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Citation: **1789**, 040002 (2016); doi: 10.1063/1.4968455
View online: http://dx.doi.org/10.1063/1.4968455
View Table of Contents: http://aip.scitation.org/toc/apc/1789/1
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Basis Properties of Eigenfunctions of the Differential Operator $-u''(-x) + q(x)u(x)$ with Cauchy Data

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Abstract. Uniform equiconvergence of spectral expansions associated with two second-order differential operators with involution $-u''(-x) + q(x)u(x)$ and the Cauchy data $u(-1) = 0$, $u'(-1) = 0$ is obtained. The proof uses the Cauchy integral method and the Green's function asymptotics of the considered operator. As a corollary, it is proved that the root functions of this operator form the basis in $L_2(-1, 1)$ for any continuous complex-valued coefficient $q(x)$.

Introduction

Consider the functional-differential operation

$$l[u] = -u''(-x) + q(x)u(x), \quad -1 < x < 1, \tag{1}$$

with a complex-valued continuous coefficient $q(x)$ subject to the Cauchy data at the end-point $x = -1$:

$$u(-1) = 0, \quad u'(-1) = 0. \tag{2}$$

Clearly, if the coefficient $q(x)$ is real-valued then the operator (1)-(2) is symmetric and has the unique self-adjoint extension L with the domain $W_2^2(-1, 1)$. As the spectrum of L is discrete and non-empty, each function $f(x) \in L_2(-1, 1)$ has the L_2-convergent eigenfunction expansion in the orthogonal series.

There are two ways to enhance this convergence – either take the expanded function $f(x)$ from some restricted class, or compare this series with a simpler one with known properties. The second approach for ordinary differential operators supposes such a comparison with the conventional Fourier series. Here, for the differential operator with involution, it is natural to study the equiconvergence of the general biorthogonal series for (1)-(2) and the orthogonal series that could be easily constructed in the case $q(x) \equiv 0$. The central result of the paper (Theorem 2) states that if $S_m(x, f)$ and $\sigma_m(x, f)$ are the partial sums of the biorthogonal series for (1)-(2) and of the orthogonal series in the unperturbed case then

$$S_m(x, f) - \sigma_m(x, f) = o(1), \quad m \to \infty, \tag{3}$$

uniformly for $x \in [-1, 1]$.

The study of convergence for eigenfunction expansions related to ordinary differential operators is the classical topic of the spectral theory (see e.g. Coddington and Levinson [1]). A different approach to this problem was proposed by V.A.Il'in (see survey in [2]). It is not connected directly with the form of the boundary conditions and could be applied in the variety of non-self-adjoint cases [3–13]. The basis property of eigenfunctions for various types of differential operators was lately discussed in [14–18].

The operation (1) under consideration has a peculiarity as it contains reflection $\nu(x) = -x$ in its main term. This is the simplest example of involution of the interval $[-1, 1]$ since $\nu(\nu(x)) = x$ for all $x \in [-1, 1]$.

Applications of Mathematics in Engineering and Economics (AMEE'16)
AIP Conf. Proc. 1789, 040002-1–040002-7; doi: 10.1063/1.4968455
Published by AIP Publishing. 978-0-7354-1453-2/$30.00

Results on the basis properties for eigen or root functions for one-dimensional differential operators with involution are eagerly applied in research of PDE. The recent papers by Aleroev, Kirane and Malik [19], Kirane and Al-Sati [20] give plausible examples. Various applications of differential operators with involutions could be found in [21].

Spectral expansions associated with differential operations with involution are less studied than the ODE case. Spectral properties of the first-order differential operators with involution are considered in [22–24]. Sample second-order differential operators with involution are discussed in [25–27]. A specific example of a boundary-value problem for the second-order differential operator with involution that produces an infinite number of associated functions is given in [28,29].

This paper consists of two parts. Its first section focuses on the properties of the unperturbed case ($q(x) \equiv 0$). It characterizes the spectrum, the eigenfunctions, constructs the Green's function (Theorem 1), and gives its estimates. The second section moves to the general case of (1)-(2). The study of its Green's function gives tools to apply the Cauchy integral technique for obtaining the equiconvergence result in Theorem 2. The important corollary on the basis property of the root functions of (1)-(2) in $L_2(-1,1)$ is given in the final Theorem 3.

The case of $q(x) = 0$

Consider the unperturbed boundary value problem (1)-(2)

$$\begin{aligned} &-u''(-x) = \lambda u(x), \quad -1 < x < 1, \\ &u(-1) = u'(-1) = 0. \end{aligned} \tag{4}$$

The straightforward calculation shows that the eigenvalue of (1)-(2) are the roots to the equation ($\lambda = \rho^2$):

$$\omega(\rho) \equiv i(1 - \exp(2\rho))(1 - \exp(2i\rho)) - (1 + \exp(2\rho))(1 + \exp(2i\rho)) = 0. \tag{5}$$

Lemma 1. *The spectrum of (4) is non-empty, all the eigenvalues are non-zero and simple and the sequence of its eigenvalues $\lambda_k = \rho_k^2$ splits into two subsequences:*

$$\rho_{k,1} = (k + (1/4))\pi i + \delta_{k,1} i, \ k = 0, 1, 2, \ldots, \quad \delta_{k,1} \sim \exp(-(\pi/2) - 2\pi k), \ k \to +\infty, \tag{6}$$

$$\rho_{k,2} = (k + (1/4))\pi + \delta_{k,2}, \ k = -1, -2, \ldots, \quad \delta_{k,2} \sim \exp((\pi/2) - 2\pi|k|), \ k \to -\infty. \tag{7}$$

The eigenfunctions of (4) constitute an orthogonal system in $L_2(-1,1)$ and has the form

$$u_{k,1}(x) = \exp\Big(\rho_{k,1}(1+x)\Big) - \exp\Big(\rho_{k,1}(1-x)\Big) + (i-1)\Big(\exp\Big(i\rho_{k,1}(1+x)\Big) + \exp\Big(i\rho_{k,1}(1-x)\Big)\Big), \ k = 0, 1, 2, \ldots, \tag{8}$$

$$u_{k,2}(x) = (1+i)\Big(\exp\Big(\rho_{k,2}(1+x)\Big) - \exp\Big(\rho_{k,2}(1-x)\Big)\Big) - \Big(\exp\Big(i\rho_{k,2}(1+x)\Big) + \exp\Big(i\rho_{k,2}(1-x)\Big)\Big), \ k = -1, -2, \ldots. \tag{9}$$

Lemma 1 unveils the main asymptotic terms of these eigenfunctions – these are the right-hand sides of (8) and (9) with $\rho_{k,1}$ and $\rho_{k,2}$ substituted by their main asymptotic terms in (6) and (7) respectively.

It also follows from (6) and (7) that the eigenfunctions (8), (9) are almost normed, i.e.

$$\exists A_1, A_2 > 0 : \quad A_1 \le \|u_{k,1}\|_2, \|u_{k,2}\|_2 \le A_2 \quad \forall k$$

(here $\|\cdot\|_2$ denotes the norm in $L_2(-1,1)$). Thus, any function $f(x) \in L_2(-1,1)$ has the Fourier-type expansion in the series and its partial sums

$$\sigma_m(x,f) = \sum_{k=0}^{m} a_k u_{k,1}(x) + \sum_{k=1}^{m} b_k u_{-k,2}(x) \tag{10}$$

with the coefficients

$$a_k = \|u_{k,1}\|_2^{-2} \int_{-1}^{1} f(t) u_{k,1}(t)\, dt, \quad b_k = \|u_{-k,2}\|_2^{-2} \int_{-1}^{1} f(t) u_{-k,2}(t)\, dt,$$

converge to $f(x)$ with respect to the $L_2(-1,1)$-norm.

Let us now construct the Green's function for the problem (4). Following the classical notion, the Green's function $G(x, t, \lambda)$ is introduced as the integral kernel in the representation

$$u(x) = \int_{-1}^{1} G(x, t, \lambda) f(t)\, dt \tag{11}$$

of the solution to the problem

$$\begin{aligned} &-u''(-x) = \lambda u(x) + f(x), \quad -1 < x < 1, \\ &u(-1) = u'(-1) = 0. \end{aligned} \tag{12}$$

If $f(x)$ is a continuous function then $u(x)$ in (11) is the classical (pointwise) solution to (12). If $f(x)$ belongs to $L_1(-1, 1)$ then the representation (11) satisfies the equation in (12) almost everywhere.

Theorem 1. *If λ is not an eigenvalue of (4) then the solution of (12) has the form*

$$u(x) = \frac{-1}{\rho\omega(\rho)} \left\{ \left[-\frac{1}{2} \int_{-1}^{1} \left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right) f(t)\, dt + \right.\right.$$

$$+ \frac{1}{8} \left(i\left(1 - e^{2i\rho}\right)\left(e^{-2\rho} + 1\right) - \left(1 + e^{2i\rho}\right)\left(e^{-2\rho} - 1\right) \right) \int_{-1}^{1} \left(e^{\rho(1+t)} - e^{\rho(1-t)} \right) f(t)\, dt \Bigg] \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right) +$$

$$+ \left[\frac{1}{8i} \left(i\left(1 - e^{2\rho}\right)\left(e^{-2i\rho} + 1\right) - \left(1 + e^{2\rho}\right)\left(e^{-2i\rho} - 1\right) \right) \int_{-1}^{1} \left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right) f(t)\, dt - \right.$$

$$\left.\left. - \frac{1}{2} \int_{-1}^{1} \left(e^{\rho(1+t)} - e^{\rho(1-t)} \right) f(t)\, dt \right] \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right) \right\} +$$

$$+ \frac{1}{8\rho} \int_{-1}^{-x} \left[-ie^{-2i\rho} \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right)\left(e^{i\rho(1+t)} - e^{i\rho(1-t)} \right) + e^{-2\rho} \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right)\left(e^{\rho(1+t)} + e^{\rho(1-t)} \right) \right] f(t)\, dt +$$

$$+ \frac{1}{8\rho} \int_{-x}^{x} \left[ie^{-2i\rho} \left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right)\left(e^{i\rho(1+x)} - e^{i\rho(1-x)} \right) - e^{-2\rho} \left(e^{\rho(1+t)} - e^{\rho(1-t)} \right)\left(e^{\rho(1+x)} + e^{\rho(1-x)} \right) \right] f(t)\, dt +$$

$$+ \frac{1}{8\rho} \int_{x}^{1} \left[ie^{-2i\rho} \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right)\left(e^{i\rho(1+t)} - e^{i\rho(1-t)} \right) - e^{-2\rho} \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right)\left(e^{\rho(1+t)} + e^{\rho(1-t)} \right) \right] f(t)\, dt.$$

Theorem 1 is proved by the direct checking of (12).

Corollary. *The Green's function of (4) has the form*

$$G(x, t, \lambda) = \frac{-1}{\rho\omega(\rho)} \left\{ \left[-\frac{1}{2}\left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right) + \frac{1}{8} \left(i\left(1 - e^{2i\rho}\right)\left(e^{-2\rho} + 1\right) - \left(1 + e^{2i\rho}\right)\left(e^{-2\rho} - 1\right) \right) \left(e^{\rho(1+t)} - e^{\rho(1-t)} \right) \right] \cdot \right.$$

$$\cdot \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right) + \left[\frac{1}{8i} \left(i\left(1 - e^{2\rho}\right)\left(e^{-2i\rho} + 1\right) - \left(1 + e^{2\rho}\right)\left(e^{-2i\rho} - 1\right) \right) \left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right) - \frac{1}{2}\left(e^{\rho(1+t)} - e^{\rho(1-t)} \right) \right] \cdot$$

$$\left. \cdot \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right) \right\} + g(x, t, \lambda)$$

where

$$g(x, t, \lambda) = (8\rho)^{-1} \cdot$$

$$\cdot \begin{cases} -ie^{-2i\rho} \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right)\left(e^{i\rho(1+t)} - e^{i\rho(1-t)} \right) + e^{-2\rho} \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right)\left(e^{\rho(1+t)} + e^{\rho(1-t)} \right), & t \le -|x|, \\ \operatorname{sgn}(x) \left[ie^{-2i\rho} \left(e^{i\rho(1+t)} + e^{i\rho(1-t)} \right)\left(e^{i\rho(1+x)} - e^{i\rho(1-x)} \right) - e^{-2\rho} \left(e^{\rho(1+t)} - e^{\rho(1-t)} \right)\left(e^{\rho(1+x)} + e^{\rho(1-x)} \right) \right], & |t| \le |x|, \\ ie^{-2i\rho} \left(e^{i\rho(1+x)} + e^{i\rho(1-x)} \right)\left(e^{i\rho(1+t)} - e^{i\rho(1-t)} \right) - e^{-2\rho} \left(e^{\rho(1+x)} - e^{\rho(1-x)} \right)\left(e^{\rho(1+t)} + e^{\rho(1-t)} \right), & t \ge |x|. \end{cases}$$

Remark. The explicit form of the Green's function for the first-order linear differential equation with involution was recently obtained by Cabada and Tojo [30].

The Green's function helps to rewrite the eigenfunction expansion related to (4) using the Cauchy contour integral.

As the poles of the Green's function of (4) are the zeroes of $\omega(\rho)$ in (5), we introduce the circles C_k in the complex λ-plane that separate these poles from each other. Due to the asymptotics in Lemma 1, the circles C_k : $|\lambda| = \Big((k + (1/4))\pi + 1\Big)^2$ do not cross the spectrum of (4) for large values of k. It follows from the general theory of self-adjoint operators that, for large values of m, the partial sum (10) can be rewritten in the form

$$\sigma_m(x, f) = -\frac{1}{2\pi i} \int\limits_{C_m} \left(\int_{-1}^{1} G(x, t, \lambda) f(t)\, dt \right) d\lambda. \tag{13}$$

In order to use (13) for the equiconvergence analysis one needs an appropriate estimate for the Green's function.

Lemma 2. *There exists a constant $C > 0$ such that, for sufficiently large $|\lambda|$, the Green's function of (4) satisfies the estimate*

$$|G(x, t, \lambda)| \le C|\rho|^{-1} r(x, t, \rho) \tag{14}$$

where $\rho = \sqrt{\lambda}$ lays outside the collection B of the circles $\Big\{|\rho - \rho_{k,j}| \le 1/16\Big\}$, $j = 1, 2$, centered at the zeroes of (5) and $r(x, t, \rho)$ denotes the function:

$$r(x, t, \rho) = \exp\Big(-\rho_0 \big||x| - |t|\big|\Big) + \exp\Big(-\rho_0 \big(2 - \big||x| - |t|\big|\big)\Big), \quad \rho_0 = \min\Big(|\mathrm{Re}\,\rho|, |\mathrm{Im}\,\rho|\Big).$$

Lemma 2 follows from the explicit form of the Green's function $G(x, t, \lambda)$.

Equiconvergence Property

Let us consider the problem of equiconvergence of the eigenfunction expansions related to the unperturbed problem (4) and to the general problem (1)-(2) with an arbitrary continuous complex-valued coefficient $q(x)$.

Denote by $G_q(x, t, \lambda)$ the Green's function of (1)-(2). It follows from the relations

$$-\frac{\partial^2 G(-x, t, \lambda)}{\partial x^2} = \lambda G(x, t, \lambda), \quad -\frac{\partial^2 G_q(-x, t, \lambda)}{\partial x^2} + q(x) G_q(x, t, \lambda) = \lambda G_q(x, t, \lambda), \quad t \ne \pm x,$$

that

$$\frac{\partial^2 \Big(G_q(\xi, t, \lambda) - G(\xi, t, \lambda)\Big)}{\partial \xi^2}\Big|_{\xi = -x} + \lambda \Big(G_q(x, t, \lambda) - G(x, t, \lambda)\Big) = -q(x) G(x, t, \lambda).$$

Since the difference $G_q(x, t, \lambda) - G(x, t, \lambda)$ satisfies the initial conditions (2), one can apply (11) outside the poles of $\omega(\rho)$:

$$G_q(x, t, \lambda) - G(x, t, \lambda) = -\int_{-1}^{1} G(x, s, \lambda) q(s) G_q(s, t, \lambda)\, ds. \tag{15}$$

Lemma 3. *For sufficiently large $|\rho|$ outside the collection B of circles in Lemma 2, the integral equation (15) has a unique solution $G_q(x, t, \lambda)$.*

In order to prove Lemma 3 we introduce the iterative kernels

$$G_q^{(0)}(x, t, \lambda) \equiv 0, \quad G_q^{(p+1)}(x, t, \lambda) = G(x, t, \lambda) - \int_{-1}^{1} G(x, s, \lambda) q(s) G_q^{(p)}(s, t, \lambda)\, ds$$

and, for a fixed $t \in [-1, 1]$, the related constants

$$\gamma_0 = \max_{-1 \le x \le 1} |G_q^{(1)}(x, t, \lambda)|\, |\rho| r^{-1}(x, t, \rho), \quad \gamma_p = \max_{-1 \le x \le 1} |G_q^{(p+1)}(x, t, \lambda) - G_q^{(p)}(x, t, \lambda)|\, |\rho| r^{-1}(x, t, \rho).$$

We prove that, for sufficiently large ρ laying outside the collection B of circles in Lemma 2, the estimate

$$\gamma_p \le 2^{-p} C \tag{16}$$

holds with the constant C from the estimate (14) of Lemma 2.

In fact, the estimate (16) with $p = 0$ repeats (14). Then, by the induction reasoning, one deduces the relations:

$$\gamma_{p+1} \leq \max_{-1\leq x\leq 1}\left\{|\rho| r^{-1}(x,t,\rho)\int_{-1}^{1}|G(x,a,\lambda)|\,|q(s)|\,|G_q^{(p+1)}(s,t,\lambda)-G_q^{(p)}(s,t,\lambda)|\,ds\right\} \leq$$

$$\leq C\gamma_p|\rho|^{-1}\max_{-1\leq x\leq 1} r^{-1}(x,t,\rho)\int_{-1}^{1} r(x,s,\rho)r(s,t,\rho)|q(s)|\,ds. \tag{17}$$

Since $r(x,s,\rho)r(s,t,\rho) \leq 2r(x,t,\rho)$, the estimate (17) yields

$$\gamma_{p+1} \leq 2C\gamma_p|\rho|^{-1}\int_{-1}^{1}|q(s)|\,ds$$

which delivers the estimate (16) if

$$4C|\rho|^{-1}\int_{-1}^{1}|q(s)|\,ds \leq 1.$$

Estimate (16) provides that the series $\sum_{p=1}^{\infty}\left(G_q^{(p+1)}(x,t,\lambda)-G_q^{(p)}(x,t,\lambda)\right)$ converges uniformly for $x\in[-1,1]$. As its m-th partial sum equals $G_q^{(m+1)}(x,t,\lambda)-G(x,t,\lambda)$, the sequence $G_q^{(m)}(x,t,\lambda)$ also converges uniformly and its limit $G_q(x,t,\lambda)$ satisfies (15).

Lemma 3 is proved.

Remark. Under the assumptions of Lemma 3, $G_q(x,t,\lambda)$ satisfies the estimate

$$|G_q(x,t,\lambda)| \leq 2C|\rho|^{-1}r(x,t,\rho). \tag{18}$$

In particular, it means that, for sufficiently large $|\rho|$, the poles of $G_q(x,t,\lambda)$ could lay solely within the collection B of circles around the poles of the Green's function $G(x,t,\lambda)$.

Let C_k be the same contours as in (13). Lemma 3 implies that the partial sum $S_m(x,f)$ of the spectral expansion related to the problem (1)-(2) has the same integral representation

$$S_m(x,f) = -\frac{1}{2\pi i}\int_{C_m}\left(\int_{-1}^{1}G_q(x,t,\lambda)f(t)\,dt\right)d\lambda. \tag{19}$$

We say that the sequence $S_m(x,f)$ equiconverges with the sequence $\sigma_m(x,f)$ on $[-1,1]$ if the difference $S_m(x,f)-\sigma_m(x,f)$ vanishes as $m\to\infty$ uniformly for $-1\leq x\leq 1$.

Theorem 2. *For an arbitrary function* $f(x)\in L_1(-1,1)$*, the sequence* $S_m(x,f)$ *equiconverges with the sequence* $\sigma_m(x,f)$ *on* $[-1,1]$.

Proof. Taking into account (13) and (19), we consider the difference $S_m(x,f)-\sigma_m(x,f)$ in the form

$$S_m(x,f)-\sigma_m(x,f) = -\frac{1}{\pi i}\int_{\Gamma_m}\left\{\int_{-1}^{1}\left[G_q(x,t,\lambda)-G(x,t,\lambda)\right]f(t)\,dt\right\}\rho\,d\rho \tag{20}$$

where Γ_m denotes the set of complex $\rho:\ \rho^2\in C_m$.

It follows from the estimates (14), (18) and the relation (15) that, for sufficiently large m and $\rho\in\Gamma_m$,

$$|G_q(x,t,\lambda)-G(x,t,\lambda)| \leq 4C^2|\rho|^{-2}r(x,t,\rho)\int_{-1}^{1}|q(s)|\,ds.$$

This inequality combined with the relation (20) gives the estimate

$$|S_m(x,f)-\sigma_m(x,f)| \leq C_1\int_{\Gamma_m}\left\{\int_{-1}^{1}r(x,t,\rho)|f(t)|\,dt\right\}\left|\frac{d\rho}{\rho}\right| \tag{21}$$

where C_1 denotes the constant $4C^2\pi^{-1}\int_{-1}^{1}|q(s)|\,ds$.

Now taking sufficiently small $\delta > 0$, we split the interval $(-1, 1)$ into two parts:

$$\Delta_1 = (-1, 1) \setminus \Delta_2, \quad \Delta_2 = (-1, -1 + \delta) \cup (-x - \delta, -x + \delta) \cup (x - \delta, x + \delta) \cup (1 - \delta, 1).$$

Therefore, the estimate (21) turns into the relation

$$|S_m(x, f) - \sigma_m(x, f)| \leq C_1 \int_{\Gamma_m} \int_{\Delta_1} \left(e^{-\rho_0||x|-|t||} + e^{-\rho_0(2-||x|-|t||)}\right)|f(t)|\, dt \left|\frac{d\rho}{\rho}\right| + 2C_1\pi \int_{\Delta_2} |f(t)|\, dt, \tag{22}$$

Since the function $f(x)$ is in $L_1(-1, 1)$, by the Lebesgue theorem, there exists $\delta > 0$ such that the second term in the right-hand side of (22) is less than $\varepsilon/2$. The first term satisfies the estimate

$$\int_{\Gamma_m} \int_{\Delta_1} \left(e^{-\rho_0||x|-|t||} + e^{-\rho_0(2-||x|-|t||)}\right)|f(t)|\, dt \left|\frac{d\rho}{\rho}\right| \leq 3 \int_{-1}^{1} |f(t)|\, dt \int_{\Gamma_m} \exp(-\rho_0\delta) \left|\frac{d\rho}{\rho}\right|.$$

If ρ_m is the radius of Γ_m then

$$\int_{\Gamma_m} \exp(-\rho_0\delta) \left|\frac{d\rho}{\rho}\right| = 4 \int_0^{\pi/4} \exp(-\delta\rho_m|\sin t|)\, dt + 2 \int_{\pi/4}^{3\pi/4} \exp(-\delta\rho_m|\cos t|)\, dt \leq C_2|\rho_m|^{-1}.$$

Therefore, the second term in the right-hand side of (22) could be also made less than $\varepsilon/2$ provided m is sufficiently large.

Theorem 2 is completely proved.

As the equiconvergence property is obtained in the uniform over $[-1, 1]$ metric for any integrable function $f(x)$ and the orthogonal series (10) converges for any function $f(x) \in L_2(-1, 1)$ in the metric of $L_2(-1, 1)$, Theorem 2 immediately delivers the following result on the basis property.

Theorem 3. *The system of root functions of the problem (1)-(2) with an arbitrary continuous complex-valued coefficient $q(x)$ constitutes the basis in $L_2(-1, 1)$.*

ACKNOWLEDGMENTS

This research is financially supported by the grant 0971/GF4 and by the target program 0085/PTSF-14 from the Ministry for Education and Science of the Republic of Kazakhstan.

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