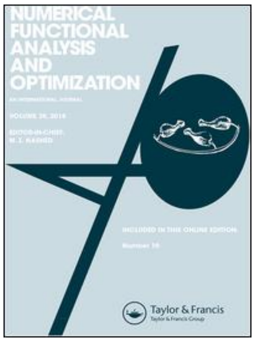

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Well-Posedness and Approximation for Nonhomogeneous Fractional Differential Equations

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ABSTRACT

In this paper, we consider the well-posedness and approximation for nonhomogeneous fractional differential equations in Banach spaces E. Firstly, we get the necessary and sufficient condition for the well-posedness of nonhomogeneous fractional Cauchy problems in the spaces $C_0^{\beta}([0,T];E)$. Secondly, by using implicit difference scheme and explicit difference scheme, we deal with the full discretization of the solutions of nonhomogeneous fractional differential equations in time variables, get the stability of the schemes and the order of convergence.

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1. Introduction

A lot of works were devoted to the approximations of C_0-semigroups, see [1–5] and the references therein. While, other mathematicians considered the discrete approximation of integrated semigroups in their papers [6–8]. We all know that Podlubny introduced fractional derivatives, fractional differential equations, some methods of their solutions and some of their applications in his book [9]. Ashyralyev and Cakir considered the numerical solutions of fractional parabolic partial differential equations [10–15]. In papers [16–19], we dealt with the discrete approximation of the homogeneous fractional differential equations and semilinear fractional differential equations in Banach spaces. Especially in [18,19], we get the stability and the order of convergence of implicit difference scheme and explicit difference scheme for homogeneous fractional differential equations. In this paper, we will consider the fulldiscrete approximation of the nonhomogeneous fractional differential equation in the space $C([0,T];E)$, which will be presented in section 3.

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Let $0<\alpha<1$, we consider the well-posed nonhomogeneous Cauchy problem:

$$(\mathbf{D}_t^\alpha u)(t) = Au(t) + f(t), \quad t \in (0,T]; \quad u(0) = x, \tag{1.1}$$

where $\mathbf{D}_t^\alpha$ is the Caputo-Dzhrbashyan derivative.

In [20], Ashyralyev and Sobolevskii indicated that in the Hölder space $C_0^\beta([0,T];E)$, the analyticity of a C_0-semigroup is equivalent to the coercive well-posedness of nonhomogeneous problem. Ashyralyev studied the well-posedness of fractional parabolic partial differential equations [14, 21], and used modified Gauss elimination method to consider their numerical solutions [14]. In [22], the authors got the maximal regularity as well as approximation for fractional Cauchy equation in space $C_0^\beta([0,T];E)$. Here $C_0^\beta([0,T];E)$ is the Banach space [20] obtained by completion of the set of E-valued smooth functions $u(\cdot)$ on [0,T] in the norm

$$||u(\cdot)||_{C_0^\beta([0,T];E)} = ||u(\cdot)||_{C([0,T];E)} + \sup_{0\le t<t+\tau\le T} \frac{t^\beta||u(t+\tau)-u(t)||_E}{\tau^\beta}.$$

So, in the second section, we concentrate on the well-posedness of (1.1) in the Hölder space $C_0^\beta([0,T];E)$ and prove that the analyticity of α-times resolvent family is the necessary and sufficient condition for the well-posedness of (1.1).

Remark 1.1. The followings are the main differences between this paper and papers [14, 22].

- In [14], the initial value of the problem is zero and the corresponding operator is positive. We do not need such assumptions in the present paper.
- The authors in [22] got the maximal regularity for fractional Cauchy equation on space $C_0^\beta([0,T];E)$ when $\beta \le \alpha$. There is no such restriction on β in this paper.
- They used the modified Gauss elimination method to study approximations [14, 22]. While, we consider the L1 difference scheme here.

It was proved in [23] that the homogeneous Cauchy problem

$$(\mathbf{D}_t^\alpha u)(t) = Au(t), \quad t>0; \quad u(0) = x, \tag{1.2}$$

is well-posed iff A generates an α-times resolvent family $S_\alpha(\cdot,A)$. We assume from the beginning that resolvent family $S_\alpha(\cdot,A)$ satisfies $||S_\alpha(t,A)|| \le Me^{\omega t}, \quad t \ge 0,$ for some $M \ge 1, \omega \ge 0$. In such case, for $\{\lambda^\alpha : Re\lambda>\omega\} \subset \rho(A)$, one has

$$\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t,A)xdt = (S_\alpha\widehat{(t,A)}x)(\lambda), \quad Re\lambda>\omega, x\in E,$$

where $\widehat{q}(\cdot)$ is denoted the Laplace transform of $q(\cdot)$. In the paper [16], we have proved that if the operator A generates an α-times resolvent family $S_\alpha(\cdot, A)$ which is satisfying $||S_\alpha(t, A)|| \le Me^{\omega t}, \quad t \ge 0,$ then the operator A is closed and densely defined.

Definition 1.1. [23] A family $\{S_\alpha(t, A)\}_{t\ge 0} \subset B(E)$ is called an α-times resolvent family generated by A if the following conditions are satisfied:

(a) $S_\alpha(t, A)$ is strongly continuous for $t \ge 0$ and $S_\alpha(0, A) = I$;

(b) $S_\alpha(t, A)D(A) \subseteq D(A)$ and $AS_\alpha(t, A)x = S_\alpha(t, A)Ax$ for all $x \in D(A), \; t \ge 0$;

(c) for $x \in D(A), S_\alpha(t, A)x$ satisfies the resolvent equation

$$S_\alpha(t, A)x = x + \int_0^t g_\alpha(t-s)S_\alpha(s, A)Axds, \quad t \ge 0.$$

Definition 1.2. An α-times resolvent family $S_\alpha(\cdot, A)$ is called analytic if $S_\alpha(\cdot, A)$ admits an analytic extension to a sector $\Sigma_{\theta_0}\backslash\{0\}$ for some $\theta_0 \in (0, \pi/2]$, where $\Sigma_{\theta_0} := \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta_0\}$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$, there is $M = M(\theta, \omega)$ such that $||S_\alpha(z, A)|| \le Me^{\omega Re z}, z \in \Sigma_\theta$.

Note that $S_\alpha(t, A)$ for bounded operator A is given by Mittag-Leffler function $E_\alpha(t^\alpha A)$, i.e. $S_\alpha(t, A) = E_\alpha(t^\alpha A) = \sum_{j=0}^{\infty} \frac{(t^\alpha A)^j}{\Gamma(\alpha j+1)}$.

Definition 1.3. [24] A family $\{P_\alpha(t, A)\}_{t\ge 0}$ of strongly continuous function $P_\alpha(\cdot, A) : (0, \infty) \to B(E)$ is called an (α, α)-times resolvent family generated by A if there exists $\omega \ge 0$, such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t}P_\alpha(t, A)xdt, Re\lambda > \omega, x \in E.$$

Remark 1.2. [23–25] If A generates an α-times resolvent family $S_\alpha(t, A)$ for the case $1 < \alpha < 2$, then A is also the generator of (α, α)-times resolvent family $P_\alpha(t, A)$ and

$$P_\alpha(t, A) = (g_{\alpha-1} * S_\alpha)(t).$$

While, when $0 < \alpha < 1$, if A generates an analytic α-times resolvent family $S_\alpha(t, A)$, then it is also the generator of analytic (α, α)-times resolvent family

$$P_\alpha(t, A) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha; A)d\lambda,$$

and

$$(g_{1-\alpha} * P_\alpha)(t) = S_\alpha(t, A). \tag{1.3}$$

For $P_\alpha(\cdot, A)$, we have the following properties [24, 25]:

$$P_\alpha(t,A)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)P_\alpha(s,A)xds, \quad t>0, \text{ for any } x \in E,$$
$$AP_\alpha(t,A)x = P_\alpha(t,A)Ax, \text{ for any } x \in D(A).$$

When $0<\alpha<1$, the following lemma holds:

Lemma 1.1. [25] *Let A be the generator of analytic resolvent family $S_\alpha(t, A)$. We have*

(1) $P_\alpha(t,A) \in B(E)$ *and* $||P_\alpha(t,A)|| \leq Me^{\omega t}(1+t^{\alpha-1})$ for any $t > 0$;
(2) For every $x \in E, P_\alpha(t,A)x \in D(A)$ *and* $||AP_\alpha(t,A)|| \leq Me^{\omega t}(1+t^{-1})$, *for any $t > 0$;*
(3) $S'_\alpha(t,A) = AP_\alpha(t,A)$ *for any $t > 0$, $R(P_\alpha^{(l)}(t,A)) \subseteq D(A)$ for any integer $l \geq 0$ and $||A^k P_\alpha^{(l)}(t,A)|| \leq Me^{\omega t}(1+t^{-l-1-\alpha(k-1)})$ for any $t > 0$, where $k = 0,\ 1$.*

Remark 1.3. If $S_\alpha(\cdot, A)$ is bounded, i.e. $||S_\alpha(t,A)|| \leq M_1, \quad t \in [0, T]$, then all the items $Me^{\omega t}$ in Lemma 1.1 can be changed by M_1.

Definition 1.4. A function $u(\cdot) \in C([0,T];E)$ is called a solution to (1.1) in $C([0,T];E)$, if $u(\cdot) \in C([0,T];D(A)), g_{1-\alpha} * (u(t)-x) \in C^1([0,T];E)$ *and $u(\cdot)$ satisfies* (1.1).

Definition 1.5. A function $u(\cdot) \in C_0^\beta([0,T];E)$ is called a solution to (1) in $C_0^\beta([0,T];E)$ if it is a solution to this problem in $C([0,T];E)$, functions $(\mathbf{D}_t^\alpha u)(\cdot)$ and $Au(\cdot)$ are belonging to $C_0^\beta([0,T];E)$.

Obviously, if $u(\cdot)$ is a solution to (1.1) in $C_0^\beta([0,T];E)$, then $x \in D(A)$ and $f(\cdot) \in C_0^\beta([0,T];E)$. Then we can define the well-posedness of the problem (1.1) in $C_0^\beta([0,T];E)$ as follows.

Definition 1.6. The problem (1.1) is well-posed in $C_0^\beta([0,T];E)$, if:

1) For any $f(\cdot) \in C_0^\beta([0,T];E)$ and $x \in D(A)$, there exists a unique solution $u(t) = u(t;f(\cdot),x)$ to (1.1) in $C_0^\beta([0,T];E)$;
2) The operator $u(t;f(\cdot),x)$ is continuous as an operator from the space $C_0^\beta([0,T];E) \times D(A)$ to the space $C_0^\beta([0,T];E)$.

Here $C_0^\beta([0,T];E) \times D(A)$ is equipped with the norm

$$||(f(\cdot),x)||_{C_0^\beta([0,T];E)\times D(A)} = ||f(\cdot)||_{C_0^\beta([0,T];E)} + ||x||_E.$$

The semidiscrete approximation on the general discretization scheme of the problem (1.1) are the Cauchy problems in Banach spaces E_n:

$$(\mathbf{D}_t^\alpha u_n)(t) = A_n u_n(t) + f_n(t), \quad t \in (0,T]; \quad u_n(0) = x_n. \tag{1.4}$$

The general approximation scheme, due to [26], can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators: $p_n : E \to E_n, p_n \in B(E, E_n), n \in \mathbb{N},$ with the property $||p_n x||_{E_n} \to ||x||_E$ as $n \to \infty$ for any $x \in E$.

Definition 1.7. The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N},$ is said to be P-convergent to $x \in E$ iff $||x_n - p_n x||_{E_n} \to 0$ as $n \to \infty$. We write this as $x_n \xrightarrow{\mathcal{P}} x$.

Definition 1.8. The sequence of bounded linear operators $B_n \in B(E_n),\ n \in \mathbb{N},$ is said to be PP-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N},$ such that $x_n \xrightarrow{\mathcal{P}} x,$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write this as $B_n \xrightarrow{\mathcal{PP}} B$.

The problem of convergence of solutions of semidiscrete approximation

$$(\mathbf{D}_t^\alpha u_n)(t) = A_n u_n(t), \quad t \in (0,T]; \quad u_n(0) = x_n,$$

to solution of problem (1.2) is completely solved by analogy of Theorem ABC from Appendix [22, 23, 24, 25]. The problem of convergence of solutions of (1.4) in $C([0,T];E_n)$ to the solution of (1.1) in $C([0,T];E)$ can also be described by ABC Theorem's terminology using the conditions (A) and (B). We will address this issue in section 3.

2. Necessary and sufficient condition for the well-posedness in $C_0^\beta([0,T];E)$

Obviously, the well-posedness of (1.2) in $C_0^\beta([0,T];E)$ imply the well-posedness of it in $C([0,T];E)$. Then A is the generator of an α-times resolvent family $S_\alpha(t,A)$, and the solution to (1.2) is $S_\alpha(t,A)x$. Furthermore, it follows from the well-posedness of (1.2) in $C_0^\beta([0,T];E)$ that

$$||S_\alpha(\cdot,A)x||_{C_0^\beta([0,T];E)} \le \bar{M}||x||_E. \tag{2.1}$$

Lemma 2.1. *Let A be a generator of analytic α-times resolvent family. For any $0<t<t+\tau \le T$ and $0 \le \beta \le 1$, one has the following inequalities:*

$$||S_\alpha(t+\tau,A)-S_\alpha(t,A)||_{E\to E}\le M_2\frac{\tau^\beta}{t^\beta};\tag{2.2}$$

$$||AP_\alpha(t+\tau,A)-AP_\alpha(t,A)||_{E\to E}\le M_2\frac{\tau^\beta}{t^{1+\beta}};\tag{2.3}$$

$$||g_\alpha*S_\alpha(t+\tau,A)-g_\alpha*S_\alpha(t,A)||_{E\to E}\le M_2\frac{\tau^\beta}{t^\beta};\tag{2.4}$$

$$||P_\alpha(t+\tau,A)-P_\alpha(t,A)||_{E\to E}\le M_2\frac{\tau^\beta}{t^{1+\beta}}.\tag{2.5}$$

Proof. We know that $S_\alpha(t,A)=I+g_\alpha*AS_\alpha(t,A)$. Then it follows from (1.3) that

$$\begin{aligned}S_\alpha(t,A)&=I+g_\alpha*Ag_{1-\alpha}*P_\alpha(t,A)=I+g_1*AP_\alpha(t,A)\\&=I+\int_0^t AP_\alpha(s,A)ds.\end{aligned}\tag{2.6}$$

And, $S_\alpha(t+\tau,A)-S_\alpha(t,A)=\int_t^{t+\tau}AP_\alpha(s,A)ds$. From Lemma 1.1, we know that $||AP_\alpha(t,A)||\le M_1(1+t^{-1})$ for any $t>0$, then

$$||S_\alpha(t+\tau,A)-S_\alpha(t,A)||_{E\to E}\le M_1\int_t^{t+\tau}\frac{ds}{s}\le\frac{M_1}{t}\int_t^{t+\tau}ds=M_1\frac{\tau}{t}\le M_2\frac{\tau}{t}.\tag{2.7}$$

We also have

$$||S_\alpha(t+\tau,A)-S_\alpha(t,A)||_{E\to E}\le 2M_1\le M_2.\tag{2.8}$$

Interpolating (2.7) and (2.8), we obtain (2.2). It follows from $||AP'_\alpha(s,A)||\le M_1(1+s^{-2})$ for any $s>0$, that

$$\begin{aligned}||AP_\alpha(t+\tau,A)-AP_\alpha(t,A)||_{E\to E}&=||\textstyle\int_t^{t+\tau}AP'_\alpha(s,A)ds||_{E\to E}\le M_1\int_t^{t+\tau}\frac{ds}{s^2}\\&\le\frac{M_1}{t^2}\int_t^{t+\tau}ds=M_1\frac{\tau}{t^2}\le M_2\frac{\tau}{t^2}.\end{aligned}\tag{2.9}$$

From Lemma 1.1 we have

$$||AP_\alpha(t+\tau,A)-AP_\alpha(t,A)||_{E\to E}\le\frac{M_1}{t+\tau}+\frac{M_1}{t}\le\frac{2M_1}{t}\le\frac{M_2}{t}.\tag{2.10}$$

Interpolating (2.9) and (2.10), we obtain (2.3). It follows from $||P_\alpha(t,A)||\le M_1(1+t^{\alpha-1})$ for any $t>0$ that

$$
\begin{aligned}
&||g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(t, A)||_{E\to E} \\
&\quad = ||\int_t^{t+\tau} P_\alpha(s, A)ds||_{E\to E} \le M_1 \int_t^{t+\tau} s^{\alpha-1} ds \le M_1 t^{\alpha-1} \int_t^{t+\tau} ds \\
&\quad = M_1 t^{\alpha-1}\tau = M_1 t^\alpha \frac{\tau}{t} \le M_2 \frac{\tau}{t}.
\end{aligned}
$$

While, $||g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(t, A)||_{E\to E} \le M_1(t+\tau)^\alpha + M_1 t^\alpha \le M_2$, then similar to the above process, one has (2.4). From the inequalities

$$
\begin{aligned}
||P_\alpha(t+\tau, A) - P_\alpha(t, A)||_{E\to E} &= ||\int_t^{t+\tau} P'_\alpha(s, A)ds||_{E\to E} \\
&\le M_1 \int_t^{t+\tau} \frac{ds}{s^{2-\alpha}} \le \frac{M_1\tau}{t^{2-\alpha}} = \frac{M_1 t^\alpha \tau}{t^2} \le M_2 \frac{\tau}{t^2}, \\
||P_\alpha(t+\tau, A) - P_\alpha(t, A)||_{E\to E} &\le M_1 \frac{(t+\tau)^\alpha}{t+\tau} + M_1 \frac{t^\alpha}{t} \le \frac{M_2}{t},
\end{aligned}
$$

we get (2.5). □

Remark 2.1. Actually, we can get (2.1) from the above lemma. In fact,

$$
\begin{aligned}
&||S_\alpha(\cdot, A)x||_{C_0^\beta([0,T];E)} \\
&\quad = \sup_{0\le t\le T} ||S_\alpha(t, A)x||_E + \sup_{0\le t<t+\tau\le T} \frac{t^\beta ||S_\alpha(t+\tau, A)x - S_\alpha(t, A)x||_E}{\tau^\beta} \\
&\quad \le M_1||x||_E + \sup_{0\le t<t+\tau\le T} \frac{M_2 t^\beta}{\tau^\beta}\frac{\tau^\beta}{t^\beta}||x||_E \le \bar{M}||x||_E.
\end{aligned}
$$

The authors in [20] has proved that when the operator A has a bounded inverse A^{-1}, the well-posedness of

$$
u'(t) = Au(t) + f(t), \quad t \in [0, T]; \quad u(0) = x, \tag{2.11}
$$

is valid in $C_0^\beta([0, T]; E)$ iff the following coercivity inequality holds

$$
||u'||_{C_0^\beta([0,T];E)} + ||Au||_{C_0^\beta([0,T];E)} \le M\left(||Ax||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\right),
$$

where M is independent of β, u_0 *and* $f(\cdot)$.

In fact, it is such a strong condition that the operator A has a bounded inverse A^{-1}. When we consider the problem (2.11), it can be replaced by the condition that the operator $(\lambda I - A)^{-1}$ is bounded for some λ. The latter one can be easily satisfied. While, when it comes to the problem (1.1), such condition can not be replaced. It means that we do not have the boundedness of the operator A^{-1} in this paper. Then

$$||\mathbf{D}_t^{\alpha}u(\cdot)||_{C_0^{\beta}([0,T];E)} + ||Au(\cdot)||_{C_0^{\beta}([0,T];E)} \leq \bar{M}\left(||Ax||_E + \frac{1}{\beta(1-\beta)}||f(\cdot)||_{C_0^{\beta}([0,T];E)}\right),$$

can not imply that the operator $u(t;f(\cdot),x)$ is continuous. From (1.5) on page 3 of [20], we see that the well-posedness of (1.1) is valid in $C_0^{\beta}([0,T];E)$ iff the following coercivity inequality holds

$$\begin{aligned} &||u||_{C_0^{\beta}([0.T];E)} + ||\mathbf{D}_t^{\alpha}u||_{C_0^{\beta}([0,T];E)} + ||Au||_{C_0^{\beta}([0,T];E)} \\ &\leq \ \bar{M}\left(||x||_E + ||Ax||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^{\beta}([0,T];E)}\right), \end{aligned} \tag{2.12}$$

where $\bar{M}$ is independent of β, u_0 and $f(\cdot)$.

We are going to show that the analyticity of $S_\alpha(t,A)$ is a necessary and sufficient condition for the well-posedness of (1) in $C_0^{\beta}([0,T];E)$.

Theorem 2.1. *If the problem (1.1) is well posed in $C_0^{\beta}([0,T];E)$, then $S_\alpha(t,A)$ is an analytic α-resolvent family.*

Proof. The problem (1.1) is considered in a complex Banach space E. By the strong continuity of $S_\alpha(t,A), \lambda^\alpha I - A$ has a bounded inverse for all complex λ with $Re\lambda > \omega$. It means that for any $\varphi \in E, \lambda^\alpha\psi - A\psi = \varphi$ has a unique solution $\psi = (\lambda^\alpha I - A)^{-1}\varphi$. Clearly, the function $u(t) = S_\alpha(t,\lambda^\alpha)\psi$ is a solution in $C_0^{\beta}([0,T];E)$ of (1) with $f(t) = S_\alpha(t,\lambda^\alpha)\varphi \mathrm{and} u(0) = \psi$. Actually, for such $u(\cdot)$ *and* $f(\cdot)$, the coercivity inequality (2.12) provides the following inequality

$$||\mathbf{D}_t^{\alpha}u||_{C_0^{\beta}([0,T];E)} + ||Au||_{C_0^{\beta}([0,T];E)} \leq \bar{M}\left(||Ax||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^{\beta}([0,T];E)}\right),$$

As the same as the discussion on the page 17 of [20], we get

$$\begin{aligned} &||\lambda^\alpha S_\alpha(\cdot,\lambda^\alpha)\psi||_{C_0^{\beta}([0,T];E)} + ||AS_\alpha(\cdot,\lambda^\alpha)\psi||_{C_0^{\beta}([0,T];E)} \\ &\leq \bar{M}(||S_\alpha(\cdot,\lambda^\alpha)\varphi||_{C_0^{\beta}([0,T];E)} + ||A\psi||_E). \end{aligned}$$

Hence,

$$|\lambda^\alpha|||\psi||_E + ||A\psi||_E \leq \bar{M}\Big[||\varphi||_E + ||S_\alpha(\cdot,\lambda^\alpha)||^{-1}_{C_0^{\beta}([0,T];\mathbb{C})}||A\psi||_E\Big],$$

where

$$||S_\alpha(\cdot,\lambda^\alpha)||_{C_0^{\beta}([0,T];\mathbb{C})} = \sup_{0\leq t\leq T}||S_\alpha(t,\lambda^\alpha)|| + \sup_{0\leq t<t+\tau\leq T}\frac{t^\beta||S_\alpha(t+\tau,\lambda^\alpha) - S_\alpha(t,\lambda^\alpha)||}{\tau^\beta}.$$

Clearly, $||S_\alpha(\cdot,\lambda^\alpha)||_{C_0^\beta([0,T];\mathbb{C})}\to\infty$, as $Re\lambda\to\infty$. Together with $\psi=(\lambda^\alpha I-A)^{-1}\varphi$, we have, for sufficiently large $\omega_1>\omega$ and any $\lambda$with $Re\lambda>\omega_1$, $||\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}||_{E\to E}\le\frac{M}{|\lambda|}\le\frac{M}{|\lambda-\omega|}$. Then, $S_\alpha(t,A)$ is analytic. □

Theorem 2.2. *Let A be the generator of an analytic α-times resolvent family. Then (1.1) is well posed in $C_0^\beta([0,T];E)$ and the coercivity inequality (2.12) holds.*

Proof. If $u(\cdot)$ is a solution to problem (1.1) in $C_0^\beta([0,T];E)$, then it is a solution in $C([0,T];E)$, too. Hence, we have the representation $u(t)=S_\alpha(t,A)x+\int_0^t P_\alpha(t-s,A)f(s)ds:=w(t)+v(t)$. We need to show that $Aw(t),Av(t),w(t),v(t)$ belongs to $C_0^\beta([0,T];E)$ and (2.12) holds.

Firstly, let us consider the estimate of $||Au(\cdot)||_{C_0^\beta([0,T];E)}$. We know that $w(t)\in D(A)$ and $||Aw(t)||_E=||S_\alpha(t,A)Ax||_E\le M_1||Ax||_E, 0\le t\le T$. Applying (2.2), we get that, for $0<t<t+\tau\le T$,

$$||Aw(t+\tau)-Aw(t)||_E=||S_\alpha(t+\tau,A)Ax-S_\alpha(t,A)Ax||_E\le M_2||Ax||_E\frac{\tau^\beta}{t^\beta}.$$

Then,

$$\begin{aligned}||Aw||_{C_0^\beta([0,T];E)}&=\sup_{0\le t\le T}||Aw(t)||_E+\sup_{0\le t<t+\tau\le T}||Aw(t+\tau)-Aw(t)||_E\frac{t^\beta}{\tau^\beta}\\&\le M_1||Ax||_E+\sup_{0\le t<t+\tau\le T}\frac{M_2t^\beta}{\tau^\beta}\frac{\tau^\beta}{t^\beta}||Ax||_E\le\bar{M}||Ax||_E.\end{aligned}$$

Since

$$\begin{aligned}Av(t)&=\int_0^t AP_\alpha(t-s,A)(f(s)-f(t))ds+\int_0^t AP_\alpha(t-s,A)dsf(t)\\&=\int_0^t AP_\alpha(t-s,A)(f(s)-f(t))ds+(S_\alpha(t,A)-I)f(t),\end{aligned}$$

we have $||Av(t)||_E\le(M_1+1)||f(t)||_E+M_1\int_0^t\frac{||f(s)-f(t)||_E}{t-s}ds$. Following from the definition $||f||_{C_0^\beta([0,T];E)}=\sup_{0\le t\le T}||f(t)||_E+\sup_{0\le s<t\le T}\frac{s^\beta||f(s)-f(t)||_E}{(t-s)^\beta}$, we can get that $||f(s)-f(t)||_E\le||f||_{C_0^\beta([0,T];E)}\frac{(t-s)^\beta}{s^\beta}$. Hence $||Av(t)||_E\le(M_1+1)||f||_{C_0^\beta([0,T];E)}+M_1\int_0^t\frac{ds}{(t-s)^{1-\beta}s^\beta}||f||_{C_0^\beta([0,T];E)}$. While,

$$\begin{aligned}\int_0^t\frac{ds}{(t-s)^{1-\beta}s^\beta}&=\int_0^1\frac{d\sigma}{(1-\sigma)^{1-\beta}\sigma^\beta}=\Gamma(1-\beta)\Gamma(\beta)=\frac{\Gamma(2-\beta)\Gamma(1+\beta)}{\beta(1-\beta)}\\&\le\frac{N}{\beta(1-\beta)},\end{aligned}$$

it follows that for $0 \le t \le T$,

$$||Av(t)||_E \le \frac{(M_1+1)\beta(1-\beta)+M_1N}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)} \le \frac{M_3}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}$$
$$\lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}.$$

Next, we shall estimate the difference $Av(t+\tau)-Av(t), 0<t<t+\tau \le T$. We consider the cases $t \le 2\tau$ and $t>2\tau$, separately. When $t \le 2\tau$,

$$\begin{aligned}||Av(t+\tau)-Av(t)||_E &\le ||Av(t+\tau)||_E+||Av(t)||_E \le \frac{2M_3}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\\ &= \frac{2M_3}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\tau^\beta\tau^{-\beta} \le \frac{2^{\beta+1}M_3}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}\\ &\lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.\end{aligned}$$

When $t>2\tau$,

$$\begin{aligned}&Av(t+\tau)-Av(t)\\ &= (S_\alpha(t+\tau,A)-I)f(t+\tau)-(S_\alpha(t,A)-I)f(t)\\ &+\int_0^{t+\tau} AP_\alpha(t+\tau-s,A)(f(s)-f(t+\tau))ds-\int_0^t AP_\alpha(t-s,A)(f(s)-f(t))ds\\ &= f(t)-f(t+\tau)+S_\alpha(t+\tau,A)f(t+\tau)-S_\alpha(t,A)f(t)\\ &+\int_{t-\tau}^{t+\tau} AP_\alpha(t+\tau-s,A)(f(s)-f(t+\tau))ds+\int_{t-\tau}^t AP_\alpha(t-s,A)(f(t)-f(s))ds\\ &+\int_0^{t-\tau} AP_\alpha(t+\tau-s,A)(f(t)-f(t+\tau))ds\\ &+\int_0^{t-\tau} A(P_\alpha(t+\tau-s,A)-P_\alpha(t-s,A))(f(s)-f(t))ds\\ &:= I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

$$\begin{aligned}||I_1||_E &= ||f(t)-f(t+\tau)+S_\alpha(t+\tau,A)f(t+\tau)-S_\alpha(t,A)f(t)||_E\\ &\le ||f(t)-f(t+\tau)||_E+||S_\alpha(t+\tau,A)||_{E\to E}||f(t+\tau)-f(t)||_E\\ &+||S_\alpha(t+\tau,A)-S_\alpha(t,A)||_{E\to E}||f(t)||_E\end{aligned}$$

From (2.2) and the definition of $||f(\cdot)||_{C_0^\beta([0,T];E)}$, we have

$$||I_1||_E \lesssim \bar{M}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.$$

$$\begin{aligned}||I_2||_E &\le \int_{t-\tau}^{t+\tau}||AP_\alpha(t+\tau-s,A)||_{E\to E}||f(s)-f(t+\tau)||_E ds\\ &\le M_1\int_{t-\tau}^{t+\tau}\frac{ds}{(t+\tau-s)^{1-\beta}s^\beta}||f||_{C_0^\beta([0,T];E)}\\ &\le M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\int_{t-\tau}^{t+\tau}\frac{ds}{(t+\tau-s)^{1-\beta}}\\ &= M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\frac{(2\tau)^\beta}{\beta}.\end{aligned}$$

Because $t-\tau=\frac{t}{2}+\frac{t}{2}-\tau>\frac{t}{2}$, then

$$||I_2||_E \le M_1||f||_{C_0^\beta([0,T];E)}\frac{(2\tau)^\beta}{\beta(\frac{t}{2})^\beta}\lesssim\frac{\bar{M}}{\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.$$

$$\begin{aligned}||I_3||_E &\le \int_{t-\tau}^{t}||AP_\alpha(t-s,A)||_{E\to E}||f(s)-f(t)||_E ds\\ &\le M_1\int_{t-\tau}^{t}\frac{ds}{(t-s)^{1-\beta}s^\beta}||f||_{C_0^\beta([0,T];E)}\\ &\le M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\int_{t-\tau}^{t}\frac{ds}{(t-s)^{1-\beta}} = M_1||f||_{C_0^\beta([0,T];E)}\frac{\tau^\beta}{\beta(t-\tau)^\beta}\\ &\le M_1||f||_{C_0^\beta([0,T];E)}\frac{\tau^\beta}{\beta(\frac{t}{2})^\beta}\lesssim\frac{\bar{M}}{\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.\end{aligned}$$

$$\begin{aligned}I_4 &= \int_0^{t-\tau}AP_\alpha(t+\tau-s,A)(f(t)-f(t+\tau))ds\\ &= \int_{2\tau}^{t+\tau}AP_\alpha(h,A)dh(f(t)-f(t+\tau))\\ &= (\int_0^{t+\tau}AP_\alpha(h,A)dh-\int_0^{2\tau}AP_\alpha(h,A)dh)(f(t)-f(t+\tau))\\ &= (S_\alpha(t+\tau,A)-I-S_\alpha(2\tau,A)+I)(f(t)-f(t+\tau))\\ &= (S_\alpha(t+\tau,A)-S_\alpha(2\tau,A))(f(t)-f(t+\tau)).\end{aligned}$$

$$||I_4||_E \le ||S_\alpha(t+\tau,A)-S_\alpha(2\tau,A)||_{E\to E}||f(t)-f(t+\tau)||_E\lesssim\bar{M}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.$$

If $\beta = 1$ in (2.3), we obtain

$$
\begin{aligned}
||I_5||_E &\le \int_0^{t-\tau} ||A(P_\alpha(t+\tau-s,A)-P_\alpha(t-s,A))||_{E\to E}||f(s)-f(t)||_E ds \\
&\le M_2 \int_0^{t-\tau} \frac{\tau}{(t-s)^2}\frac{(t-s)^\beta}{s^\beta}||f||_{C_0^\beta([0,T];E)} ds \\
&= M_2 ||f||_{C_0^\beta([0,T];E)} \int_0^{t-\tau} \frac{\tau}{(t-s)^{2-\beta}s^\beta} ds.
\end{aligned}
$$

While,

$$
\begin{aligned}
\int_0^{t-\tau} \frac{\tau}{(t-s)^{2-\beta}s^\beta} ds &= \int_0^{\frac{t}{2}} \frac{\tau}{(t-s)^{2-\beta}s^\beta} ds + \int_{\frac{t}{2}}^{t-\tau} \frac{\tau}{(t-s)^{2-\beta}s^\beta} ds \\
&\le \frac{\tau}{(\frac{t}{2})^{2-\beta}}\frac{s^{1-\beta}}{1-\beta}\Big|_0^{\frac{t}{2}} + \frac{\tau}{(\frac{t}{2})^\beta}\frac{(t-s)^{\beta-1}}{1-\beta}\Big|_{\frac{t}{2}}^{t-\tau} \\
&= \frac{2\tau}{(1-\beta)t} + \frac{2^\beta\tau^\beta}{(1-\beta)t^\beta} - \frac{2\tau}{(1-\beta)t} = \frac{2^\beta\tau^\beta}{(1-\beta)t^\beta}.
\end{aligned}
$$

Then we get

$$
||I_5||_E \lesssim \frac{\bar{M}}{1-\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
$$

It means that we have proven that for any $0<t<t+\tau \le T$,

$$
||Av(t+\tau)-Av(t)||_E \lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
$$

Consequently,

$$
||Av||_{C_0^\beta([0,T];E)} \lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}.
$$

Hence,

$$
||Au||_{C_0^\beta([0,T];E)} \lesssim \bar{M}\left(||Ax||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\right). \tag{2.13}
$$

From (1.1), (2.13) and the triangle inequality, we know that

$$
||\mathbf{D}_t^\alpha u||_{C_0^\beta([0,T];E)} \lesssim \bar{M}\left(||Ax||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\right). \tag{2.14}
$$

Secondly, let us consider the estimate of $||u||_{C_0^\beta([0,T];E)}$. We can easily get the estimate $||w||_{C_0^\beta([0,T];E)} \lesssim \bar{M}||x||_E$. Since

$$v(t) = \int_0^t P_\alpha(t-s,A)(f(s)-f(t))ds + f(t)\int_0^t P_\alpha(t-s,A)ds$$
$$= \int_0^t P_\alpha(t-s,A)(f(s)-f(t))ds + g_\alpha * S_\alpha(t,A)f(t),$$

it follows from $||P_\alpha(t,A)|| \le M_1(1+t^{\alpha-1})$ for any $t>0$ that

$$||v(t)||_E \le M_1 t^\alpha ||f(t)||_E + M_1 \int_0^t \frac{||f(s)-f(t)||_E}{(t-s)^{1-\alpha}} ds$$
$$\le M_1 t^\alpha ||f(t)||_E + M_1 \int_0^t \frac{ds}{(t-s)^{1-\alpha-\beta} s^\beta} ||f||_{C_0^\beta([0,T];E)}.$$

While, $\int_0^t \frac{ds}{(t-s)^{1-\alpha-\beta} s^\beta} = t^\alpha \int_0^1 (1-\sigma)^{\alpha+\beta-1} \sigma^{-\beta} d\sigma = t^\alpha \frac{\Gamma(\alpha+\beta)\Gamma(1-\beta)}{\Gamma(1+\alpha)} = t^\alpha \frac{\Gamma(\alpha+\beta+1)\Gamma(2-\beta)}{\Gamma(1+\alpha)(\alpha+\beta)(1-\beta)} \le \frac{Lt^\alpha}{\beta(1-\beta)}$. Then,

$$||v(t)||_E \le \frac{M_1(\beta(1-\beta)+L)t^\alpha}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)} \le \frac{M_4}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)}.$$

Then we consider $v(t+\tau)-v(t), 0<t<t+\tau \le T$ under the case $t \le 2\tau$ and $t>2\tau$, separately. For the case $t \le 2\tau$ one has

$$||v(t+\tau)-v(t)||_E \le ||v(t+\tau)||_E + ||v(t)||_E \le \frac{2M_4}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)}$$
$$= \frac{2M_4}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)} \tau^\beta \tau^{-\beta} \le \frac{2^{\beta+1}M_4}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)} \tau^\beta t^{-\beta}$$
$$\lesssim \frac{\bar{M}}{\beta(1-\beta)} ||f||_{C_0^\beta([0,T];E)} \tau^\beta t^{-\beta}.$$

When $t>2\tau$,

$$\begin{aligned}
&v(t+\tau)-v(t)\\
&= g_\alpha * S_\alpha(t+\tau,A)f(t+\tau) - g_\alpha * S_\alpha(t,A)f(t)\\
&\quad + \int_0^{t+\tau} P_\alpha(t+\tau-s,A)(f(s)-f(t+\tau))ds - \int_0^t P_\alpha(t-s,A)(f(s)-f(t))ds\\
&= \left[g_\alpha * S_\alpha(t+\tau,A)(f(t+\tau)-f(t)) + (g_\alpha * S_\alpha(t+\tau,A) - g_\alpha * S_\alpha(t,A))f(t)\right]\\
&\quad + \int_{t-\tau}^{t+\tau} P_\alpha(t+\tau-s,A)(f(s)-f(t+\tau))ds + \int_{t-\tau}^t P_\alpha(t-s,A)(f(t)-f(s))ds\\
&\quad + \int_0^{t-\tau} P_\alpha(t+\tau-s,A)(f(t)-f(t+\tau))ds\\
&\quad + \int_0^{t-\tau} (P_\alpha(t+\tau-s,A) - P_\alpha(t-s,A))(f(s)-f(t))ds\\
&:= \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4 + \bar{I}_5.
\end{aligned}$$

Applying (2.4), we get that

$$\begin{aligned}
||\bar{I}_1||_E &= ||g_\alpha * S_\alpha(t+\tau, A)(f(t+\tau) - f(t)) \\
&\quad + (g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(t, A))f(t)||_E \\
&\le ||g_\alpha * S_\alpha(t+\tau, A)||_{E\to E}||f(t+\tau) - f(t)||_E \\
&\quad + ||g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(t, A)||_{E\to E}||f(t)||_E \\
&\le M_1(t+\tau)^\alpha ||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta} + M_2||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta} \\
&\lesssim \bar{M}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
\end{aligned}$$

$$\begin{aligned}
||\bar{I}_2||_E &\le \int_{t-\tau}^{t+\tau} ||P_\alpha(t+\tau-s, A)||_{E\to E}||f(s) - f(t+\tau)||_E ds \\
&\le M_1 \int_{t-\tau}^{t+\tau} \frac{ds}{(t+\tau-s)^{1-\alpha-\beta}s^\beta}||f||_{C_0^\beta([0,T];E)} \\
&\le M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\int_{t-\tau}^{t+\tau}\frac{ds}{(t+\tau-s)^{1-\alpha-\beta}} \\
&= M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\frac{(2\tau)^{\alpha+\beta}}{\alpha+\beta} \\
&\le M_1||f||_{C_0^\beta([0,T];E)}\frac{2^{\alpha+\beta}\tau^\alpha\tau^\beta}{\beta(\frac{t}{2})^\beta} \lesssim \frac{\bar{M}}{\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
\end{aligned}$$

$$\begin{aligned}
||\bar{I}_3||_E &\le \int_{t-\tau}^{t} ||P_\alpha(t-s, A)||_{E\to E}||f(s) - f(t)||_E ds \\
&\le M_1 \int_{t-\tau}^{t} \frac{ds}{(t-s)^{1-\alpha-\beta}s^\beta}||f||_{C_0^\beta([0,T];E)} \\
&\le M_1||f||_{C_0^\beta([0,T];E)}\frac{1}{(t-\tau)^\beta}\int_{t-\tau}^{t}\frac{ds}{(t-s)^{1-\alpha-\beta}} \\
&= M_1||f||_{C_0^\beta([0,T];E)}\frac{\tau^{\alpha+\beta}}{(\alpha+\beta)(t-\tau)^\beta} \\
&\le M_1||f||_{C_0^\beta([0,T];E)}\frac{\tau^\alpha\tau^\beta}{\beta(\frac{t}{2})^\beta} \lesssim \frac{\bar{M}}{\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
\end{aligned}$$

$$\begin{aligned}
\bar{I}_4 &= \int_0^{t-\tau} P_\alpha(t+\tau-s, A)(f(t) - f(t+\tau))ds = \int_{2\tau}^{t+\tau} P_\alpha(h, A)dh(f(t) - f(t+\tau)) \\
&= \left(\int_0^{t+\tau} P_\alpha(h, A)dh - \int_0^{2\tau} P_\alpha(h, A)dh)(f(t) - f(t+\tau)\right) \\
&= (g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(2\tau, A))(f(t) - f(t+\tau)).
\end{aligned}$$

$$\begin{aligned}
||\bar{I}_4||_E &\le ||g_\alpha * S_\alpha(t+\tau, A) - g_\alpha * S_\alpha(2\tau, A)||_{E\to E}||f(t) - f(t+\tau)||_E \\
&\le (||g_\alpha * S_\alpha(t+\tau, A)||_{E\to E} + ||g_\alpha * S_\alpha(2\tau, A)||_{E\to E})||f(t) - f(t+\tau)||_E \\
&\le M_1((t+\tau)^\alpha + (2\tau)^\alpha)||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta} \lesssim \bar{M}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.
\end{aligned}$$

Putting $\beta = 1$ in (2.5) and using the method similar to what we used to estimate I_5, we obtain

$$||\bar{I}_5||_E \lesssim \frac{\bar{M}}{1-\beta}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.$$

It means that we have proved that for any $0 < t < t+\tau \le T$,

$$||v(t+\tau) - v(t)||_E \lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}\tau^\beta t^{-\beta}.$$

Consequently,

$$||v||_{C_0^\beta([0,T];E)} \lesssim \frac{\bar{M}}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}.$$

Hence,

$$||u||_{C_0^\beta([0,T];E)} \lesssim \bar{M}(||x||_E + \frac{1}{\beta(1-\beta)}||f||_{C_0^\beta([0,T];E)}). \tag{2.15}$$

It follows from (2.13); (2.14); (2.15) that (2.12) holds. We proved the theorem. □

3. The fulldiscrete approximation in $C([0,T];E)$

Assume that the functions $f_n(\cdot) \in C([0,T];E_n)$ converge to the function $f(\cdot) \in C([0,T];E)$ in the sense $\sup_{t\in[0,T]}||f_n(t) - p_n f(t)||_{E_n} \to 0 \; as \; n \to \infty$. Although Theorems 4.2 and 4.3 hold, when consider the full discretization, we will impose stronger conditions on operators A, A_n. First of all, this is due to the fact that a bounded linear perturbation $A+B$, for bounded B, remove the problem, in general, from the class of well-posed problems. Therefore, we assume that the operators A, A_n generate C_0-semigroups. In such situation, as was shown in [17], one has $P_\alpha(t, A_n)x_n \xrightarrow{\mathcal{P}} P_\alpha(t, A)x$ for $t > 0$ as $n \to \infty$, whenever $x_n \xrightarrow{\mathcal{P}} x$ for any $x_n \in E_n, x \in E$. Then under conditions (A) and (B_1) from Appendix, one can get the convergence of solutions of problems (1.4) to solution of problem (1.1) by major convergence Theorem. So in this section, we investigate approximation of problems (1.4) and we consider the full discretization of the problem (1.1) by using the following implicit difference scheme [18]

$$\Delta_{t_k}^\alpha \bar{U}_n(\cdot) = A_n\bar{U}_n(t_k) + f_n(t_k), \quad t_k = k\tau_n, \quad \bar{U}_n(0) = x_n, \tag{3.1}$$

and explicit difference scheme [18]

$$\Delta^{\alpha}_{t_k} U_n(\cdot) = A_n U_n(t_{k-1}) + f_n(t_{k-1}), \quad t_k = k\tau_n, \quad U_n(0) = x_n. \tag{3.2}$$

For any grid function $\Theta_n(\cdot)$ the finite difference approximation is defined by

$$\Delta^{\alpha}_{t_k}\Theta_n(\cdot) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} (t^{1-\alpha}_{j+1} - t^{1-\alpha}_{j}) \frac{\Theta_n(t_{k-j}) - \Theta_n(t_{k-j-1})}{\tau_n}.$$

We can split the solution of (1.4) by the equation $u_n(t) = S_\alpha(t, A_n)x_n + \int_0^t P_\alpha(t-s, A_n) f_n(s)ds := w_n(t) + v_n(t)$.

Lemma 3.1. *Let $x_n \in D(A_n)$ and $f_n(\cdot) \in C^2([0,T];E_n)$. Then*

$$\begin{aligned} ||u'_n(t)|| &\le C(\alpha)(1+t^{\alpha-1})\max\{||A_n x_n||, ||f_n(\cdot)||_{C^1([0,T];E_n)}\}, \\ ||u''_n(t)|| &\le C(\alpha)(1+t^{\alpha-2})\max\{||A_n x_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}\}. \end{aligned} \tag{3.3}$$

Proof. Indeed, from Lemma 1.1, we know that $||w'_n(t)|| = ||S'_\alpha(t, A_n)x_n|| \le C(\alpha)(1+t^{\alpha-1})||A_n x_n||, ||w''_n(t)|| = ||S''_\alpha(t, A_n)x_n|| \le C(\alpha)(1+t^{\alpha-2})||A_n x_n||$.

One can write,

$$\begin{aligned} v'_n(t) &= P_\alpha(t, A_n) f_n(0) + \int_0^t P_\alpha(s, A_n) f'_n(t-s)ds, \\ v''_n(t) &= P'_\alpha(t, A_n) f_n(0) + P_\alpha(t, A_n) f'_n(0) + \int_0^t P_\alpha(s, A_n) f''_n(t-s)ds. \end{aligned}$$

Then we can get

$$\begin{aligned} ||v'_n(t)|| &\le C(\alpha)(1+t^{\alpha-1})||f_n(0)|| + C(\alpha)(1+t^{\alpha})||f'_n(\cdot)||_{C([0,T];E_n)}, \\ ||v''_n(t)|| &\le C(\alpha)(1+t^{\alpha-2})||f_n(0)|| + C(\alpha)(1+t^{\alpha-1})||f'_n(0)|| + C(\alpha)(1+t^{\alpha})||f''_n(\cdot)||_{C([0,T];E_n)}. \end{aligned}$$

Hence, we have (3.3). □

From the above Lemma, we know that when $x_n \in D(A_n)$ and $f_n(\cdot) \in C^2([0,T];E_n)$, the solution $u_n(t)$ satisfies the estimates (3.3) which is as the same as it in [19]. From the analysis in [19] and from the estimates (3.3), one has that $\bar{r}_{u_n}(k\tau_n) := (\mathbf{D}^{\alpha}_t u_n)(t_k) - \Delta^{\alpha}_{t_k} u_n(\cdot) = O(\tau_n^{\alpha})$ and

$$||\bar{r}_{u_n}(k\tau_n)|| \le C(\alpha)\tau_n^{\alpha}\max\{||A_n x_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}\}.$$

First, we approximate the problem (1.4) by implicit scheme (3.1). We know that (3.1) can be split into

$$\Delta^{\alpha}_{t_k}\bar{W}_n(\cdot) = A_n \bar{W}_n(t_k), \quad \bar{W}_n(0) = x_n,$$

and

$$\Delta^{\alpha}_{t_k}\bar{V}_n(\cdot) = A_n \bar{V}_n(t_k) + f_n(t_k), \quad \bar{V}_n(0) = 0.$$

It is clear that $\bar{U}_n(t_k) = \bar{W}_n(t_k) + \bar{V}_n(t_k)$. Then from Proposition 3.1 and Theorem 3.2 in [18], we have the following theorem.

Theorem 3.1. *For the implicit difference scheme (3.1), i.e. for the system*

$$\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k-1} b_j \frac{\bar{U}_n((k-j)\tau_n)-\bar{U}_n((k-j-1)\tau_n)}{\tau_n^\alpha} = A_n\bar{U}_n(k\tau_n)+f_n(k\tau_n),$$
$$\bar{U}_n(0)=x_n,$$

we have that

$$\bar{U}_n(k\tau_n)=\sum_{j=1}^{k} c_j^{(k)}R^j x_n + \Gamma(2-\alpha)\tau_n^\alpha \sum_{j=1}^{k}\sum_{i=1}^{k-j+1} d_{i,j}^{(k)} R^i f_n(j\tau_n) \text{ for any } k\in I\!N,$$

where $c_j^{(k)}, d_{i,j}^{(k)}$ are the same as them in [18, 19]. Moreover, for every $k\in\mathbb{N}, c_j^{(k)}>0, j=1,2,\ldots,k,\quad d_{i,j}^{(k)}\geq 0, i=1,\ldots,k-j+1, j=1,\ldots,k,$ and $\sum_{j=1}^{k} c_j^{(k)}=1$, $\sum_{j=1}^{k}\sum_{i=1}^{k-j+1} d_{i,j}^{(k)} b_{j-1}=1$.

Now from Theorem 3.1 and Theorem 3.3 in [18], we can easily get the stability of implicit difference scheme (3.1).

Theorem 3.2. *Assume that C_0-semigroups e^{tA_n} satisfy condition (B) with $\omega=0$. Then the implicit difference scheme (3.1) is stable, i.e.*

$$||\bar{U}_n(k\tau_n)|| \leq M||x_n|| + M\Gamma(1-\alpha)(k\tau_n)^\alpha \sup_{1\leq j\leq k} ||f_n(j\tau_n)||.$$

The next theorem gives us the order of convergence of the implicit difference scheme (3.1). Set $\bar{z}_{u_n}(k\tau_n)=u_n(k\tau_n)-\bar{U}_n(k\tau_n)$.

Theorem 3.3. *The representation of $\bar{z}_{u_n}(k\tau_n)$ is:*

$$\bar{z}_{u_n}(k\tau_n)=\Gamma(2-\alpha)\tau_n^\alpha\sum_{j=1}^{k}\sum_{i=1}^{k-j+1} d_{i,j}^{(k)} R^i \bar{r}_{u_n}(j\tau_n),$$

where $d_{i,j}^{(k)}$ are the same as they in [18, 19]. Moreover, under conditions of Theorem 3.2 and $x_n\in D(A_n), f_n(\cdot)\in C^2([0,T];E_n)$, we have

$$||\bar{z}_{u_n}(k\tau_n)|| \leq C(\alpha)\tau_n^\alpha \max\{||A_n x_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}\}.$$

Proof. Since $\bar{z}_{u_n}(k\tau_n)=u_n(k\tau_n)-\bar{U}_n(k\tau_n)$, one has $\bar{U}_n(k\tau_n)=u_n(k\tau_n)-\bar{z}_{u_n}(k\tau_n)$. Then $\Delta_{k\tau_n}^\alpha(u_n(\cdot)-\bar{z}_{u_n}(\cdot))=A_n(u_n(k\tau_n)-\bar{z}_{u_n}(k\tau_n))+f_n(k\tau_n)$, and $\Delta_{t_k}^\alpha\bar{z}_{u_n}(\cdot)=A_n\bar{z}_{u_n}(t_k)+\Delta_{t_k}^\alpha u_n(\cdot)-A_nu_n(k\tau_n)-f_n(k\tau_n)$. While $(\mathbf{D}_t^\alpha u_n)(t_k)=A_nu_n(k\tau_n)+f_n(k\tau_n)$, we get the following equation

$$\Delta_{t_k}^\alpha\bar{z}_{u_n}(\cdot)=A_n\bar{z}_{u_n}(t_k)+\Delta_{t_k}^\alpha u_n(\cdot)-(\mathbf{D}_t^\alpha u_n)(t_k)=A_n\bar{z}_{u_n}(k\tau_n)+\bar{r}_{u_n}(k\tau_n).$$

Then we can get the conclusion of Theorem from the proof of Theorem 3.3 in [19]. □

Next, the explicit scheme (3.2) can be split into

$$\Delta_{t_k}^{\alpha} W_n(\cdot) = A_n W_n(t_{k-1}), \quad W_n(0) = x_n,$$

and

$$\Delta_{t_k}^{\alpha} V_n(\cdot) = A_n V_n(t_{k-1}) + f_n(t_{k-1}), \quad V_n(0) = 0.$$

Here, $U_n(t_k) = W_n(t_k) + V_n(t_k)$. Then we have the following theorem.

Theorem 3.4. *For the explicit scheme (3.2), i.e. for the scheme*

$$\frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} b_j \frac{U_n((k-j)\tau_n) - U_n((k-j-1)\tau_n)}{\tau_n^{\alpha}} = A_n U_n((k-1)\tau_n) + f_n((k-1)\tau_n),$$
$$U_n(0) = x_n,$$

we have that

$$U_n(k\tau_n) = \sum_{j=0}^{k} \bar{c}_j^{(k)} \bar{R}^j x_n + \Gamma(2-\alpha)\tau_n^{\alpha} \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} \bar{R}^i f_n(j\tau_n) \text{ for any } k \in I\!N,$$

where $\bar{c}_j^{(k)}, \bar{d}_{i,j}^{(k)}$ are the same as them in [18, 19]. Moreover, for any $k = 1, 2, 3, \ldots, \bar{c}_j^{(k)} > 0, j = 0, 1, \ldots, k$, $\bar{d}_{i,j+1}^{(k+1)} \geq 0, i = 0, \ldots, k-j-1, j = 0, \ldots, k-1$, and $\sum_{j=0}^{k} \bar{c}_j^{(k)} = 1$, $\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} b_j = 1$.

Proof. We only need to show that $V_n(k\tau_n) = \Gamma(2-\alpha)\ \tau_n^{\alpha} \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} \bar{R}^i f_n(j\tau_n)$.

(1) For $k = 1$, we have $V_n(\tau_n) = \Gamma(2-\alpha)\tau_n^{\alpha} f_n(0), \bar{d}_{0,1}^{(2)} = 1$;
(2) For $k = 2$, we have $V_n(2\tau_n) = \Gamma(2-\alpha)\tau_n^{\alpha}((1-b_1)\bar{R} f_n(0) + f_n(\tau_n)), \bar{d}_{0,1}^{(3)} = 0, \bar{d}_{1,1}^{(3)} = 1-b_1 \geq 0, \bar{d}_{0,2}^{(3)} = 1$ and $(1-b_1)b_0 + b_1 = 1$;

While, we know that, when $k \geq 3$,

$$V_n(k\tau_n)$$
$$= (1-b_1)\bar{R} V_n((k-1)\tau_n) + \sum_{j=2}^{k-1} (b_{j-1} - b_j) V_n((k-j)\tau_n) + \Gamma(2-\alpha)\tau_n^{\alpha} f_n(k-1\tau_n).$$

(3) For $k = 3$, we have $V_n(3\tau_n) = \Gamma(2-\alpha)\tau_n^{\alpha}((b_1-b_2) f_n(0) + (1-b_1)^2 \bar{R}^2 f_n(0) + (1-b_1)\bar{R} f_n(\tau_n) + f_n(2\tau_n)), \bar{d}_{0,1}^{(4)} = b_1 - b_2 \geq 0, \bar{d}_{1,1}^{(4)} = 0,$

$\bar{d}^{(4)}_{2,1} = (b_1 - b_2)^2$, $\bar{d}^{(4)}_{0,2} = 0$, $\bar{d}^{(4)}_{1,2} = 1 - b_1 \geq 0$, $\bar{d}^{(4)}_{0,3} = 1$ and $(b_1 - b_2 + (1 - b_1)^2)b_0 + (1 - b_1)b_1 + b_2 = 1$;

(4) Assume that $V_n(k\tau_n) = \Gamma(2-\alpha)\tau_n^\alpha \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}^{(k+1)}_{i,j+1}\bar{R}^i f_n(j\tau_n)$ holds for all $k \leq K-1$, $\bar{d}^{(k+1)}_{i,j+1} \geq 0, i = 0, \ldots, k-j-1, j = 0, \ldots, k-1$, and $\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}^{(k+1)}_{i,j+1} b_j = 1$. Then for $k = K$,

$$\begin{aligned}
&V_n(K\tau_n)\\
&= (1-b_1)\bar{R}V_n((K-1)\tau_n) + \sum_{l=2}^{K-1}(b_{j-1} - b_j)V_n((K-l)\tau_n) + \Gamma(2-\alpha)\tau_n^\alpha f_n((K-1)\tau_n)\\
&= \Gamma(2-\alpha)\tau_n^\alpha((1-b_1)\bar{R}\sum_{j=0}^{K-2}\sum_{i=0}^{K-j-2} \bar{d}^{(K)}_{i,j+1}\bar{R}^i f_n(j\tau_n)\\
&+ \sum_{l=2}^{K-2}(b_{l-1} - b_l)\sum_{j=0}^{K-l-1}\sum_{i=0}^{K-l-j-1} \bar{d}^{(K-l+1)}_{i,j+1}\bar{R}^i f_n(j\tau_n) + f_n((K-1)\tau_n))\\
&= \Gamma(2-\alpha)\tau_n^\alpha\Bigg(\sum_{j=0}^{K-2}\sum_{i=1}^{K-j-1}(1-b_1)\bar{d}^{(K)}_{i-1,j+1}\bar{R}^i f_n(j\tau_n)\\
&+ \sum_{j=0}^{K-3}\sum_{i=0}^{K-j-3}\sum_{l=2}^{K-i-j-1}(b_{l-1} - b_l)\bar{d}^{(K-l+1)}_{i,j+1}\bar{R}^i f_n(j\tau_n) + f_n((K-1)\tau_n)\Bigg)\\
&= \Gamma(2-\alpha)\tau_n^\alpha\Bigg(\sum_{j=0}^{K-3}\Bigg(\sum_{l=2}^{K-j-1}(b_{l-1} - b_l)\bar{d}^{(K-l+1)}_{0,j+1} + \sum_{i=K-j-2}^{K-j-1}(1-b_1)\bar{d}^{(K)}_{i-1,j+1}\bar{R}^i\\
&+ \sum_{i=1}^{K-j-3}((1-b_1)\bar{d}^{(K)}_{i-1,j+1} + \sum_{l=2}^{K-i-j-1}(b_{l-1} - b_l)\bar{d}^{(K-l+1)}_{i,j+1})\bar{R}^i\Bigg)f_n(j\tau_n) + f_n((K-1)\tau_n)\Bigg).
\end{aligned}$$

It means that $\bar{d}^{(K+1)}_{0,j+1} = \sum_{l=2}^{K-j-1}(b_{l-1} - b_l)\bar{d}^{(K)}_{0,j+1} \geq 0, \bar{d}^{(K+1)}_{i,j+1} = (1-b_1)\bar{d}^{(K)}_{i-1,j+1} + \sum_{l=2}^{K-i-j-1}(b_{l-1} - b_l)\bar{d}^{(K-l+1)}_{i,j+1} \geq 0, i = 1 , \ldots, K-j-3, \bar{d}^{(K+1)}_{i,j+1} = (1-b_1)\bar{d}^{(K)}_{i-1,j+1} \geq 0, i = K-j-2, K-j-1, j = 0, \ldots, K-1$; $\bar{d}^{(K+1)}_{0,K-1} = 0, \bar{d}^{(K+1)}_{1,K-1} = (1-b_1)\bar{d}^{(K)}_{0,K-1} \geq 0$; $\bar{d}^{(K+1)}_{0,K} = 1$; and

$$\begin{aligned}
&\sum_{j=0}^{K-1}\sum_{i=0}^{K-j-1} \bar{d}^{(K+1)}_{i,j+1} b_j\\
&= (1-b_1)\sum_{j=0}^{K-2}\sum_{i=0}^{K-j-2} \bar{d}^{(K)}_{i,j+1} b_j + \sum_{l=2}^{K-2}(b_{l-1} - b_l)\sum_{j=0}^{K-l-1}\sum_{i=0}^{K-l-j-1} \bar{d}^{(K-l+1)}_{i,j+1} b_j + b_{K-1} = 1.
\end{aligned}$$

Thus, by induction, we can obtain our conclusion. □

Remark 3.1. Here, the solution $V_n(k\tau_n)$ is a little bit different from it in [18,19]. It is because in general that $f_n(0) \neq 0$ in the present paper, but $f_n(0) = 0$ in [18, 19].

Theorem 3.5. *Let $\alpha > \frac{1}{2}$ and let e^{tA_n} be the C_0-semigroups generated by the operators A_n. Assume that e^{tA_n} satisfy condition $(B)$and$\omega = 0$. If we have $||\tau_n^{2\alpha-1}A_n^2|| \leq c$, where c is independent of n, then for the scheme (3.2), we have*

$$||U_n(k\tau_n)|| \leq \bar{M}||x_n|| + \bar{M}\Gamma(1-\alpha)(k\tau_n)^\alpha \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)||.$$

Proof. From Theorem 3.5 in [18, 19], we have $||W_n(k\tau_n)|| \leq \bar{M}||x_n||$. We just need to show $||V_n(k\tau_n)|| \leq \bar{M}\Gamma(1-\alpha)(k\tau_n)^\alpha \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)||$.

Since $\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} b_j = 1$, then we have $\sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} b_j \leq 1$, $\sum_{i=0}^{k-1} \bar{d}_{i,1}^{(k+1)} \leq 1$. We also know that $b_j b_{k-1}^{-1} \geq 1$, when $j \leq k-1, b_{k-1}^{-1} \leq \frac{k^\alpha}{1-\alpha}$ and $b_{k-1}^{-1} \approx \frac{k^\alpha}{1-\alpha}$ as$k \to \infty$ (see [27]). Then together with $||\bar{R}^i|| \leq \bar{M}$ from [18], we obtain

$$\begin{aligned}
||V_n(k\tau_n)|| &\leq \bar{M}\Gamma(2-\alpha)\tau_n^\alpha \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)|| \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} \\
&\leq \bar{M}\Gamma(2-\alpha)\tau_n^\alpha b_{k-1}^{-1} \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)|| \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} \bar{d}_{i,j+1}^{(k+1)} b_j \\
&= \bar{M}\Gamma(2-\alpha)\tau_n^\alpha b_{k-1}^{-1} \sup_{1\leq j\leq k} ||f_n(j\tau_n)|| \\
&\leq \bar{M}\Gamma(2-\alpha)\tau_n^\alpha \frac{k^\alpha}{1-\alpha} \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)|| \\
&= \bar{M}\Gamma(1-\alpha)(k\tau_n)^\alpha \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)||.
\end{aligned}$$

□

From the theorem above and Theorem 3.10 in [18], we have the following conclusion.

Theorem 3.6. *Assume that analytic C_0-semigroups e^{tA_n} satisfy condition $(B_1)$with$\omega = 0$ and $||\frac{\Gamma(2-\alpha)}{(1-b_1)}\tau_n^\alpha A_n|| \leq c$. Then there exists a constant $\bar{M}$, such that*

$$||U_n(k\tau_n)|| \leq \bar{M}||x_n|| + \bar{M}\Gamma(1-\alpha)(k\tau_n)^\alpha \sup_{0\leq j\leq k-1} ||f_n(j\tau_n)||.$$

Now we get the order of convergence of the explicit difference scheme (3.2).

Theorem 3.7. *The representation of the difference* $z_{u_n}(k\tau_n) = u_n(k\tau_n) - U_n(k\tau_n)$ *is:*

$$z_{u_n}(k\tau_n) = \Gamma(2-\alpha)\tau_n^\alpha \sum_{j=1}^{k}\sum_{i=0}^{k-j} \bar{d}_{i,j}^{(k+1)} \bar{R}^i r_{u_n}(j\tau_n), \tag{3.4}$$

where $r_{u_n}(k\tau_n) = \Delta_{t_k}^\alpha u_n(\cdot) - (\mathbf{D}_t^\alpha u_n)(t_{k-1})$ *and* $\bar{d}_{i,j}^{(k)}$ *are the same as them in [18, 19]. Then under the assumption of Theorem 3.5 and* $x_n \in D(A_n^2), f_n(\cdot) \in C^2([0,T];E_n)$, $A_n f_n(\cdot) \in C^1([0,T];E_n)$, *there are the following estimate for* $z_{u_n}(k\tau_n)$,

$$||z_{u_n}(k\tau_n)|| \le C(\alpha)\tau_n^\alpha \max\{||A_n x_n||, ||A_n^2 x_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}, ||A_n f_n(\cdot)||_{C^1([0,T];E_n)}\}.$$

Proof. Since $z_{u_n}(k\tau_n) = u_n(k\tau_n) - U_n(k\tau_n)$, one has that $U_n(k\tau_n) = u_n(k\tau_n) - z_{u_n}(k\tau_n)$. Then $\Delta_{k\tau_n}^\alpha(u_n(\cdot) - z_{u_n}(\cdot)) = A_n(u_n((k-1)\tau_n) - z_{u_n}((k-1)\tau_n)) + f_n((k-1)\tau_n)$, and

$$\begin{aligned}\Delta_{t_k}^\alpha z_{u_n}(\cdot) &= A_n z_{u_n}(t_{k-1}) + \Delta_{t_k}^\alpha u_n(\cdot) - (\mathbf{D}_t^\alpha u_n)(t_{k-1}) \\ &= A_n z_{u_n}((k-1)\tau_n) + r_{u_n}(k\tau_n).\end{aligned} \tag{3.5}$$

Here

$$\begin{aligned} r_{u_n}(\tau_n) &= \Delta_{\tau_n}^\alpha u_n(\cdot) - A_n u_n(0) - f_n(0) \\ &= \frac{\tau_n^{1-\alpha}}{\Gamma(2-\alpha)} \frac{u_n(\tau_n) - u_n(0)}{\tau_n} - A_n x_n - f_n(0) \\ &= \frac{u_n(\tau_n) - x_n}{\Gamma(2-\alpha)\tau_n^\alpha} - A_n x_n - f_n(0). \end{aligned}$$

While we know that $u_n(\tau_n) = S_\alpha(\tau_n, A_n)x_n + \int_0^{\tau_n} P_\alpha(\tau_n - s, A_n) f_n(s) ds$. Together with equation (2.6), one has that $u_n(\tau_n) = x_n + \int_0^{\tau_n} AP_\alpha(s, A_n)x_n ds + \int_0^{\tau_n} P_\alpha(\tau_n - s, A_n) f_n(s) ds$. Hence from Lemma 1.1, $||r_{u_n}(\tau_n)|| \le \mathrm{C}\max\{||A_n x_n||, ||f_n(\cdot)||_{C([0,T];E_n)}\}$.

For $k \ge 2$, one has

$$\begin{aligned} r_{u_n}(k\tau_n) &= \Delta_{t_k}^\alpha u_n(\cdot) - (\mathbf{D}_t^\alpha u_n)(t_{k-1}) \\ &= \Delta_{t_k}^\alpha u_n(\cdot) - (\mathbf{D}_t^\alpha u_n)(t_k) + (\mathbf{D}_t^\alpha u_n)(t_k) - (\mathbf{D}_t^\alpha u_n)(t_{k-1}) \\ &= \bar{r}_{u_n}(k\tau_n) + A_n u_n(k\tau_n) + f_n(k\tau_n) - A_n u_n((k-1)\tau_n) - f_n((k-1)\tau_n) \\ &= \bar{r}_{u_n}(k\tau_n) + (A_n u_n'(\xi\tau_n) + f_n'(\zeta\tau_n))\tau_n, \quad \xi, \zeta \in ((k-1), k) \end{aligned}$$

if $u_n(\cdot) \in C^1([0,T];D(A_n))$ and $f_n(\cdot) \in C^1([0,T];E_n)$

We know from (3.3) that $||u_n'(t)|| \le C(1+t^{\alpha-1})\max\{||A_nx_n||, ||f_n(\cdot)||_{C^1([0,T];E_n)}\}$. Therefore $||A_nu_n'(t)|| \le C(1+t^{\alpha-1})\max\{||A_n^2x_n||, ||A_nf_n(\cdot)||_{C^1([0,T];E_n)}\}$. Together with $||\bar{r}_{u_n}(k\tau_n)|| \le C\tau_n^\alpha\max\{||A_nx_n||, ||f_n||_{C^2([0,T];E_n)}\}$, we can get

$$r_{u_n}(k\tau_n) \le C\tau_n^\alpha\max\{||A_nx_n||,||A_n^2x_n||,||f_n(\cdot)||_{C^2([0,T];E_n)},||A_nf_n(\cdot)||_{C^1([0,T];E_n)}\}, k\ge 2.$$

Similar to the proof of Theorem 3.7 in [19], we can get our conclusion. □

Theorem 3.8. *Under the assumptions of Theorem 3.6 and $x_n \in D(A_n^2), f_n(\cdot) \in C^2([0,T];E_n), A_nf_n(\cdot) \in C^1([0,T];E_n)$, one has the following estimate for $z_{u_n}(k\tau_n)$:*

$$||z_{u_n}(k\tau_n)|| \le C(\alpha)\tau_n^\alpha\max\{||A_nx_n||,||A_n^2x_n||,||f_n(\cdot)||_{C^2([0,T];E_n)},||A_nf_n(\cdot)||_{C^1([0,T];E_n)}\}.$$

The proof is the same as in Theorem 3.7 just with correspondent changes of the assumption for stability.

Remark 3.2. In the implicit difference scheme, the error which is $\bar{z}_{u_n}(k\tau_n) = u_n(k\tau_n) - \bar{U}_n(k\tau_n)$, satisfies the estimate

$$||\bar{z}_{u_n}(k\tau_n)|| \le C(\alpha)\tau_n^\alpha\max\{||A_nx_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}\}.$$

While in the explicit difference scheme, $z_{u_n}(k\tau_n) = u_n(k\tau_n) - U_n(k\tau_n)$ satisfies the estimate

$$||z_{u_n}(k\tau_n)|| \le C(\alpha)\tau_n^\alpha\max\{||A_nx_n||, ||A_n^2x_n||, ||f_n(\cdot)||_{C^2([0,T];E_n)}, ||A_nf_n(\cdot)||_{C^1([0,T];E_n)}\}.$$

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Appendix

In this part, we recall the following version of Trotter-Kato's Theorem [28, 29] on general approximation scheme.

Theorem 4.1. [4] *(Theorem ABC) Assume that $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ and they generate C_0-semigroups. The following conditions (A) and (B) are equivalent to condition (C).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B) Stability. There are some constants $M \geq 1$ and ω, which are not depending on n and such that $\|\exp(tA_n)\| \leq M \exp(\omega t)$ for $t \geq 0$ and any $n \in \mathbb{N}$;

(C) Convergence. For any finite $T>0$ one has

$$\max_{t\in[0,T]} \|\exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$$

as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Remark 4.1. In case of approximation of analytic semigroups they have some changes in formulation of Theorem 4.1:

(B_1) Stability. There exist constants $M \geq 1$ and ω independent of n such that for any $Re\lambda > \omega$, $\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda - \omega|}$ for all $n \in \mathbb{N}$;

(C_1) Convergence. For any finite $\mu>0$ and some $0<\theta<\frac{\pi}{2}$ we have $\max_{\eta\in\Sigma(\theta,\mu)} \|\exp(\eta A_n)u_n^0 - p_n \exp(\eta A)u^0\| \to 0$ as $n \to \infty$ whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$. Here we denote $\Sigma(\theta,\mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ and $\Sigma(\theta) = \{z \in \mathbb{C} \ : \ |arg\ z| \leq \theta\}$.

For the semidiscrete approximation of α-times resolvent family, we have the following ABC Theorems:

Theorem 4.2. [16] *Suppose that* $0<\alpha\leq 2$ *and* A, A_n *generate exponentially bounded* α*-times resolvent families* $S_\alpha(\cdot,A), S_\alpha(\cdot,A_n)$ *in the Banach spaces* E, E_n*, respectively. The following conditions (A) and* $(\tilde{B})$ *are equivalent to condition* $(\tilde{C})$.

(A) Consistency. There exists $\lambda\in\rho(A)\cap\cap_n\rho(A_n)$ *such that the resolvents converge* $(\lambda I_n-A_n)^{-1}\xrightarrow{\mathcal{PP}}(\lambda I-A)^{-1}$;

$(\tilde{B})$ *Stability. There are some constants* $M\geq 1$ *and* ω*, which are not depending on* n *and such that* $||S_\alpha(t,A_n)||_{B(E_n)}\leq Me^{\omega t}$ for $t\geq 0, n\in\mathbb{N}$;

$(\tilde{C})$ Convergence. For some finite $\omega_1>0$ one has $\max_{t\in[0,\infty)}e^{-\omega_1 t}||S_\alpha(t,A_n)x_n-p_nS_\alpha(t,A)x||_{E_n}\to 0$ as $n\to\infty$, whenever $x_n\xrightarrow{\mathcal{P}}x$ for any $x_n\in E_n, x\in E$.

Theorem 4.3. [17] *Suppose that* $0<\alpha\leq 2$ *and* A, A_n *generate exponentially bounded analytic* α*-times resolvent families* $S_\alpha(\cdot,A), S_\alpha(\cdot,A_n)$ *in the Banach spaces* E, E_n*, respectively. The following conditions (A) and* (B') *are equivalent to condition* (C').

(A) Consistency. There exists $\lambda\in\rho(A)\cap\cap_n\rho(A_n)$ *such that the resolvents converge* $(\lambda I_n-A_n)^{-1}\xrightarrow{\mathcal{PP}}(\lambda I-A)^{-1}$;

(B') *Stability. There are some constants* $M\geq 1$, $0<\theta\leq\pi/2$ *and* ω *which are independent of* n*, such that the sector* $(\omega+\Sigma_{\theta+\pi/2})^\alpha$ *is included in* $\rho(A_n)$ *and*

$$\sup_{\lambda\in\omega+\Sigma_{\beta+\pi/2}}||\lambda^{\alpha-1}R(\lambda^\alpha;A_n)||_{B(E_n)}\leq M/|\lambda-\omega| \text{ for any } n\in\mathbb{N} \text{ and for any } 0<\beta<\theta.$$

(C') *Convergence. For some finite* $\omega_1>0$ *one has*

$$\sup_{z\in\Sigma_\beta}e^{-\omega_1 Rez}||S_\alpha(z,A_n)x_n-p_nS_\alpha(z,A)x||_{E_n}\to 0 \text{ as } n\to\infty,$$

whenever $x_n\xrightarrow{\mathcal{P}}x$ *for any* $x_n\in E_n, x\in E$ *and for any* $0<\beta<\theta$.