition

$$r_0(H) > r_{ma}(G) + 3. \tag{7}$$

If such a subgroup does not exist then, for each finitely generated subgroup of G, the 0-rank is at most $r_{ma}(G) + 3$. Let A be an Abelian subgroup of H of finite index (in H). From (7) and Lemma 3 it follows that A is not central in H. Hence, there exists an element $h \in H$ such that the subgroup $A\langle h\rangle$ is not Abelian; moreover, we have $h^n \in A$ for some $n > 0$. The subgroup $\langle K, H\rangle = H_1$ is the finite extension of a suitable Abelian subgroup A_1. We show that the subgroup $A_1\langle h^n\rangle$ is Abelian. We assume the contrary, i.e., let $A_1\langle h^n\rangle$ be a non-Abelian subgroup. Since $h^n \in A$, we have $A \cap A_1 \le Z\bigl(A_1\langle h^n\rangle\bigr)$. By Lemma 3, we find that $r_0(A \cap A_1) \le r_{ma}(G) + 3$ because $A_1\langle h^n\rangle$ is not Abelian. Since the indices $|A : A \cap A_1|$ and $|H : A|$ are finite, we obtain the inequality $r_0(H) \le r_{ma}(G) + 3$, which contradicts (7). It is not difficult to see that the subgroup $A_1\langle h\rangle$ cannot be Abelian. Indeed, if it is Abelian then $A \cap A_1 \le Z\bigl(A\langle h\rangle\bigr)$, where the subgroup $A\langle h\rangle$ is not Abelian by construction. It remains to take into account Lemma 3 and conclude that $r_0(A \cap A_1) \le r_{ma}(G) + 3$, which (as is already mentioned) is impossible.

We apply Lemma 5 to the non-Abelian subgroup $A_1\langle h\rangle$ and take into account the fact that $A_1\langle h^n\rangle$ is an Abelian subgroup. We obtain the inequality

$$r(A_1) \le r_{ma}(G)n + 1,$$

which implies the following relations:

$$r_0(K) \le r_0(H_1) = r_0(A_1) \le r(A_1) \le r_{ma}(G)n + 1.$$

Thus, we have proven that the set of the 0-ranks of finitely generated subgroups of G is bounded by some natural number t.

We prove that the rank of the quotient group G/T is finite. We begin with the case in which the group G/T is Abelian. Notice that G/T is a torsion-free group. We consider a finitely generated subgroup B/T of the group G/T. We represent the subgroup B in the form $B = B_1T$, where B_1 is a finitely generated subgroup of G. Since B_1 is the finitely generated extension of the locally finite group $B_1 \cap T$ of finite rank by the Abelian group $B_1/B_1 \cap T$, from Lemma 7 it follows that B_1 is a nilpotent-by-finite subgroup. Hence, there exists a normal nilpotent torsion-free subgroup B_2 of B_1 of finite index (in B_1). Since the set of the 0-ranks of finitely generated subgroups of G is bounded, we find that $r_0(B_2) \le t$; hence, $r(B_2) \le t$. Since B_1T/T is an Abelian torsion-free group and the index $|B_1 : B_2|$ is finite, we obtain

$$r(B_1T/T) = r(B_2T/T).$$

Since $r(B_2) \le t$ and $B = B_1T$, we obtain the inequality $r(B/T) \le t$. Since the finitely generated subgroup B/T is arbitrary, the rank $r(G/T)$ is at most t.

We turn to the case in which the quotient group G/T is not Abelian and the set of the 0-ranks of finitely generated subgroups of G/T is bounded by t. By [38, Lemma 5], for every finitely generated subgroup S/T of G/T, there exists a periodic normal subgroup F/T of S/T such that $r(S/F)$ is bounded by some number $f(t)$ that depends on t only. Since the subgroup S/T is almost torsion free, the subgroup F/T is finite; moreover, the set of finite normal subgroups of S/T is finite. This allows us to apply the well-known method of projections (see, for example, [25, Sec. 55]) and find a periodic normal subgroup P/T of G/T with $r(G/P) \le f(t)$. Since the periodic radical of the group G/T is the trivial subgroup, we conclude that $r(G/T) \le f(t)$. □

Corollary 2. *If a group G is the extension of a locally finite group by a locally nilpotent-by-finite group and $r_{ma}(G) < \infty$ then the rank of G is finite.*

Proof. The rank of the periodic radical of the group G is finite. This fact is established by the arguments from the proof of Theorem 3. By Lemma 7, the group G is locally nilpotent-by-finite. Therefore, the fact that the rank $r(G)$ is finite is immediate from Theorem 3. □

Notice that, even for periodic nilpotent groups, there is no function describing the dependence of the rank of a group on the metabelian rank of that group. For example, consider the wreath product of two cyclic groups of prime order p. The metabelian rank of this group is 2 while the rank is equal to p.

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