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Linear Transverse Oscillations of the Space Elevator Cable

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Abstract—Small transverse oscillations near the equilibrium vertical position are studied for an extra-long flexible inextensible cable of a space elevator. The linear approximation shows that the oscillations in the two planes, equatorial and meridional, are separated and have identical sets of eigenfunctions, while their eigenfrequencies are connected by a simple relation. A Sturm–Liouville problem for determining the frequencies and modes of the cable's natural oscillations is stated. A transformation of the problem aimed at simplifying computations is described.

Keywords: space elevator, flexible inextensible cable, uniformly stressed cable, small oscillations, Sturm–Liouville problem, Prüfer substitution

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1. SPACE ELEVATOR CABLE

An extra-long and extra-strong cable is not only the supporting part of the entire structure of the space elevator (SE), but is also its most massive and extended part. For this reason, the cable determines the main features of the dynamics of the entire system. The cable is a nearly one-dimensional, absolutely flexible, and highly-stressed thread. Dynamic problems related to it do not look complicated. Difficulties arise when the motions strongly deviate from the vertical due to the great flexibility of the system. A systematic study of such motions is still problematic; As is customary in such cases, it is reasonable to start the study with a linear setting regarding small oscillations of the system near a stationary position. In essence, this amounts to studying small oscillations of a long flexible pendulum. On the other hand, the prospects for the implementation of the SE design are so appealing that it is necessary to reproduce, verify, and develop the few available results.

The first data on linear SE oscillations were obtained back in 1975 by Pearson [1] using simple dynamic models. On the basis of an analogy with a centrifugal mathematical pendulum, the period of pendulum oscillations (with a straight cable) was estimated, which turned out to be of the order of several days; as a result, it was concluded that there was no resonances between such oscillations and main lunisolar disturbances.

More detailed calculations were carried out on the basis of Beletskii's dynamic model [2], in which the elevator is modeled by a point mass connected by a weightless cable to the Earth's surface. Linearized equations of such oscillations were obtained [3] and a simple formula relating the frequencies of pendulum oscillations in two planes, meridional (T^{mer}) and equatorial (T^{eq}), was derived.

As immediately follows trom that formula [3], in the single-mass model T^{mer} is always shorter than a day; T^{eq}, as shown by the calculations, is greater than a day. Periods of bending oscillations cannot be estimated within a simple single-mass SE model, but the same formula (see (4.9) below) remains valid in the cable model with a continuously distributed mass studied here; moreover, it also holds for other oscillation modes.

An intermediate place is occupied by the two-mass model of the system studied in [3], where one mass is placed on the geostationary orbit, and the other is at the end of the tether. Two oscillation modes, in-phase and antiphase, were analyzed. In the in-phase mode, both masses deviate simultaneously in one direction from the vertical; in the antiphase mode, the masses deviate in opposite directions. It is shown

† Deceased.

that in-phase oscillations, which are analogous to pendulum oscillations, have periods greater than a day in the equatorial plane. The oscillation periods of all other types are shorter than a day.

Oscillation equations of the SE cable with a distributed mass are derived in [4]; the derivation is briefly reproduced here. The study [5] is close to this paper, overlapping with it in some results and also using the methodology described in [6] for analyzing small vibrations of a cable with a load at the end.

Using a discrete, multi-mass model of the tether, numerical data were obtained [7] on the periods and some other characteristics of low-order bending modes (linear elastic vibrations were also considered on the basis of assumed little-known data on the elastic properties of the cable material). A generalization of this model, which allows for an off-equatorial starting position of the SE cable, is considered in [8]. The same paper also contains a fairly complete bibliography of studies on the SE dynamics in peer-reviewed journals.

The present paper is concerned with an in-depth study of an important, but specific problem related to the subject of a SE. A broader perspective is outlined in our other study [9]. Of interest are publications by the American organization ISEC (International Space Elevator Consortium), which discuss a variety of scientific and engineering problems related to SE.

The research regarding space cable systems (SCSs) has been underway at the Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, since the late 1960s. The study of issues related to the SE was initiated by V.V. Beletskii as a branch-off of this direction. Several pages of the monograph [10] are dedicated to the SE. Despite the similarity between the basic equations of the dynamics of a flexible cable in the SCS and SE models, the quantitative and qualitative characteristics of these systems differ significantly.

2. CABLE MOTION EQUATIONS

We consider the motion of an absolutely flexible, weighty, inextensible thread, fixed at one end at the Earth's equator. A heavy rigid body (material point) is fixed at the other end. The entire system moves in the field of gravity and inertial centrifugal force. To describe the motion of a thread in a continuous model, we use a coordinate system associated with the rotating Earth; the system's center O coincides with the center of the Earth, axis x is oriented from point O to the anchor point of the thread at the equator, axis z is parallel to the axis of the Earth's rotation and oriented northward, axis y is parallel to the equator line at the anchor point and oriented to the east.

The position of the thread's point at a distance s along the thread from the anchor point is specified by vector $\mathbf{r} = (x(s,t), y(s,t), y(s,t))$, where t is time. Let $\rho(s)$ be the local linear density of the thread at point s, and $P(s,t)$ be the tension force acting along the thread on the lower part (with smaller s values) of the cable from the upper part and directed at point s in the positive s-direction. The equation of motion for the current point of the cable, involving the force of gravity, centrifugal and Coriolis forces in the coordinate system rotating with the Earth, and the tension force of the cable, is then written as follows:

$$\ddot{\mathbf{r}} = -2[\boldsymbol{\omega}, \dot{\mathbf{r}}] - [\boldsymbol{\omega}, [\boldsymbol{\omega}, \mathbf{r}]] - \frac{\mu}{r^3}\mathbf{r} + \frac{1}{\rho}(P\mathbf{r}')', \tag{2.1}$$

where μ is the gravitational parameter, and $\boldsymbol{\omega}$ is the vector of the sidereal angular velocity of the Earth. The dot denotes the derivative with respect to t; the prime denotes the derivative with respect to s.

This equation needs to be supplemented with the condition for determining the tension $P(s,t)$ (in this case, the condition of the thread's inextensibility):

$$|\mathbf{r}'| = 1. \tag{2.2}$$

The system of partial differential equations (2.1), (2.2) should be supplemented with initial and boundary conditions. The initial data are the $\mathbf{r}(s,0)$ and $\dot{\mathbf{r}}(s,0)$ values.

Let L denote the cable length, and $\mathbf{R}$ be the radius vector of its anchor point; $R = |\mathbf{R}|$ is the equatorial radius of the Earth.

The boundary conditions at the lower end of the cable express the stationarity of its attachment at the lower point. The conditions at the upper end, where a massive point (counterweight) is attached to the cable, express the consistency of the motion of this point with the motion of the cable's upper end. The equation of motion for the point at the upper end is

$$\ddot{\mathbf{r}}_L = -2[\vec{\omega}, \dot{\mathbf{r}}_L] - [\vec{\omega}, [\vec{\omega}, \mathbf{r}_L]] - \frac{\mu}{r_L^3}\mathbf{r}_L + \frac{1}{M}P_L\mathbf{r}'_L, \tag{2.3}$$

where $\mathbf{r}_L$ is the position of the top point, P_L is the tensile force at the upper end of the cable, and M is the mass of the counterweight.

As a result, the boundary conditions for Eq. (2.1) take the form

$$\mathbf{r}(0,t) = \mathbf{R}, \quad \dot{\mathbf{r}}(0,t) = 0, \quad \mathbf{r}(L,t) = \mathbf{r}_L(t), \quad \dot{\mathbf{r}}(L,t) = \dot{\mathbf{r}}_L(t). \tag{2.4}$$

Let us differentiate the boundary conditions at point L in time, replace the second derivatives with their expressions from the equations of motion (2.1) and (2.3), and project the resulting relations onto the tangent vector to the cable $\mathbf{r}'(L,t)$ at its end. We obtain the boundary condition for the cable tension at the end point:

$$P'(L,t) + \frac{\rho(L)P_L(t)}{M} = 0. \tag{2.5}$$

To complete the statement of the problem, one needs to specify the linear density distribution function $\rho(s)$ along the cable. Its inhomogeneity is essential for the dynamics of the system, but the explicit form of $\rho(s)$ is not required for the analysis below. For several SE models, it is given in [9, Eqs. (2), (8), (11)].

Vertical equilibrium configuration. Eqs. (2.1)–(2.4) have a particular stationary solution corresponding to the equilibrium of a cable stretched along the radius vector:

$$\begin{gathered} x_0(s) = r, \quad y_0(s,t) = 0, \quad z_0(s) = 0 \\ x_0(L) = \ell, \quad y_0(L) = 0, \quad z_0(L) = 0. \end{gathered} \tag{2.6}$$

In accordance with the previously introduced notations, $r = r(s) = s + R$. In addition, we introduce the notation $\ell = L + R$ (the distance from the center of the Earth to the end of the cable).

The subscript "0" denotes the values related to the equilibrium state.

The cable tension $P_0(s)$ in equilibrium, according to Eqs. (2.1) and (2.5), is determined by the differential relation

$$P_0'(s) = -\rho(s)\left(\omega^2 r - \frac{\mu}{r^2}\right) \tag{2.7}$$

with the boundary condition

$$P_{0L} = M\left(\omega^2 \ell - \frac{\mu}{\ell^2}\right). \tag{2.8}$$

Here, $\omega = |\boldsymbol{\omega}|$ is the frequency of the Earth's daily rotation. Note that expression (2.8) does not include the density $\rho(L)$ of the cable at the end.

Let L_{gs} be the height of the geostationary orbit. Then $P_0'(L_{gs}) = 0$, and the condition $P_{0L} > 0$ is equivalent to $L > L_{gs}$. This condition is well known [2]: to ensure that the cable is taut, the counterweight must be beyond the geostationary.

3. LINEARIZATION OF EQUATIONS

Considering now the equilibrium state as unperturbed, let us write the equations of motions close to equilibrium in the linear approximation. To do this, we introduce small variations and write the variables in the original equations as $\mathbf{r}(s,t) = \mathbf{r}_0(s) + \delta\mathbf{r}(s,t)$ and (considering that $\dot{\mathbf{r}}_0 = 0$) $\dot{\mathbf{r}}(s,t) = \delta\dot{\mathbf{r}}(s,t)$.

Taking into account the condition of the cable's inextensibility (2.2), and the boundary conditions (2.4), we obtain $\delta x(s,t) = 0$, i.e., in the linear approximation, the variations in both the radial displacement and the radial velocity relative to the vertical equilibrium position vanish. We introduce the vector of variations in the direction transverse to the cable

$$Y(s,t) = (\delta y(s,t), \delta z(s,t))$$

and set

$$Y_L(t) = (\delta y_L(t), \delta z_L(t)), \quad Y'_L(t) = Y'(L,t)$$

Let us also introduce the notations (recall that $r = s + R$)

$$F_z(s) = -\frac{\mu}{r^3}, \quad F_y(s) = F_z(s) + \omega^2 \tag{3.1}$$

and $F(s) = \mathrm{diag}(F_y, F_z)$ (a diagonal 2×2 matrix).

Linearization of Eqs. (2.1) and (2.3) yields the system

$$\ddot{Y} = F(s)Y + \frac{1}{\rho(s)}(P_0 Y')', \quad \ddot{Y}_L = F_L Y_L - \frac{P_{0L}}{M} Y'_L \tag{3.2}$$

with boundary conditions corresponding to (2.4):

$$Y(0,t) = \dot{Y}(0,t) = 0, \quad Y(L,t) = Y_L(t). \tag{3.3}$$

The coefficients $P_0(s)$ and P_{0L} are defined by Eqs. (2.7) and (2.8).

Thus, the displacements $\delta y(s,t)$ and $\delta z(s,t)$ are separated in the linear setting. Scalar linear boundary-value problems for δy and δz differ only in the constant shift of the coefficients due to (3.1). To be specific, we will analyse in detail the problem for the component δy.

4. STURM–LIOUVILLE PROBLEM

4.1. Separation of Variables

We seek particular solutions to the boundary-value problem (3.2), (3.3) for the equatorial component δy as

$$\delta y(s,t) = S(s)T(t). \tag{4.1}$$

By means of the usual procedure for separating variables, we find

$$\frac{\ddot{T}}{T} = F_y(s) + \frac{(P_0(s)S')'}{\rho(s)S} = -\lambda, \quad -F_y(L) + \frac{P_{0L}S'(L)}{MS(L)} = \lambda, \tag{4.2}$$

where λ is some constant. It turns out (see below) that $\lambda > 0$, so the parameter λ is the square of the oscillation mode frequency corresponding to the particular solution (4.1). The shape of such a mode with respect to s is described by the second-order linear differential equation

$$(P_0(s)S')' + F_y(s)\rho(s)S = -\lambda\rho(s)S \tag{4.3}$$

with uniform boundary conditions

$$S(0) = 0, \quad P_{0L}S'(L) = M(F_y(L) + \lambda)S(L). \tag{4.4}$$

As a result, to define the function $S(s)$, we obtain the one-dimensional Sturm–Liouville problem (4.3), (4.4) with the spectral parameter λ. Its formulation differs from the classical one in that the spectral parameter here is also present in the second boundary condition (4.4).

The classical Sturm–Liouville theory provides a justification for the method of separation of variables: it is proved that, under certain conditions, eigenfunctions satisfying the given homogeneous relations at the boundary exist for a countable discrete set of parameter values λ and form an orthogonal basis in the corresponding Hilbert space. This is also true in the case under consideration.

4.2. Spectral Properties of the Problem

We will interpret the Sturm–Liouville problem (4.3), (4.4) as determination of the spectrum of the differential operator D:

$$Du = -\frac{1}{\rho(s)}(P_0(s)u')' - F_y(s)u, \tag{4.5}$$

which acts on functions from $C^2[0, L]$ satisfying the boundary conditions, cf. (3.3), (3.2):

$$u(0) = 0, \quad \frac{1}{\rho(L)}(P_0(s)u')'(L) = -\frac{1}{M}P_{0L}u'(L). \tag{4.6}$$

Denote the set (real linear space) of such functions by C_0^2. (In order to obtain an operator on a domain that does not depend on λ, one has to use a boundary condition with the second derivative.) Let us transform C_0^2 into a pre-Hilbert space by introducing the inner product on it similarly to [11, Chapter II, Appendix III]:

$$(u,v) = \int_0^L u(s)v(s)ds + Mu(L)v(L). \tag{4.7}$$

Integrating by parts and taking into account (4.6), we obtain

$$(Du,v) = \int_0^L (P_0u'v' - F_y\rho uv)ds - MF_y(L)u(L)v(L). \tag{4.8}$$

Hence, the operator D is symmetric in C_0^2, so the eigenfunctions of this problem with different eigenvalues λ are orthogonal with respect to the introduced scalar product.

The countability of the set of eigenvalues follows from the algorithm for their calculation described below.

In a mathematically rigorous analysis of the problem, it is necessary to prove the completeness of the system of eigenfunctions of the operator D in the Hilbert space which is the completion of the space C_0^2 in the norm corresponding to the inner product (4.7). For Sturm–Liouville problems of this type, this is done, for example, in [12].

Let us verify that all eigenvalues of the operator D are positive. For this, we prove that $(Du,u) > 0$ for a nonzero function u.

Assuming $v = u$ in (4.8) and using relations (2.8) and (3.1), we obtain

$$(Du,u) = \int_0^L (P_0u'^2 - F_y\rho u^2 - (r^{-1}P_0u^2)')ds.$$

We transform the integrand using (2.7), (3.1), and the identity $u'^2 - (u^2/r)' = (u' - u/r)^2$:

$$P_0u'^2 - F_y\rho u^2 - \left(\frac{P_0u^2}{r}\right)' = P_0u'^2 + \frac{P'_0u^2}{r} - \left(\frac{P_0u^2}{r}\right)' = P_0\left(u' - \frac{u}{r}\right)^2.$$

Since $P_0(s) > 0$ at $0 < s \le L$ (the tension force is positive), we have $(Du,u) \ge 0$. The equality $(Du,u) = 0$ could only take place if $ru' - u \equiv 0$, i.e., for $u/r = \mathrm{const}$, but then the boundary conditions $u(0) = 0$ and $r(0) = R$ imply $u \equiv 0$.

The above refers to the Sturm–Liouville problem for the component δy of the small deviation of the cable in the equatorial plane. For the component δz, by virtue of (3.1), the following is true. The Sturm–Liouville problems for both components have a common set of eigenfunctions, and if $\omega_n^{\mathrm{eq}} = \sqrt{\lambda_n}$ is the eigenfrequency of the n-th equatorial mode, the eigenfrequency of the corresponding meridional mode (whose spatial shape is the same) is

$$\omega_n^{\mathrm{mer}} = \sqrt{(\omega_n^{\mathrm{eq}})^2 + \omega^2}. \tag{4.9}$$

In particular, for any distribution of the cable mass, $\omega_n^{\mathrm{mer}} > \omega$ and, therefore, the periods of all natural oscillations in the meridional plane are shorter than a day.

5. CALCULATION OF NATURAL FREQUENCIES AND MODES

Let us note the features of the problem that affect both the methods of its solution and the formulation of computational questions:

(1) the large geometric size of the system and great inhomogeneity of the cable, which cause a strong variation in the coefficients $P_e(s)$, $\rho(s)$, and $F_Y(s)$;

(2) the practical absence of both external (vacuum of space) and internal (almost one-dimensional configuration with zero bending stiffness) dissipation, which, together with a great length, leads to an increase in the relative role of high-mode oscillations in the dynamics and the need to adequately take them into account.

5.1. Prüfer's Substitution

Let us describe a method for solving the boundary-value problem using the transition to polar coordinates in the phase plane. A similar method for the standard Sturm–Liouville problem, called the trigonometric run, is considered in [13].

For the convenience of calculations that follow, we make the substitution $S(s) = rV(s)$ proposed by A.V. Chernov and introduce the notations $Q = P_0 r^2$, $J = \rho r^2$. The physical meaning of the variable V is the angle of deviation of the cable's point from the vertical (measured from the center of the Earth). The function $J(s)$ is the moment of inertia of the cable's element, and $(QV')'$ is the moment of force (relative to the center of the Earth) created by the tension of the cable. The Sturm–Liouville problem (4.3), (4.4) takes the form

$$(Q(s)V')' = -\lambda J(s)V, \quad V(0) = 0, \quad P_{0L}V'(L) = \lambda MV(L). \tag{5.1}$$

Let us introduce polar coordinates q and φ (Prüfer's substitution):

$$q\cos\varphi = f(s)V', \quad q\sin\varphi = g(s)V. \tag{5.2}$$

Functions f and g are not specified yet. Differentiating with respect to s and in view of (5.1), we have

$$q'\cos\varphi - q\varphi'\sin\varphi = \left(\frac{f(s)}{Q(s)}\right)'\frac{Q(s)}{f(s)}q\cos\varphi - \frac{f(s)}{Q(s)}\frac{J(s)}{g(s)}\lambda q\sin\varphi$$

$$q'\sin\varphi + q\varphi'\cos\varphi = \frac{g'(s)}{g(s)}q\sin\varphi + \frac{g(s)}{f(s)}q\cos\varphi,$$

whence

$$\frac{q'}{q} = \left(\log\frac{f}{Q}\right)'\cos^2\varphi + (\log g)'\sin^2\varphi + \left(\frac{g}{f} - \lambda\frac{fJ}{gQ}\right)\cos\varphi\sin\varphi \tag{5.3}$$

and

$$\varphi' = \lambda\frac{fJ}{gQ}\sin^2\varphi + \frac{g}{f}\cos^2\varphi + \left(\log\frac{Qg}{f}\right)'\sin\varphi\cos\varphi. \tag{5.4}$$

By virtue of (5.1), the boundary conditions for Eq. (5.4) have the form

$$\varphi(0) = 0, \quad \tan\varphi(L) = \frac{P_{0L}g(L)}{\lambda Mf(L)}. \tag{5.5}$$

Let us call Eq. (5.4) the *defining equation*. The spectrum of the Sturm–Liouville problem consists of such values λ for which the boundary value problem (5.4), (5.5) has a solution. When λ and $\varphi(s)$ are already calculated, the function $q(s)$ is found by integrating Eq. (5.3); the corresponding eigenfunction $V(s)$ is then determined by means of the second equation (5.2).

A freedom in the choice of the functions f and g in Prüfer's substitution can be used to obtain the boundary-value problem (5.4), (5.5) in a form best suited to defined purposes. Let us consider two particular versions.

Version I. $f(s) = Q(s)$, $g = \text{const} = \lambda M\ell^2$. The defining equation looks as follows:

$$\varphi' = \lambda a(s)\cos^2\varphi + b(s)\sin^2\varphi, \tag{5.6}$$

where $a(s) = M\ell^2/Q(s)$, $b(s) = J(s)/(M\ell^2)$. This form is convenient in that the right-hand boundary condition $\tan\varphi(L) = 1$ does not depend on λ.

To calculate the frequency of the main mode, we divide both sides of Eq. (5.6) by $\cos^2\varphi$ and obtain a boundary-value problem for the Riccati equation for $\tau(s) = \tan\varphi(s)$:

$$\tau' = \lambda a(s) + b(s)\tau^2, \quad \tau(0) = 0, \quad \tau(L) = 1.$$

Hence, assuming $A = \int_0^L a(s)ds$, $B = \int_0^L b(s)ds$, we obtain the estimate

$$\frac{1-B}{A} < \lambda_0 < \frac{1}{A}. \tag{5.7}$$

From the computational point of view, a drawback of this variant of the Prüfer transformation, especially at large values of λ, is that the function $\varphi(s)$ defined by Eq. (5.6) has a characteristic staircase structure with alternating sections of large and small tilt to the abscissa axis. (The behavior of $\varphi(s)$ can be visualized by considering the solution of Eq. (5.6) with constant coefficients a and b.) In addition, the coefficients in our problem vary by several orders of magnitude over the integration interval.

Version II. $f = \sqrt{Q/\lambda J}$, $g = 1$. The defining equation is

$$\varphi' = \sqrt{\frac{\lambda J}{Q}} + \frac{1}{4}(\log(JQ))' \sin 2\varphi, \tag{5.8}$$

with the right-end boundary condition

$$\tan\varphi(L) = \eta/\sqrt{\lambda}, \tag{5.9}$$

where $\eta = M^{-1}\sqrt{P_0(L)\rho(L)}$.

The boundary-value problem can be further simplified by replacing the independent variable s with $z = \int_0^s \sqrt{J/Q}\,ds$ and designating $\varphi(z(s)) = \varphi(s)$ and $Z = z(L)$. We obtain

$$\frac{d\varphi}{dz} = \sqrt{\lambda} + \frac{1}{4}\frac{d}{dz}\ln(JQ)\sin 2\varphi, \quad \varphi(Z) = \pi n + \arctan(\eta/\sqrt{\lambda}). \tag{5.10}$$

If λ is large, the angular variable φ varies (in the zero approximation) uniformly in z, and on the interval of periodicity of the right-hand side with respect to φ, the variation in the coefficient at $\sin 2\varphi$ has magnitude of the order $\lambda^{-1/2}$. This consideration can be used to construct a semi-analytical algorithm for calculating high-order eigenmodes. It is also easy to deduce from (5.10) that the asymptotic behavior of the natural frequencies in the equatorial plane has the form

$$\omega_n^{\text{eq}} = \frac{\pi n}{Z} + O\left(\frac{1}{n}\right). \tag{5.11}$$

From expressions (4.9) and (5.11), it follows that the frequency difference for the meridional and equatorial modes of the same order has the asymptotics

$$\omega_n^{\text{mer}} - \omega_n^{\text{eq}} \sim \frac{\omega^2 Z}{2\pi n},$$

so the right-hand side of formula (5.11) describes the asymptotics of the eigenfrequencies in the meridional mode with the same accuracy.

5.2. The Case of an Uniformly Stressed Cable

The model of an uniformly stressed cable [1] is characterized by the relation

$$P_0(s) = \tau\rho(s), \tag{5.12}$$

where $\tau = \text{const}$. In addition to physical constants, the model is completely determined by two parameters: cable length L and cable tension τ, which has the dimension of speed squared.

Under condition (5.12), Eq. (2.7) takes the form

$$\tau(\log\rho)' = -\omega^2 r + \mu r^{-2}.$$

For the coefficients in Eq. (5.8), we obtain simple expressions: $J/Q = 1/\tau$ and

$$\frac{1}{4}(\log(JQ))' = \frac{1}{2}(\log\rho r^2)' = \frac{1}{r} + \frac{1}{2\tau}\left(\frac{\mu}{r^2} - \omega^2 r\right). \tag{5.13}$$

From relation (2.8), we find the expression for the parameter η in (5.9):

$$\eta = \tau^{-1/2}(P_{0L}/M) = \tau^{-1/2}(\omega^2\ell - \mu/\ell^2), \tag{5.14}$$

By way of illustration, instead of parameter τ, we can use a parameter that has the dimension of length, specifically, the *equivalent breaking length* L_p related to τ by the formula $\tau = g_0 L_p$, where $g_0 = 10$ m/s^2 is an upper estimate for the acceleration of gravity near the Earth's surface. There is a relation

Table 1.

Physical parameters of the Earth		
Gravitational parameter of the Earth	μ	3.986×10^{14} m^3/s^2
Earth rotation speed	ω	7.292×10^{-5} s^{-1}
Earth radius	R	6.378×10^5 m
Defining parameters of the model		
Cable length	L	8.0×10^7 m
Cable tension	τ	3×10^7 m^2/s^2
Auxiliary (dependent) parameters		
Equivalent breaking length	L_p	3×10^6 m
Cable-to-balance mass ratio	M_{cable}/M	1.288
Parameter in boundary conditions (5.9)	η	7.41×10^{-5}
Coefficient in the asymptotics of frequencies (5.11)	π/Z	2.15×10^{-4} s^{-1}

Table 2.

n	T_n^{eq}, h	T_n^{mer}, h	n	T_n^{eq}, h	n	T_n^{eq}, s
0	138.25	23.58	5	1.615	20	1460
1	7.818	7.431	6	1.347	40	730.2
2	3.996	3.941	7	1.155	60	486.8
3	2.679	2.662	8	1.012	80	365.1
4	2.015	2.008	9	0.8996	100	292.1

$k\tau = \tau_b = g_0 L_b$, where τ_b is the specific tensile strength of the cable material, $k > 1$ is the coefficient called the *safety factor*, and L_b is the so-called *breaking length*. For details, refer to article [9].

5.3. An Example of Calculating Natural Frequencies

The sample values are given in Table 1.

Table 2 shows the periods for the first 10 oscillation modes in the equatorial plane $T_n^{\text{eq}} = 2\pi/\omega_n^{\text{eq}}$ in hours and the periods for some higher-order modes in seconds. The periods of the first 5 oscillation modes in the meridional plane are also given; further the difference $T_n^{\text{eq}} - T_n^{\text{mer}}$ becomes very small.

Note that the estimate of the period (in hours) of the main mode by formula (5.7) gives $114 < T_0^{\text{eq}} < 158$.

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