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Dynamics of Evolutionary PDE Systems

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Abstract—The article is devoted to a method for constructing exact and approximate solutions of systems of partial differential evolution equations (PDE). The basis of this method is the concept of finite-dimensional dynamics, introduced for scalar equations by B. Kruglikov, O. Lychagina and V. Lychagin. The basic idea is that a system of evolution equations generates a flow in the space of solutions of some systems of ordinary differential equations. These ordinary differential equations have symmetries whose generating functions are generated by the right-hand sides of the evolutionary system. Dynamics and exact solutions for a system of evolution equations that describes processes of deep filtration of a suspension and the telegraph equation are constructed.

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1. INTRODUCTION

The theory of finite-dimensional dynamics is a natural development of the theory of dynamical systems. Dynamics makes it possible to find families of solutions depending on a finite number of parameters among all solutions of evolutionary differential equations. The main ideas and methods of this theory were formulated in [4, 8].

This article is devoted to the extension of the theory of finite-dimensional dynamics to systems of partial differential evolution equations.

We note that dynamics make it possible to construct exact solutions of systems of evolution equations, even if they do not have the necessary set of symmetries.

A method for constructing attractors for second-order evolutionary differential equations was proposed in [1]. Based on this method, an algorithm for the numerical solution of such equations was developed in [9].

As an example, we construct dynamics and exact solutions for a system of evolution equations that describes the deep filtration of a suspension in which small solid particles are suspended. This model takes into account the clogging of pores with such sediment. The exact solutions obtained have a clear physical interpretation.

Another example relates to linear second-order partial differential equations. It is shown how exact solutions can be constructed for some of them using dynamics. In particular, the dynamics and exact solutions are constructed for the classical telegraph equation.

When finding dynamics, we have to carry out calculations in jet spaces. This leads to cumbersome formulas. To facilitate calculations and avoid errors, we use the packages DifferentialGeometry and JetCalculus of the system of symbolic calculations Maple. A description of the basics of working with these packages can be found in [10].

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2. SYMMETRIES OF ODE SYSTEMS

Consider the following system of ordinary differential equations of order $k+1$:

$$\mathbf{y}^{(k+1)} = \mathbf{f}\left(x, \mathbf{y}, \ldots, \mathbf{y}^{(k)}\right). \tag{1}$$

Here $\mathbf{y} = \left(y^1, \ldots, y^n\right)^T$ is an unknown vector function of an independent variable x and the vector-valued function $\mathbf{f} = \left(f^1, \ldots, f^n\right)^T$ belong to the class $C^\infty(\mathbb{R}^{nk+n+1})$.

Let $J^k := J^k(1;n)$ be the k-jet space of functions from $\mathbb{R}$ to $\mathbb{R}^n$ with canonical coordinates x, y_i^j $(i = 0, \ldots, k; j = 1, \ldots, n)$. System (1) generates a distribution P of lines on the jet space J^k such that its integral curves are prolongations of the solutions graphs into the space J^k. The distribution P is generated by the vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + \sum_{\substack{i=0,\ldots,k-1\\ j=1,\ldots,n}} y_{i+1}^j \frac{\partial}{\partial y_i^j} + \sum_{j=1}^{n} f^j \frac{\partial}{\partial y_k^j} \tag{2}$$

or, equivalently, by the set of differential 1-forms

$$\omega_i^j = dy_i^j - y_{i+1}^j dx, \quad \omega_k^j = dy_k^j - f^j dx \quad (i = 0, \ldots, k-1; j = 1, \ldots, n). \tag{3}$$

This means that each of the differential 1-forms vanishes on the vector field $\mathcal{D}$: $\omega_i^j(\mathcal{D}) = 0$ $(i = 0, \ldots, k; j = 1, \ldots, n)$.

Remark 1. *In case when $k = 1$ the vector field $\mathcal{D}$ has the form $\mathcal{D} = \frac{\partial}{\partial x} + \sum_{j=1}^n f^j \frac{\partial}{\partial y_0^j}$ and the distribution P is generated by the following differential 1-forms: $\omega^j = dy_0^j - f^j dx \quad (j = 1, \ldots, n)$.*

A vector field X on J^k is called an *infinitesimal symmetry* of system (1) if translations along X preserve P. All infinitesimal symmetries form a Lie algebra with respect to the Lie bracket. We denote this algebra by $\mathsf{Symm}\,\mathsf{P}$. An infinitesimal symmetry is called *characteristic* if translations along it preserve each integral curve of the distribution P.

Characteristic symmetries form an ideal in $\mathsf{Symm}\,\mathsf{P}$ which we denote by $\mathsf{Char}\,\mathsf{P}$ (see [7]). That is $\mathsf{Char}\,\mathsf{P}$ is a subspace of $\mathsf{Symm}\,\mathsf{P}$ and the following properties hold:

- if $X \in \mathsf{Char}\,\mathsf{P}$ and $Y \in \mathsf{Symm}\,\mathsf{P}$ then $[X, Y] \in \mathsf{Char}\,\mathsf{P}$;
- if $X \in \mathsf{Char}\,\mathsf{P}$ and $f \in C^\infty(J^k)$ then $fX \in \mathsf{Char}\,\mathsf{P}$.

The quotient Lie algebra $\mathsf{Shuff}\,\mathsf{P} := \mathsf{Symm}\,\mathsf{P}/\mathsf{Char}\,\mathsf{P}$ is called the Lie algebra of *shuffling* symmetries of system (1). The vector field

$$S = \sum_{\substack{i=0,\ldots,k\\ j=1,\ldots,n}} a_i^j \frac{\partial}{\partial y_i^j}, \tag{4}$$

can be chosen as a representative of the equivalence class of shuffling symmetries. Here a_i^j $(i = 0, \ldots, k; j = 0, \ldots, n)$ are smooth functions on J^k. This vector field is a symmetry of the system if differential forms (3) and the form $\Phi_t^*(\omega_p^q)$ are linearly dependent for any $p = 0, \ldots, k; q = 1, \ldots, n$:

$$\Phi_t^*(\omega_p^q) = \sum_{\substack{i=0,\ldots,k\\ j=1,\ldots,n}} \alpha_j^i(t) \omega_i^j,$$

where $\alpha_j^i(t)$ are smooth functions on J^k depending on the parameter t. After differentiating this equality with respect to the parameter t at the point $t = 0$, we obtain

$$\left.\frac{d\Phi_t^*(\omega_p^q)}{dt}\right|_{t=0} = \sum_{\substack{i=0,\ldots,k\\ j=1,\ldots,n}} \left.\frac{d\alpha_j^i(t)}{dt}\right|_{t=0} \omega_i^j, \quad \text{or equivalently} \quad L_S(\omega_p^q) = \sum_{\substack{i=0,\ldots,k\\ j=1,\ldots,n}} \beta_j^i \omega_i^j.$$

The left side here is the Lie derivative of the form ω_p^q and $\beta_j^i = \left.\frac{d\alpha_j^i(t)}{dt}\right|_{t=0}$. This conditions is equivalent to the following equality:

$$L_S(\omega_p^q) \wedge \left(\bigwedge_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \omega_i^j \right) = 0 \tag{5}$$

for any $p = 0, \dots, k; q = 1, \dots, n$. Using the properties of the Lie derivative, we obtain

$$L_S(\omega_i^j) = L_S(dy_i^j) - L_S(y_{i+1}^j dx) = dS(y_i^j) - S(y_{i+1}^j)dx - y_{i+1}^j dS(x) = da_i^j - a_{i+1}^j dx. \tag{6}$$

Lemma 1. *The exterior differential of a function $a \in C^\infty(J^k)$ can be presented in the following form:*

$$da = \mathcal{D}(a)dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j}\omega_i^j. \tag{7}$$

Proof. Due to formulas (3) $dy_i^j = \omega_i^j + y_{i+1}^j dx$, $dy_k^j = \omega_k^j + f^j dx$, where $i = 0, \dots, k-1; j = 1, \dots, n$. Therefore

$$da = \frac{\partial a}{\partial x}dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j} dy_i^j = \left(\frac{\partial a}{\partial x} + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j} y_{i+1}^j \right) dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j}\omega_i^j$$

$$= \left(\frac{\partial a}{\partial x} + \sum_{\substack{i=0,\dots,k-1\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j} y_{i+1}^j + \sum_{j=1}^n f^j \frac{\partial a}{\partial y_k^j} \right) dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j}\omega_i^j = \mathcal{D}(a)dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a}{\partial y_i^j}\omega_i^j.$$

□

Due to formula (6) we get

$$L_S(\omega_i^j) = \mathcal{D}(a_i^j)dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \left(\frac{\partial a_i^j}{\partial y_i^j}\omega_i^j - a_{i+1}^j dx \right) = \left(\mathcal{D}(a_i^j) - a_{i+1}^j \right) dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a_i^j}{\partial y_i^j}\omega_i^j.$$

Then formula (5) takes the form $\left(\mathcal{D}(a_i^j) - a_{i+1}^j \right) \mu = 0$, where $\mu = dx \wedge dy_0^1 \wedge \dots \wedge dy_k^n$ is a volume form. Therefore $a_{i+1}^j = \mathcal{D}(a_i^j)$ for $i = 0, \dots, k-1$; $j = 1, \dots, n$. If we denote a_0^j by φ^j we get vector field (4) in the following form:

$$S = \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \mathcal{D}^i(\varphi^j) \frac{\partial}{\partial y_i^j}.$$

Here $\mathcal{D}^0 = \mathrm{id}$ and $\mathcal{D}^i = \underbrace{\mathcal{D} \circ \dots \circ \mathcal{D}}_{i \text{ times}}$. The vector-valued function $\boldsymbol{\varphi} = \left(\varphi^1, \dots, \varphi^n \right)^T$ is called the *generating function* of the vector field S.

Theorem 2. *A generating function of a symmetry of system (1) satisfies to the following differential equation:*

$$\mathcal{D}^{k+1}(\varphi^j) - \sum_{i=0,\dots,k} \mathcal{D}^i(\varphi^j) \frac{\partial f^j}{\partial y_i^j} = 0. \tag{8}$$

Proof. For any differential 1-form ω_k^q $(q = 1, \dots, n)$ we have

$$L_S\left(\omega_k^q\right) = dS(y_k^q) - S(f^q)dx = da_k^q - S(f^q)dx = \left(\mathcal{D}^k(a_k^q) - S(f^q)\right)dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a_k^q}{\partial y_i^j}\omega_i^j$$
$$= \left(\mathcal{D}^{k+1}(\varphi^q) - \sum_{i=0,\dots,k} \mathcal{D}^i(\varphi^q)\frac{\partial f^q}{\partial y_i^j}\right)dx + \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \frac{\partial a_k^q}{\partial y_i^j}\omega_i^j.$$

After applying the exterior multiplication of both parts by the differential form $\bigwedge\limits_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \omega_i^j$ due to (5) we get formula (8). □

3. DYNAMICS ON SOLUTIONS OF ODE SYSTEMS AND SOLUTIONS OF EVOLUTIONARY PDE SYSTEMS

Consider the following system of evolutionary partial differential equation with two independent variables t, x:

$$\frac{\partial \mathbf{u}}{\partial t} = \boldsymbol{\varphi}\left(x, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \dots, \frac{\partial^k \mathbf{u}}{\partial x^k}\right), \tag{9}$$

where $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)^T$ are vector-valued function of the class C^∞, $\mathbf{u} = (u_1, \dots, u_n)^T$.

Let $\boldsymbol{\varphi}(x, \mathbf{y}_0, \dots, \mathbf{y}_k)$ be a generating vector-valued function of the shuffling symmetry S of (1) and let Φ_t be the translation along the vector field S from $t = 0$ to t.

If $\boldsymbol{\varphi}$ is a generating vector-valued function of a shuffling symmetry of system (1), then system (1) is called a (finite-dimensional) *dynamics* of PDE system (9). The number $k+1$ is called an *order* of the dynamics.

We describe two approaches to construct solutions of system (9) using dynamics.

1. Let $L_{\mathbf{y}(x)} = \{\mathbf{y}_0 = \mathbf{y}(x)\}$ be the graph of a solution $\mathbf{y} = \mathbf{y}(x)$ of system (1) and let

$$L_{\mathbf{y}(x)}^{(k)} = \left\{\mathbf{y}_0 = \mathbf{y}(x), \mathbf{y}_1 = \mathbf{y}'(x), \dots, \mathbf{y}_k = \mathbf{y}^{(k)}(x)\right\} \tag{10}$$

be its prolongation into the space $J^k(1;n)$.

Shifting the curve $L_{\mathbf{y}(x)}^{(k)}$ along the trajectories of the vector field S, we get the surface $\Phi_t\left(L_{\mathbf{y}(x)}^{(k)}\right) \subset J^k(2;n)$ that is a prolongation of the graph of a solution $\mathbf{u}(t,x)$ of evolutionary equation (9). Here $J^k(2;n)$ is the k-jets space of functions with two independent variables t, x. Since system (1) is solvable with respect to higher derivatives, then its solution space could be identified with the space $\mathbb{R}^{n(k+1)}$ by taking the initial data at a point x_0. Then, instead of the vector field S, we can use the vector field

$$E = \sum_{\substack{i=0,\dots,k\\ j=1,\dots,n}} \mathcal{D}^i(\overline{\varphi^j})\frac{\partial}{\partial y_i^j} \tag{11}$$

on the space of initial data. Here $\overline{\varphi^j}$ is a restriction of the function φ^j to system (1). Therefore, we can use transformations of the space of initial data $\mathbb{R}^{n(k+1)}$ instead of transforming curves. Such transformations are given by shifts $\overline{\Phi}_t$ along the vector field E.

Let $\mathbf{y} = \mathbf{y}(x;\mathbf{a})$ be the solution of equation (1) with initial data $\mathbf{y}(x_0) = \mathbf{a}_0$, $\mathbf{y}'(x_0) = \mathbf{a}_1$, ... $\mathbf{y}^{(k)}(x_0) = \mathbf{a}_k$. Applying the transformation $\overline{\Phi}_t$ to the point $\mathbf{a} = (\mathbf{a}_0, \dots, \mathbf{a}_k)$ (where $\mathbf{a}_i = (a_i^1, \dots, a_i^n)$) we obtain a one-parameter family $\mathbf{y}(x; \overline{\Phi}_t(\mathbf{a}))$ of solutions of equation (1). Then the function $\mathbf{u}(t,x) = \mathbf{y}(x, \overline{\Phi}_t(\mathbf{a}))$ is a solution of the evolutionary equation (9) with the initial data $\mathbf{u}(0,x) = \mathbf{y}(x;\mathbf{a})$.

2. A different approach can be used to construct solutions of evolutionary PDE. This approach is more preferable when the solution of the system of ODE and the shift transformation Φ_t are known explicitly. We describe this approach.

The transformation Φ_t acting on the jet space J^k generates the transformation Φ_t^* acting on functions: $\Phi_t^*(f) := f \circ \Phi_t$. Let Φ_t^{-1} be the inverse transformation for Φ_t. Curve (10) is generating by the system of equalities

$$\mathbf{y}_0 - \mathbf{y}(x) = 0, \quad \mathbf{y}_1 - \mathbf{y}'(x) = 0, \dots, \mathbf{y}_k - \mathbf{y}^{(k)}(x) = 0. \tag{12}$$

Applying the transformation $\left(\Phi_t^{-1}\right)^*$ to (12) we obtain the following systems:

$$\mathbf{F}^0(t, x, \mathbf{y}_0, \dots, \mathbf{y}_k) = 0, \quad \mathbf{F}^1(t, x, \mathbf{y}_0, \dots, \mathbf{y}_k) = 0, \dots, \mathbf{F}^k(t, x, \mathbf{y}_0, \dots, \mathbf{y}_k) = 0, \tag{13}$$

where $\mathbf{F}^i$ are some vector-valued functions. Solving equations (13) with respect to $\mathbf{y}_0, \dots, \mathbf{y}_k$ we find a coordinate representation of the curve $\Phi_t\left(L^{(k)}_{\mathbf{y}(x)}\right)$:

$$\mathbf{y}_0 = \mathbf{Y}_0(t, x), \quad \mathbf{y}_1 = \mathbf{Y}_1(t, x), \dots, \mathbf{y}_k = \mathbf{Y}_k(t, x). \tag{14}$$

The vector-valued function $\mathbf{u}(t, x) = \mathbf{Y}_0(t, x)$ is a solution of equation (9) (see [7]).

For example, for first order ODE system (1), i.e. when $k = 0$, the vector field S has the form

$$S = \varphi^1 \frac{\partial}{\partial y_0^1} + \dots + \varphi^j \frac{\partial}{\partial y_0^j}.$$

4. FLOW OF SUSPENSIONS THROUGH POROUS MEDIA

Deep filtration of a single-component suspension of particles in a porous medium is described by the following first order evolutionary PDE system [3]:

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - h(v)u, \\ \frac{\partial v}{\partial t} = h(v)u. \end{cases} \tag{15}$$

This model takes into account the change in the permeability of the medium due to the capture of particles by the porous medium. Here u, v are a suspension concentration and a sediment concentration, $h(v)$ is a filtration coefficient.

We use the following notation: $u^1 = u, u^2 = v$. The corresponding generating vector valued function for dynamics has the form

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} -y_1^1 - h(y_0^2)y_0^1 \\ h(y_0^2)y_0^1 \end{pmatrix}.$$

First-order dynamics will be sought in the form

$$\begin{cases} y_1^1 = f^1(y_0^1, y_0^2), \\ y_1^2 = f^2(y_0^1, y_0^2). \end{cases} \tag{16}$$

The vector field (2) has the form

$$\mathcal{D} = \frac{\partial}{\partial x} + f^1(y_0^1, y_0^2)\frac{\partial}{\partial y_0^1} + f^2(y_0^1, y_0^2)\frac{\partial}{\partial y_0^2}.$$

Then vector field (4) is

$$\begin{aligned} S = \varphi^1 \frac{\partial}{\partial y_0^1} + \varphi^2 \frac{\partial}{\partial y_0^2} &= \left(-y_1^1 - h(y_0^2)y_0^1\right) \frac{\partial}{\partial y_0^1} + f(y_0^2)y_0^1 \frac{\partial}{\partial y_0^2} \\ &= \left(-f^1(y_0^1, y_0^2) - h(y_0^2)y_0^1\right) \frac{\partial}{\partial y_0^1} + h(y_0^2)y_0^1 \frac{\partial}{\partial y_0^2}. \end{aligned}$$

Using Theorem 2, we obtain a system of differential equations for finding functions f^1, f^2 (see formula (8)):

$$\begin{cases} y_0^1 h(y_0^2)\frac{\partial f^1}{\partial y_0^1} - \left(f^2 + y_0^1 h(y_0^2)\right)\frac{\partial f^1}{\partial y_0^2} - h(y_0^2) f^1 - y_0^1 h'(y_0^2) f^2 = 0, \\ \left(f^1 + y_0^1 h(y_0^2)\right)\frac{\partial f^2}{\partial y_0^1} - y_0^1 h(y_0^2)\frac{\partial f^2}{\partial y_0^2} + h(y_0^2) f^1 + y_0^1 h'(y_0^2) f^2 = 0. \end{cases} \tag{17}$$

Consider case when the function h is linear i.e. $h(y_0^2) = \alpha y_0^2 + \beta$, where α, β are constant. Then functions

$$\begin{cases} f^1 = (\delta - \alpha(y_0^1 + y_0^2))y_0^1, \\ f^2 = \dfrac{1}{\beta + \xi - \alpha y_0^1}\Big((\beta + \xi - \alpha y_0^1)^2 H\left(\frac{\alpha y_0^2 + \beta}{\alpha(\beta + \xi - \alpha y_0^1)}\right) \\ -y_0^1(\alpha y_0^2 + \beta)(\xi - \alpha(y_0^1 + y_0^2))\Big) \end{cases} \tag{18}$$

are solutions of system (17). Here H is arbitrary function and δ, ξ are arbitrary constant.

If we choose $H(z) = z$ then functions (18) take forms

$$\begin{cases} f^1 = (\delta - \alpha(y_0^1 + y_0^2))y_0^1, \\ f^2 = \dfrac{(\alpha y_0^2 + \beta)(-\alpha^2 y_0^1(y_0^1 + y_0^2) + \alpha(1 + \xi)y_0^1 - \beta - \xi)}{\alpha(\beta + \xi - \alpha y_0^1)}. \end{cases} \tag{19}$$

Suppose that $\alpha = -1$, $\beta = 1$, $\delta = \xi = 0$. In this case system (15) takes the form

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - (1 - v)u, \\ \frac{\partial v}{\partial t} = (1 - v)u. \end{cases} \tag{20}$$

Then the general solution of system (16) is

$$\begin{cases} y^1(x) = \dfrac{C_2}{\exp(C_1 \exp x - x - 1) - C_2}, \\ y^2(x) = \dfrac{(C_1 \exp x - 1)\exp(C_1 \exp x - 1) + C_2 \exp x}{C_2 \exp x - \exp(C_1 \exp x - 1)}, \end{cases} \tag{21}$$

where C_1, C_2 are arbitrary constants. The vector field S has the form

$$S = -y_0^1(1 + y_0^1)\frac{\partial}{\partial y_0^1} + y_0^1(1 - y_0^2)\frac{\partial}{\partial y_0^2}$$

and then the transformation is

$$\Phi_t : (x, y_0^1, y_0^2) \longmapsto \left(x, \frac{y_0^1}{e^t(1 + y_0^1) - y_0^1}, -\frac{e^t((e^{-t} - 1)y_0^1 - y_0^2)}{e^t(1 + y_0^1) - y_0^1}\right).$$

Applying the inverse transformation Φ_t^{-1} to obtained general solution (21), we get the following solution of system (20):

$$\begin{cases} u(t, x) = \dfrac{C_2 e^{x+1}}{e^{C_1 x + t} - C_2 e^{x+1}}, \\ \\ v(t, x) = -\dfrac{C_2 e^{x+1} + (C_1 e^x - 1)e^{C_1 e^x + t}}{e^{C_1 x + t} - C_2 e^{x+1}}. \end{cases} \tag{22}$$

Graphs of functions (22) are shown in Figs. 1, 2 for constant values $C_1 = C_2 = 0.1$. Analysis of the graphs allows us to conclude that the filtration coefficient and suspension concentration decrease over time (see Fig. 3). This is due to the fact that over time, the pores are filled with sediment and their permeability decreases. On the other hand, the concentration of sediment increases with time. An evolution of the filtration coefficient is shown in the Fig. 4. We see that the filtration coefficient decreases with time. These observations are in good agreement with the physics of the process.

Remark 3. *System (16) has one more irregular solution $y_0^1 = 0$, $y_0^2 = 1 + C_1 e^x$ which we do not consider because it is not physical.*

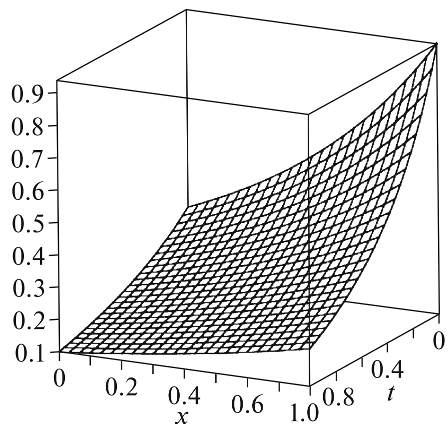

Fig. 1. Graph of the function $u = \dfrac{0.1e^{x+1}}{e^{0.1x+t} - 0.1e^{x+1}}$.

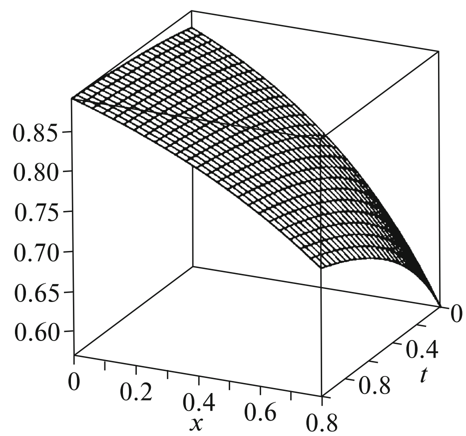

Fig. 2. Graph of the function $v = -\dfrac{0.1e^{x+1} + (0.1e^{x} - 1)e^{0.1e^{x}+t}}{e^{0.1x+t} - 0.1e^{x+1}}$.

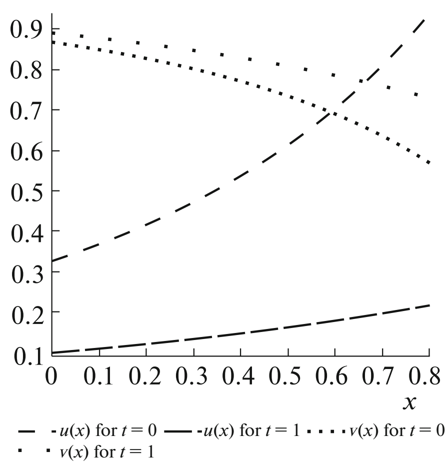

Fig. 3. Evolution of the concentrations of suspension $u(x)$ and sediment $v(x)$ from $t = 0$ to $t = 1$.

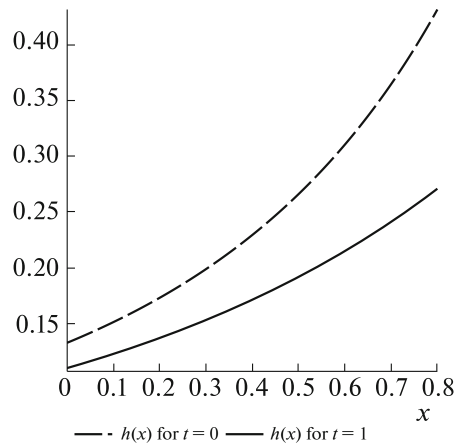

Fig. 4. Evolution of the filtration coefficient h from $t = 0$ to $t = 1$.

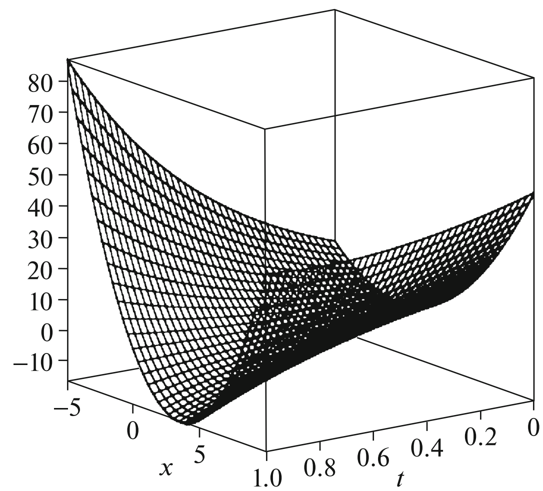

Fig. 5. Graph of solution (30).

5. SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Some scalar equations that are not formally evolutionary can be reduced to systems of evolutionary equations. Consider the following class of second order linear partial differential equations

$$u_{tt} + 2b(x)u_{tx} + c(x)u_{xx} + h(x)u_t + g(x)u_x + f(x) = 0, \tag{23}$$

where b, c, h, g, f are functions of the class C^∞. Such equations describe, for example, processes in media whose characteristics do not change with time.

Equations (23) are equivalent to the following evolutionary systems:

$$\begin{cases} u_t = v, \\ v_t = -2b(x)v_x - c(x)u_{xx} - h(x)v - g(x)u_x - f(x). \end{cases} \tag{24}$$

As above, we use the following notation: $u^1 = u, u^2 = v$. The corresponding generating vector valued function for dynamics has the form

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} y_0^2 \\ -2b(x)y_1^2 - c(x)y_2^1 - h(x)y_0^2 - g(x)y_1^1 - f(x) \end{pmatrix}.$$

Consider, for example, the telegraph equation

$$u_{tt} - u_{xx} = au + bu_t + c, \tag{25}$$

where a, b, c are constants. This equation admits two types of linear second order dynamics:

$$\begin{cases} y_2^1 = \dfrac{y_1^1}{x+\alpha}, \\ y_2^2 = \dfrac{y_1^2}{x+\alpha} \end{cases} \tag{26}$$

and

$$\begin{cases} y_2^1 = \dfrac{2b\alpha-(x+\beta)\alpha^2}{4b^2+16a-\alpha^2(x+\beta)^2} \times y_1^1 - \dfrac{4\alpha}{4b^2+16a-\alpha^2(x+\beta)^2} \times y_1^2, \\ y_2^2 = -\dfrac{4a\alpha}{4b^2+16a-\alpha^2(x+\beta)^2} \times y_1^1 - \dfrac{2b\alpha+\alpha^2(x+\beta)}{4b^2+16a-\alpha^2(x+\beta)^2} \times y_1^2. \end{cases} \tag{27}$$

Here α, β are arbitrary constants. The general solution of system (26) is

$$\begin{cases} y^1(x) = C_3 + C_4(x+\alpha)^2, \\ y^2(x) = C_1 + C_2(x+\alpha)^2, \end{cases} \tag{28}$$

and the general solution of system (27) is

$$\begin{cases} y^1(x) = \dfrac{1}{2}C_2x^2 + C_3x + C_4, \\ y^2(x) = \dfrac{1}{8\alpha}\left(x(C_2\beta - C_3)(2\beta+x)\alpha^2 + (8C_1 + 2bx^2C_2 + 4bC_3x)\alpha - 32\left(a+\frac{b^2}{4}\right)C_2x\right). \end{cases} \tag{29}$$

Here $C_1, \ldots, C_4$ are arbitrary constants. Applying the shift transformations Φ_t to the obtained general solutions, we obtain particular solutions of equation (25). For example, the function

$$u(t,x) = -1 + \frac{1}{10}\left(\frac{5}{2}x^2 + 5 + (10x+1-t)\sqrt{5}\right)e^{-\frac{t}{2}(\sqrt{5}-1)} + \frac{1}{10}\left(\frac{5}{2}x^2 + 5 + (-10x-1+t)\sqrt{5}\right)e^{\frac{t}{2}(\sqrt{5}+1)} \tag{30}$$

is a solution of equation (25). It corresponds to solution (29) with $a=b=c=1$, $\alpha=1, \beta=0$ and $C_1=0$, $C_2=1$, $C_3=0$, $C_4=0$, $C_5=0$. In this case

$$S = y_0^2\frac{\partial}{\partial y_0^1} + \frac{(x^2-20)(y_0^1+y_0^2)+(x+2)y_1^1+x^2-20+4y_1^2}{x^2-20}\frac{\partial}{\partial y_0^2} + y_1^2\frac{\partial}{\partial y_1^1} + (y_1^1+y_1^2)\frac{\partial}{\partial y_1^2}.$$

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