# New Reductions of the Unsteady Axisymmetric Boundary Layer Equation to ODEs and Simpler PDEs 

Alexander V. Aksenov 1,*(D) and Anatoly A. Kozyrev 2,* (D)<br>1 Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, 1 Leninskie Gory, Main Building, 119991 Moscow, Russia<br>2 Dukhov All-Russia Research Institute of Automatics, Rosatom, 22 Suschevskaya St., 127055 Moscow, Russia<br>* Correspondence: aksenov@mech.math.msu.su or aksenov.av@gmail.com (A.V.A.); anatoly.kozyrev@gmail.com (A.A.K.)


#### Abstract

Reductions make it possible to reduce the solution of a PDE to solving an ODE. The best known are the traveling wave, self-similar and symmetry reductions. Classical and non-classical symmetries are also used to construct reductions, as is the Clarkson-Kruskal direct method. Recently, authors have proposed a method for constructing reductions of PDEs with two independent variables based on the idea of invariance. The proposed method in this work is a modification of the ClarksonKruskal direct method and expands the possibilities for its application. The main result of this article consists of a method for constructing reductions that generalizes the previously proposed approach to the case of three independent variables. The proposed method is used to construct reductions of the unsteady axisymmetric boundary layer equation to ODEs and simpler PDEs. All reductions of this equation were obtained.


Keywords: steady-state plane boundary layer; unsteady axisymmetric boundary layer; self-similar solution; invariant solution; symmetry reduction; invariant auxiliary functions; one- and twodimensional reductions

MSC: 35B06; 35B07; 35C06; 35-G20; 35Q35; 76D10; 76M60

## Contents

1. Introduction
1.1. Preliminary Remarks
1.2. The Main Results
2. Reductions of the Burgers Equation
3. Reductions of the Steady-State Plane Boundary Layer Equation
3.1. Basic Equation
3.2. Construction of Reductions
3.3. The Existence of non-Symmetry Reductions
4. Reductions of the Unsteady Axisymmetric Boundary Layer Equation
4.1. Basic Equation
4.2. Reductions to simpler PDEs (Two-dimensional Reductions)
4.3. Reductions to ODEs (One-dimensional Reductions)
5. Conclusions

References

## 1. Introduction

### 1.1. Preliminary Remarks

Constructing reductions of nonlinear PDEs is of fundamental importance for finding exact solutions. Reductions make it possible to reduce the solution of a PDE to solving an ODE. The best known are self-similar reductions [1]

$$
\begin{equation*}
u=t^{v} \varphi(\zeta), \quad \zeta=x t^{-\delta} \tag{1}
\end{equation*}
$$

where $u$ is a dependent variable, $x$ and $t$ are independent variables, and $v$ and $\delta$ are constants. Self-similar reductions are a special case of symmetry reductions. Symmetry reductions are found by applying group analysis methods [2]. Traveling wave reductions have the form [3]

$$
u=\varphi(\zeta)+V_{0}(t), \quad \zeta=x-V(t)
$$

In [4], the reductions

$$
\begin{equation*}
u=\lambda(t)+\mu(t) \varphi(\zeta), \quad \zeta=x p(t)+q(t) \tag{2}
\end{equation*}
$$

were considered; they generalize the reductions found in (1).
A method for finding reductions of PDEs with two independent variables was proposed in [5]; this is also known as the Clarkson-Kruskal direct method. For the Boussinesq equation

$$
\begin{equation*}
u_{t t}+\frac{1}{2}\left(u^{2}\right)_{x x}+u_{x x x x}=0 \tag{3}
\end{equation*}
$$

all reductions of the form

$$
\begin{equation*}
u=U(x, t, w(z)) \tag{4}
\end{equation*}
$$

were obtained, where $z=z(x, t)$ and the function $w(z)$ is a solution of an ODE. It was shown that there are reductions other than those obtained using the Lie group method for finding group-invariant solutions of PDEs.

All reductions of the Burgers equation

$$
\begin{equation*}
u_{t}+u u_{x}=u_{x x} \tag{5}
\end{equation*}
$$

the Korteweg-de Vries equation

$$
\begin{equation*}
u_{t}+u u_{x}=u_{x x x}, \tag{6}
\end{equation*}
$$

and the modified Korteweg-de Vries equation

$$
\begin{equation*}
u_{t}+u^{2} u_{x}=u_{x x x} \tag{7}
\end{equation*}
$$

were found in [5]. It was shown that reductions of these equations coincide with the symmetry reductions. It was also shown that for Equations (3) and (5)-(7), reductions (4) have the form

$$
\begin{equation*}
u=\alpha(x, t)+\beta(x, t) w(z) \tag{8}
\end{equation*}
$$

In the works of $[6,7]$, a method for constructing reductions of PDEs with two independent variables was proposed, based on the idea of invariance. In these articles, based on the proposed approach, all reductions of the equation of a steady-state plane boundary layer with a pressure gradient were found.

### 1.2. The Main Results

The main results of this article are the following:

- a method of constructing reductions, generalizing the previously proposed approach [6,7] to the case of three independent variables;
- the construction of all reductions of the unsteady axisymmetric boundary layer equation.


## 2. Reductions of the Burgers Equation

We demonstrate the effectiveness of the method presented in [6,7] by finding the reductions of the Burgers Equation (5).

Substituting (4) into Equation (5), we obtain the relation

$$
\begin{align*}
U_{w} z_{x}^{2} w^{\prime \prime}+U_{w w} z_{x}^{2}\left(w^{\prime}\right)^{2}+\left(2 U_{x w} z_{x}+U_{w} z_{x x}\right. & \left.-U U_{w} z_{x}-U_{w} z_{t}\right) w^{\prime} \\
& +U_{x x}-U U_{x}-U_{t}=0 \tag{9}
\end{align*}
$$

We divide both sides of Relation (9) by the coefficient at the highest derivative by $U_{w} z_{x}^{2}$ (normalization of the coefficients). The normalized coefficient at the term with $\left(w^{\prime}\right)^{2}$ has the form

$$
\begin{equation*}
\frac{U_{w w}}{U_{w}}=\Gamma(z, w) \tag{10}
\end{equation*}
$$

Integrating Relation (10), we obtain

$$
U(x, t, w)=\beta(x, t) \Gamma_{1}(z, w)+\alpha(x, t) .
$$

The transformation $w \rightarrow F(z, w)$ maps ODE into ODE. Then, the reductions of the Burgers equation can be sought in the form presented in (8).

Substituting (8) into Equation (5), we obtain the relation

$$
\begin{align*}
\beta z_{x}^{2} w^{\prime \prime} & -\beta^{2} z_{x} w w^{\prime}+\left[\beta z_{x x}+\left(2 \beta_{x}-\alpha \beta\right) z_{x}-\beta z_{t}\right] w^{\prime} \\
& +\beta \beta_{x} w^{2}+\left(\beta_{x x}-\beta \alpha_{x}-\alpha \beta_{x}-\beta_{t}\right) w+\alpha_{x x}-\alpha \alpha_{x}-\alpha_{t}=0 . \tag{11}
\end{align*}
$$

Relation (11) is an ODE if the normalized coefficients depend on $\alpha(x, t), \beta(x, t), z(x, t)$ and their derivatives are functions of $z$. From this condition, we obtain the overdetermined system of equations

$$
\begin{align*}
& \frac{\beta}{z_{x}}=\Gamma_{1}(z), \quad \frac{\beta_{x}}{z_{x}^{2}}=\Gamma_{2}(z) \\
& \frac{\beta z_{x x}+\left(2 \beta_{x}-\alpha \beta\right) z_{x}-\beta z_{t}}{\beta z_{x}^{2}}=\Gamma_{3}(z), \\
& \frac{\beta \beta_{x x}-\beta \alpha_{x}-\alpha \beta_{x}-\beta_{t}}{\beta z_{x}^{2}}=\Gamma_{4}(z)  \tag{12}\\
& \frac{\alpha_{x x}-\alpha \alpha_{x}-\alpha_{t}}{\beta z_{x}^{2}}=\Gamma_{5}(z)
\end{align*}
$$

for finding the unknown functions $\alpha(x, t), \beta(x, t), z(x, t)$ and $\Gamma_{1}(z), \ldots, \Gamma_{5}(z)$.
Let us state the main considerations underlying the method for constructing reductions for PDEs with two independent variables [6,7] as applied to the system of Equation (12):

1. Each equation of the system of Equation (12) is equivalent to the condition that the Jacobian of its left-hand side and the function $z(x, t)$ is equal to zero. As a result, we can derive an overdetermined system of equations for determining the unknown functions $\alpha(x, t), \beta(x, t)$ and $z(x, t)(\mathcal{A}$-system $)$. $\mathcal{A}$-system is not presented here because of its cumbersomeness.
2. Consider the auxiliary functions $\mu_{1}=\mu_{1}(x, t), \mu_{2}=\mu_{2}(x, t)$ and $\mu_{3}=\mu_{3}(x, t)$ defined by the relations

$$
\begin{align*}
& \mu_{1} z_{x}+z_{t}=0 \\
& \mu_{1} \beta_{x}+\beta_{y}+\mu_{2} \beta=0  \tag{13}\\
& \mu_{1} \alpha_{x}+\alpha_{t}+\mu_{2} \alpha+\mu_{3}=0 .
\end{align*}
$$

Finding the auxiliary functions, we find the reductions. As was noted in [5], the following transformations map a reduction of the form found in (8) into a reduction:

$$
\begin{align*}
& z \rightarrow F_{1}(z) \\
& \beta \rightarrow \frac{\beta}{F_{2}(z)} \tag{14}
\end{align*}
$$

$$
\alpha \rightarrow \alpha+\frac{\beta}{F_{3}(z)}
$$

where $F_{1}(z), F_{2}(z)$ and $F_{3}(z)$ are arbitrary functions. These transformations are associated with the arbitrariness of $w(z)$ in finding an ODE. It can be shown that the auxiliary functions $\mu_{1}, \mu_{2}$ and $\mu_{3}$ are invariant under the transformations in (14). It can also be shown that the $\mathcal{A}$-system admits transformations (14).
3. Finding the derivatives $\alpha_{x}, \beta_{x}$ and $z_{x}$ from the relations in (13) and substituting them into the $\mathcal{A}$-system, we obtain the following overdetermined system of equations for determining the auxiliary functions:

$$
\begin{align*}
& \mu_{1_{x}}-\mu_{2}=0, \quad \mu_{2 x}=0 \\
& 2 \mu_{1} \mu_{1_{x}}+\mu_{1_{t}}+\mu_{3}=0  \tag{15}\\
& 2 \mu_{3} \mu_{1_{x}}+\mu_{3_{t}}=0
\end{align*}
$$

Thus, the $\mathcal{A}$-system is reduced to the simpler system of Equation (15). The $\mathcal{A}$-system can be written only in terms of the invariants $\mu_{1}, \mu_{2}$ and $\mu_{3}$ of transformations (14) because it is invariant under these transformations.
4. The overdetermined system of Equation (15) is easy to solve. The auxiliary functions $\mu_{1}, \mu_{2}$ and $\mu_{3}$ are used to construct reductions.

Remark 1. According to the approach described in [5], the overdetermined system of Equation (12) is solved directly. Reductions are found using modulo transformations (14).

## 3. Reductions of the Steady-State Plane Boundary Layer Equation

### 3.1. Basic Equation

We also demonstrate the effectiveness of the method for constructing reductions of PDEs with two independent variables based on the idea of invariance by finding the reductions of steady-state plane boundary layer equation [6,7].

We consider the equation

$$
\begin{equation*}
u_{y y y}-u_{y} u_{x y}+u_{x} u_{y y}-P(x)=0, \tag{16}
\end{equation*}
$$

which describes the steady motion of a viscous incompressible fluid in a laminar plane boundary layer with a pressure gradient [8]. Equation (16) is written in dimensionless variables. Here, $u$ is the stream function, $P(x)=\partial p / \partial x$ is a given function and $p$ is the pressure. Self-similar solutions of Equation (16) were discussed in the classical monographs [1,8-11]. Symmetry reductions of Equation (16) can be obtained using the results of [2]. Reductions of Form (2) were found and studied in [4]. In [12], the method of non-classical symmetries [13] and its generalization were used to obtain new reductions of Form (8) for Equation (16). For invariant and noninvariant exact solutions, see also [10,11,14-20]. An extensive list of exact solutions to the boundary layer equation on a flat plane as well as related hydrodynamic equations can be found in the handbook [19].

Remark 2. For some exact solutions and transformations of the unsteady plane boundary layer equations [8,11], see [2,19,21-28].

Remark 3. The studies [17,18,29-32] present exact solutions and transformations for the steadystate and unsteady equations of non-Newtonian fluids.

### 3.2. Construction of Reductions

We look for reductions of Equation (16) in the form

$$
\begin{equation*}
u=U(x, y, w(z)) \tag{17}
\end{equation*}
$$

where $z=z(x, y)$ and the function $w(z)$ is a solution of an ODE. Substituting (17) into Equation (16), we obtain the relation

$$
\begin{align*}
U_{w} z_{y}^{3} w^{\prime \prime \prime} & +3 U_{w w} z_{y}^{3} w^{\prime} w^{\prime \prime}+z_{y}\left(3 U_{y w} z_{y}+U_{x} U_{w} z_{y}-U_{y} U_{w} z_{x}+3 U_{w} z_{y y}\right) w^{\prime \prime} \\
& +U_{w w w} z_{y}^{3}\left(w^{\prime}\right)^{3}+\left(3 U_{y w w} z_{y}^{2}-U_{w} U_{x w} z_{y}^{2}+U_{w} U_{y w} z_{x} z_{y}+U_{x} U_{w w} z_{y}^{2}\right. \\
& \left.-U_{y} U_{w w} z_{x} z_{y}+U_{w}^{2} z_{x} z_{y y}-U_{w}^{2} z_{y} z_{x y}+3 U_{w w} z_{y} z_{y y}\right)\left(w^{\prime}\right)^{2}+\left(3 U_{y y w} z_{y}\right.  \tag{18}\\
& -U_{w} U_{x y} z_{y}+U_{w} U_{y y} z_{x}-U_{y} U_{x w} z_{y}-U_{y} U_{y w} z_{x}+2 U_{x} U_{y w} z_{y}+U_{x} U_{w w} z_{y y} \\
& \left.-U_{y} U_{w} z_{x y}+3 U_{y w} z_{y y}+U_{w} z_{y y y}\right) w^{\prime}+U_{y y y}-U_{y} U_{x y}+U_{x} U_{y y}-P(x)=0 .
\end{align*}
$$

Both sides of Relation (18) are divided by the coefficient at the highest derivative by $U_{w} z_{y}^{3}$. Consider the normalized coefficient at the term with $w^{\prime} w^{\prime \prime}$. It has the form

$$
\begin{equation*}
\frac{3 U_{w w}}{U_{w}}=\Gamma(z, w) . \tag{19}
\end{equation*}
$$

Remark 4. While deriving Relation (19), we assumed that $z_{y} \neq 0$. The case $z_{y}=0$ corresponds to a degenerate reduction. It is of no interest, and its detailed consideration is omitted.

The transformation $w \rightarrow F(z, w)$ maps ODE into ODE. Then, the reductions of Equation (16) can be sought in the form

$$
\begin{equation*}
u=\alpha(x, y)+\beta(x, y) w(z), \tag{20}
\end{equation*}
$$

where $z=z(x, y)$. Substituting (20) into Equation (16), we obtain the relation

$$
\begin{align*}
\beta z_{y}^{3} w^{\prime \prime \prime} & +\beta z_{y}\left(\beta_{x} z_{y}-\beta_{y} z_{x}\right) w w^{\prime \prime}+z_{y}\left(3 \beta z_{y y}+3 \beta_{y} z_{y}+\alpha_{x} \beta z_{y}-\alpha_{y} \beta z_{x}\right) w^{\prime \prime} \\
& +\beta\left(\beta z_{x} z_{y y}-\beta z_{y} z_{x y}+\beta_{y} z_{x} z_{y}-z_{y}^{2} \beta_{x}\right)\left(w^{\prime}\right)^{2} \\
& +\left(\beta_{x} \beta z_{y y}+\beta z_{x} \beta_{y y}-\beta_{y}^{2} z_{x}+\beta_{y} \beta_{x} z_{y}-\beta_{y} \beta z_{x y}-\beta z_{y} \beta_{x y}\right) w w^{\prime} \\
& +\left(\beta z_{y y y}-\alpha_{y} \beta_{x} z_{y}-\alpha_{y} \beta_{y} z_{x}-\alpha_{y} \beta z_{x y}-\beta z_{y} \alpha_{x y}+2 \alpha_{x} \beta_{y} z_{y}\right.  \tag{21}\\
& \left.+\alpha_{x} \beta z_{y y}+\beta z_{x} \alpha_{y y}+3 \beta_{y} z_{y y}+3 \beta_{y y} z_{y}\right) w^{\prime} \\
& +\left(\beta_{x} \beta_{y y}-\beta_{y} \beta_{x y}\right) w^{2}+\left(\beta_{y y y}+\alpha_{x} \beta_{y y}-\beta_{y} \alpha_{x y}-\alpha_{y} \beta_{x y}+\beta_{x} \alpha_{y y}\right) w \\
& +\alpha_{y y y}-\alpha_{y} \alpha_{x y}+\alpha_{x} \alpha_{y y}-P(x)=0
\end{align*}
$$

Relation (21) is an ODE if the normalized coefficients depend on $\alpha(x, y), \beta(x, y)$ and $z(x, y)$ and their derivatives are functions of $z$. From this condition, we obtain the overdetermined system of equations

$$
\begin{align*}
& \frac{\beta_{x} z_{y}-\beta_{y} z_{x}}{z_{y}^{2}}=\Gamma_{1}(z), \\
& \frac{3 \beta z_{y y}+3 \beta_{y} z_{y}+\alpha_{x} \beta z_{y}-\alpha_{y} \beta z_{x}}{\beta z_{y}^{2}}=\Gamma_{2}(z), \\
& \frac{\beta\left(\beta z_{x} z_{y y}-\beta z_{y} z_{x y}+\beta_{y} z_{x} z_{y}-z_{y}^{2} \beta_{x}\right)}{\beta z_{y}^{3}}=\Gamma_{3}(z), \\
& \frac{\beta \beta_{x} \beta z_{y y}+\beta z_{x} \beta_{y y}-\beta_{y}^{2} z_{x}+\beta_{y} \beta z_{x} z_{y}-\beta_{y} \beta z_{x y}-\beta z_{y} \beta_{x y}}{\beta z_{y}^{3}}=\Gamma_{4}(z), \\
& \frac{\beta z_{y y y}-\alpha_{y} \beta_{x} z_{y}-\alpha_{y} \beta_{y} z_{x}-\alpha_{y} \beta z_{x y}-\beta z_{y} \alpha_{x y}}{\beta z_{y}^{3}}  \tag{22}\\
& \quad+\frac{2 \alpha_{x} \beta z_{y} z_{y}+\alpha_{x} \beta z_{y y}+\beta z_{x} \alpha_{y y}+3 \beta_{y} z_{y y}+3 \beta_{y y} z_{y}}{\beta z_{y}^{3}}=\Gamma_{5}(z),
\end{align*}
$$

$$
\begin{aligned}
& \frac{\beta_{x} \beta_{y y}-\beta_{y} \beta_{x y}}{\beta z_{y}^{3}}=\Gamma_{6}(z), \\
& \frac{\beta_{y y y}+\alpha_{x} \beta_{y y}-\beta_{y} \alpha_{x y}-\alpha_{y} \beta_{x y}+\beta_{x} \alpha_{y y}}{\beta z_{y}^{3}}=\Gamma_{7}(z), \\
& \frac{\alpha_{y y y}-\alpha_{y} \alpha_{x y}+\alpha_{x} \alpha_{y y}-P(x)}{\beta z_{y}^{3}}=\Gamma_{8}(z) .
\end{aligned}
$$

Each of the equations in Equation (22) is equivalent to the condition that the Jacobian of its left-hand side and $z(x, y)$ is equal to zero. As a result, we can derive an $\mathcal{A}$-system for determining the unknown functions $\alpha(x, y), \beta(x, y)$ and $z(x, y)$ (the system is not presented because of its cumbersomeness).

Consider the auxiliary functions $\mu_{1}=\mu_{1}(x, y), \mu_{2}=\mu_{2}(x, y)$ and $\mu_{3}=\mu_{3}(x, y)$ defined by the relations

$$
\begin{align*}
& z_{x}+\mu_{1}(x, y) z_{y}=0 \\
& \beta_{x}+\mu_{1}(x, y) \beta_{y}+\mu_{2}(x, y) \beta=0  \tag{23}\\
& \alpha_{x}+\mu_{1}(x, y) \alpha_{y}+\mu_{2}(x, y) \alpha+\mu_{3}(x, y)=0 .
\end{align*}
$$

Finding the partial derivatives $\alpha_{x}, \beta_{x}$ and $z_{x}$ from (23) and substituting them into the $\mathcal{A}$-system, we obtain the following overdetermined system of equations for finding the auxiliary functions:

$$
\begin{align*}
& \mu_{1 y} \mu_{2}-\mu_{1 x y}-\mu_{1 y}^{2}-\mu_{1} \mu_{1 y y}=0 \\
& \mu_{2}^{2}-\mu_{2 x}-\mu_{2} \mu_{1 y}-\mu_{1} \mu_{2 y}=0 \\
& \mu_{2} \mu_{3}-3 \mu_{1 y y}-3 \mu_{2 y}-\mu_{3 x}-\mu_{1} \mu_{3 y}-\mu_{1 y} \mu_{3}=0 \\
& \mu_{2 y}\left(2 \mu_{1 y}-\mu_{2}\right)=0  \tag{24}\\
& \mu_{2} \mu_{2 y y}-\mu_{2 y}^{2}=0 \\
& 4 \mu_{1 y y y}+6 \mu_{2 y y}+\mu_{2} \mu_{3 y}-2 \mu_{2 y} \mu_{3}+2 \mu_{1 y} \mu_{3 y}=0 \\
& \mu_{2 y y y}+2 \mu_{2 y} \mu_{3 y}-\mu_{2 y y} \mu_{3}-\mu_{2} \mu_{3 y y}=0 \\
& \mu_{3 y y y}+\mu_{3 y}^{2}-\mu_{3} \mu_{3 y y}+3 P(x) \mu_{1 y}+P(x) \mu_{2}+P^{\prime}(x)=0
\end{align*}
$$

The first seven equations of the overdetermined system of Equation (24) are easy to solve in explicit form. To solve them, it is enough to consider three cases: (1) $\mu_{2 y}=0$, $\mu_{2}=\mu_{2}(x) \neq 0$; (2) $\mu_{2}=0$; (3) $2 \mu_{1 y}-\mu_{2}=0$. The solution of the system of Equation (24) for functions $\mu_{1}, \mu_{2}$ and $\mu_{3}$ is not presented. The system of Equation (24) has a solution only for the following functions $P(x)$ :

$$
\begin{align*}
& P(x)=\lambda(x+\tau)^{v}, \quad \lambda \neq 0, \quad v \neq 0 \\
& P(x)=\sigma e^{\gamma x}, \quad \sigma \neq 0, \quad \gamma \neq 0  \tag{25}\\
& P(x)=\sigma_{0} \\
& P(x)=\lambda_{1}\left(x+\tau_{1}\right)^{-5 / 3}+\lambda_{2}\left(x+\tau_{1}\right)^{-1 / 3}
\end{align*}
$$

Here, $\lambda, \tau, v, \sigma, \gamma, \sigma_{0}, \lambda_{1}, \lambda_{2}, \tau_{1}$ and $\tau_{2}$ are arbitrary constants.

### 3.3. The Existence of Non-Symmetry Reductions

Let us find the symmetries of Equation (16). A symmetry operator is sought in the form [2]

$$
X=\xi^{1}(x, y, u) \frac{\partial}{\partial x}+\xi^{2}(x, y, u) \frac{\partial}{\partial y}+\eta(x, y, u) \frac{\partial}{\partial u} .
$$

The system of equations for determining the components of the symmetry operator is given by

$$
\begin{aligned}
& \xi_{y}^{1}=0, \quad \xi_{u}^{1}=0, \quad \xi_{u}^{2}=0, \quad \xi_{x y}^{2}=0, \\
& \eta_{x}=0, \quad \eta_{y}=0, \quad \xi_{x}^{1}-\xi_{y}^{2}-\eta_{u}=0, \\
& \xi^{1} P^{\prime}(x)+\left(3 \xi_{y}^{2}-\eta_{u}\right) P(x)=0 .
\end{aligned}
$$

The symmetries of Equation (16) depend on the form of the specified function $P(x)$. Let us present the result of the group classification:

1. for an arbitrary function $P(x)$, the basis of symmetry operators is given by

$$
X_{1}=b(x) \frac{\partial}{\partial y}, \quad X_{2}=\frac{\partial}{\partial u},
$$

where $b(x)$ is an arbitrary function;
2. for $P(x)=\lambda(x+\tau)^{v}$, where $\lambda \neq 0$ and $v \neq 0$, the basis of symmetry operators is given by

$$
X_{1}, \quad X_{2}, \quad Y_{1}=4(x+\tau) \frac{\partial}{\partial x}+(1-v) y \frac{\partial}{\partial y}+(v+3) u \frac{\partial}{\partial u} ;
$$

3. for $P(x)=\sigma e^{\gamma x}$, where $\sigma \neq 0$ and $\gamma \neq 0$, the basis of symmetry operators is given by

$$
X_{1}, \quad X_{2}, \quad Y_{2}=4 \frac{\partial}{\partial x}-\gamma y \frac{\partial}{\partial y}+\gamma u \frac{\partial}{\partial u} ;
$$

4. for $P(x)=\sigma_{0}$, where $\sigma_{0} \neq 0$, the basis of symmetry operators is given by

$$
X_{1}, \quad X_{2}, \quad Z_{1}=\frac{\partial}{\partial x}, \quad Z_{2}=4 x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}+3 u \frac{\partial}{\partial u}
$$

5. for $P(x)=0$, the basis of symmetry operators is given by

$$
X_{1}, \quad X_{2}, \quad Z_{1}, \quad Z_{3}=x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}, \quad Z_{4}=x \frac{\partial}{\partial x}+u \frac{\partial}{\partial u}
$$

Remark 5. The results of the group classification of Equation (16) correspond to those of the system of equations describing a laminar steady-state plane boundary layer with a pressure gradient [2].

From the group classification, it follows that the last of the types of the function $P(x)$ presented in Formula (25) leads to reductions other than those obtained using symmetries.

In conclusion, we note that the proposed method for finding reductions associated with the construction of a $\mathcal{A}$-system and the introduction of invariant auxiliary functions is general and applicable to PDEs with two independent variables. The proposed method is a modification of the Clarkson-Kruskal direct method and expands the possibilities for its application.

## 4. Reductions of the Unsteady Axisymmetric Boundary Layer Equation

### 4.1. Basic Equation

Let us consider an equation describing the unsteady flow of a viscous incompressible fluid in a laminar boundary layer on the surface of an axisymmetric body of rotation $[8,10]$

$$
\begin{equation*}
u_{y t}-u_{y y y}+u_{y} u_{x y}-\left(u_{x}+\frac{r_{0}^{\prime}(x)}{r_{0}(x)} u\right) u_{y y}-f(x, t)=0 . \tag{26}
\end{equation*}
$$

Equation (26) is written in dimensionless variables. Here $u(x, y, t)$ is a stream function and a given function $f(x, t)=-\partial p / \partial x$ is a pressure gradient. Function $r_{0}(x)$ describes the shape of the streamlined body. For the simplicity of further calculations, it is convenient to use Stepanov-Mangler variables $[8,10,11]$ by using the following formulas:

$$
\begin{equation*}
\bar{x}=\int_{0}^{x} r_{0}^{2}(\tau) d \tau, \quad \bar{y}=r_{0}(x) y, \quad \bar{u}=r_{0}(x) u \tag{27}
\end{equation*}
$$

This reversible transformation of variables maps the equation of a steady-state axisymmetric boundary layer into the equation of a steady-state plane boundary layer [8,10,11].

Applying the transformation of the variables in (27) to Equation (26), we obtain

$$
\frac{1}{r_{0}^{2}(x)} u_{\bar{y} t}-u_{\bar{y} \bar{y} \bar{y}}+u_{\bar{y}} u_{u_{\bar{x}} \bar{y}}-u_{\bar{x}} u_{\bar{y} \bar{y}}-\frac{f(x, t)}{r_{0}^{2}(x)}=0
$$

Expressing $x$ by $\bar{x}$ and omitting the bars, we have

$$
\begin{equation*}
r(x) u_{y t}-u_{y y y}+u_{y} u_{x y}-u_{x} u_{y y}-F(x, t)=0 \tag{28}
\end{equation*}
$$

where $r(\bar{x})=1 / r_{o}^{2}(x(\bar{x})), F(\bar{x}, t)=f(x(\bar{x}), t) / r_{o}^{2}(x(\bar{x}))$. We consider Equation (28) as the basic one. In the following, all two-dimensional and one-dimensional reductions of this equation are found.

Remark 6. Some exact solutions to unsteady axisymmetric boundary layer equations are found in [24,33,34].

Remark 7. Some exact solutions of axisymmetric boundary layer equations for non-Newtonian fluids are found in [24].

### 4.2. Reductions to Simpler PDEs (Two-Dimensional Reductions)

Let us find all two-dimensional reductions of Equation (28), i.e., reductions to a PDE with two independent variables. We are looking for two-dimensional reductions in the following form:

$$
\begin{equation*}
u=U(x, y, t, w(s(x, y, t), q(x, y, t))) . \tag{29}
\end{equation*}
$$

Substituting Expression (29) into Equation (28), we have

$$
\begin{equation*}
-U_{w} s_{y}^{3} w_{s s s}-U_{w} q_{y}^{3} w_{q q q}-3 U_{w w} s_{y}^{3} w_{s} w_{s s}-3 U_{w w} q_{y}^{3} w_{q} w_{q q}+\cdots=0 \tag{30}
\end{equation*}
$$

Here, for simplicity, only four terms are written out.
Next, we divide both parts of Equation (30) by a coefficient with a derivative of $w_{\text {sss }}$, i.e., by $-U_{w} s_{y}^{3}$ (the procedure for normalizing coefficients).

The condition that the resulting equation is a PDE for the function $w(s, q)$ is the dependence of each of the normalized coefficients for derivatives of the function $w(s, q)$ only on the variables $s, q$ and $w$. Consider the normalized coefficient for the term $w_{s} w_{s s}$. This term has the following form:

$$
\begin{equation*}
\frac{3 U_{w w}}{U_{w}}=\Gamma(s, q, w) . \tag{31}
\end{equation*}
$$

Integrating (31) twice by $w$ gives the following:

$$
U(x, y, w)=\beta(x, y, t) \tilde{\Gamma}(s, q, w)+\alpha(x, y, t)
$$

Since an arbitrary function of the variables $s, q$ and $w$ can be taken as a function of $w(s, q)$, then two-dimensional reductions of Equation (28) can be searched in the form of a linear function of $w(s, q)$, so we have

$$
\begin{equation*}
u=\beta(x, y, t) w(s, q)+\alpha(x, y, t) \tag{32}
\end{equation*}
$$

Remark 8. When deriving Expression (31), it was assumed that $s_{y} \neq 0$. If $s_{y}=0$, then with $q_{y} \neq 0$ from Expression (30), similarly, we obtain Form (32). Case $s_{y}^{2}+q_{y}^{2}=0$ corresponds to degenerate reduction that does not contain partial derivatives of the third and second orders. This case is not of particular interest, and its detailed consideration is not given here.

Let us show that one of the independent variables of the function $w(s, q)$ can, without loss of generality, be used independently of the variable $y$. Indeed, considering in Ratio (30) the normalized coefficient for the partial derivative $w_{q q q}$, we obtain $q_{y} / s_{y}=G(s, q)$. We can present the function $G(s, q)$ as $G(s, q)=-\Gamma_{s}(s, q) / \Gamma_{q}(s, q)$. From which it follows that

$$
\begin{equation*}
\Gamma_{s} s_{y}+\Gamma_{q} q_{y}=0 \tag{33}
\end{equation*}
$$

From Expression (33), it follows that $\Gamma(s, q)=h(x, t)$. Let us take, instead of the independent variable $s$, a new independent variable $\bar{s}=\Gamma(s, q)$, i.e., let us move on to the new independent variables $\bar{s}, q$. Then, in Expression (32), we assume $s=s(x, t)$.

Let us now show that it is possible, without loss of generality, to state $\beta=\beta(x, t)$. Substituting Expression (32) into Equation (28), we have

$$
-\beta q_{y}^{3} w_{q q q}+\beta^{2} s_{x} q_{y}^{2}\left(w_{q} w_{s q}-w_{s} w_{q q}\right)+\beta \beta_{y} s_{x} q_{y} w w_{s q}+\cdots=0
$$

Here, for simplicity of recording, only three terms are written out. Considering the ratio of coefficients for derivatives of the function $w$ in the third and second terms, we obtain $\beta_{y} / \beta=q_{y} \Gamma(s, q)$. Whence it follows that $\ln \beta=\tilde{\Gamma}(s, q)+g(x, t)$. Since the function $\beta(x, y, t)$ is defined up to an arbitrary multiplier depending on the variables $s$ and $q$, then we can state $\beta=\beta(x, t)$.

Thus, we have the result that two-dimensional reductions of Equation (28) can be searched in the following form:

$$
\begin{equation*}
u=\beta(x, t) w(s(x, t), q(x, y, t))+\alpha(x, y, t) . \tag{34}
\end{equation*}
$$

Substituting Expression (34) into Equation (28), we have

$$
\begin{align*}
&-\beta q_{y}^{3} w_{q q q}+\beta^{2} s_{x} q_{y}^{2}\left(w_{q} w_{s q}-w_{s} w_{q q}\right)-\beta \beta_{x} q_{y}^{2} w w_{q q}+\beta q_{y}\left[\alpha_{y} s_{x}+r(x) s_{t}\right] w_{s q} \\
&-\beta q_{y}\left[3 q_{y y}-\alpha_{y} q_{x}+\alpha_{x} q_{y}-r(x) q_{t}\right] w_{q q}-\beta^{2} s_{x} q_{y y} w_{s} w_{q} \\
&+\beta\left(\beta q_{y} q_{x y}-\beta q_{x} q_{y y}+\beta_{x} q_{y}^{2}\right) w_{q}^{2}-\beta \beta_{x} q_{y y} w w_{q}-\beta \alpha_{y y} s_{x} w_{s}  \tag{35}\\
&-\left[\beta\left(q_{y y y}-\alpha_{y} q_{x y}+\alpha_{x} q_{y y}+\alpha_{y y} q_{x}\right)-\left(\beta \alpha_{x y}+\beta_{x} \alpha_{y}\right) q_{y}\right. \\
&\left.-r(x)\left(\beta q_{y t}+\beta_{t} q_{y}\right)\right] w_{q} \\
&-\beta_{x} \alpha_{y y} w-\alpha_{y y y}+\alpha_{y} \alpha_{x y}-\alpha_{x} \alpha_{y y}+r(x) \alpha_{y t}-F(x, t)=0 .
\end{align*}
$$

From the condition that Relation (35) is a PDE, we obtain the following overdetermided system of differential equations:

$$
\begin{array}{ll}
\frac{\beta s_{x}}{q_{y}}=\Gamma_{1}(s, q), \quad \frac{\beta_{x}}{q_{y}}=\Gamma_{2}(s, q), \quad \frac{\alpha_{y} s_{x}+r(x) s_{t}}{q_{y}^{2}}=\Gamma_{3}(s, q), \\
\frac{3 q_{y y}-\alpha_{y} q_{x}+\alpha_{x} q_{y}-r(x) q_{t}}{q_{y}^{2}}=\Gamma_{4}(s, q), \quad \frac{\beta s_{x} q_{y y}}{q_{y}^{3}}=\Gamma_{5}(s, q),
\end{array}
$$

$$
\begin{align*}
& \frac{\beta q_{y} q_{x y}-\beta q_{x} q_{y y}+\beta_{x} q_{y}^{2}}{q_{y}^{3}}=\Gamma_{6}(s, q), \quad \frac{\beta_{x} q_{y y}}{q_{y}^{3}}=\Gamma_{7}(s, q), \quad \frac{\alpha_{y y} s_{x}}{q_{y}^{3}}=\Gamma_{8}(s, q),  \tag{36}\\
& \frac{\beta\left(q_{y y y}-\alpha_{y} q_{x y}+\alpha_{x} q_{y y}+\alpha_{y y} q_{x}\right)-\left(\beta \alpha_{x y}+\beta_{x} \alpha_{y}\right) q_{y}-r(x)\left(\beta q_{y t}+\beta_{t} q_{y}\right)}{\beta q_{y}^{3}}=\Gamma_{9}(s, q), \\
& \frac{\beta, \alpha_{y y}}{\beta q_{y}^{3}}=\Gamma_{10}(s, q), \quad \frac{\alpha_{y y y}-\alpha_{y} \alpha_{x y}+\alpha_{x} \alpha_{y y}-r(x) \alpha_{y t}+F(x, t)}{\beta q_{y}^{3}}=\Gamma_{11}(s, q) .
\end{align*}
$$

The system of Equation (36) allows the following transformations:

$$
\begin{array}{lr}
s \rightarrow G_{1}(s), & q \rightarrow G_{2}(s, q) \\
\beta \rightarrow G_{3}(s) \beta, & \alpha \rightarrow \alpha+G_{4}(s, q) \beta, \tag{37}
\end{array}
$$

where $G_{1}(s), G_{2}(s, q), G_{3}(s)$ and $G_{4}(s, q)$ are arbitrary functions. These transformations are associated with arbitrariness in finding a reduction of Form (34). Let us introduce the auxiliary functions $\mu_{1}=\mu_{1}(x, t), \mu_{2}=\mu_{2}(x, y, t), \mu_{3}=\mu_{3}(x, t)$ and $\mu_{4}=\mu_{4}(x, y, t)$, determined from the following relations:

$$
\begin{align*}
& s_{x}-\mu_{1} s_{t}=0, \quad q_{x}-\mu_{2} q_{y}-\mu_{1} q_{t}=0,  \tag{38}\\
& \beta_{x}-\mu_{1} \beta_{t}-\mu_{3} \beta=0, \quad \alpha_{x}-\mu_{2} \alpha_{y}-\mu_{1} \alpha_{t}-\mu_{3} \alpha-\mu_{4}=0 .
\end{align*}
$$

It can be shown that the introduced auxiliary functions are invariants of the transformations in (37).

From the first two relations in (38), it follows that $X s=0, X q=0$, where the operator $X$ has the form

$$
X=\frac{\partial}{\partial x}-\mu_{2} \frac{\partial}{\partial y}-\mu_{1} \frac{\partial}{\partial t} .
$$

By acting with the operator $X$ on the left and right sides of Equation (36), one can obtain a system of equations that does not contain unknown functions $\Gamma_{1}(s, q), \ldots, \Gamma_{11}(s, q)$. Further, in the resulting system of equations, the corresponding partial derivatives of the form $s_{x}$, $q_{x}, \beta_{x}, \alpha_{x}, \ldots$ can be excluded using the relations in (38). So, it is possible to obtain a system of equations containing only auxiliary functions. This system of equations has the form

$$
\begin{align*}
& \mu_{1 x}-\mu_{1} \mu_{2 y}+\mu_{1} \mu_{3}=0, \quad \mu_{1} \mu_{2 y y}=0, \quad \mu_{1} \mu_{4 y y}=0, \\
& \mu_{2 x y}-\mu_{2} \mu_{2 y y}+\mu_{3 x}-\mu_{2 y}^{2}+\mu_{3}^{2}=0, \quad \mu_{3 x}-\mu_{3} \mu_{2 y}+\mu_{3}^{2}=0, \\
& 2 r(x) \mu_{2 y}-\mu_{1} \mu_{4 y}-r(x) \mu_{1 t}-r^{\prime}(x)=0, \quad \mu_{3} \mu_{2 y y}=0, \quad \mu_{3} \mu_{4 y y}=0, \\
& 3 \mu_{2 y y}+\mu_{4 x}-\mu_{4} \mu_{2 y}-\mu_{2} \mu_{4 y}-r(x) \mu_{2 t}+\mu_{3} \mu_{4}=0,  \tag{39}\\
& \mu_{2 y y y}-\mu_{4 x y}+\mu_{4} \mu_{2 y y}+\mu_{2} \mu_{4 y y}-r(x) \mu_{2 y t}-2 \mu_{3} \mu_{4 y}-r(x) \mu_{3 t}=0, \\
& \mu_{4 y y y}+\mu_{4} \mu_{4 y y}-r(x) \mu_{4 y t}-\mu_{4 y}^{2}-3 F(x, t) \mu_{2 y}+F_{x}(x, t) \\
& \quad-\mu_{1} F_{t}(x, t)-F(x, t) \mu_{3}=0 .
\end{align*}
$$

The overdetermined system of Equation (39) is solved explicitly. To solve it, it is enough to consider four cases: (1) $\mu_{1}=\mu_{3}=0$; (2) $\mu_{1}=0, \mu_{3} \neq 0$; (3) $\mu_{1} \neq 0, \mu_{3}=0$; (4) $\mu_{1} \mu_{3} \neq 0$. The system of Equation (39) has a solution only for certain types of functions $r(x)$ and $F(x, t)$. Let us write out these solutions.

1. $\quad r(x)=\alpha_{0}\left(x+\beta_{0}\right)^{\gamma}, F(x, t)=\left(x+\beta_{0}\right)^{2(\gamma+\delta)+1} G\left[(t+\varepsilon)\left(x+\beta_{0}\right)^{\delta}\right]$.

In this case, we have

$$
\begin{array}{lc}
\mu_{1}=\frac{\delta(t+\varepsilon)}{x+\beta_{0}}, & \mu_{2}=\frac{y(\gamma+\delta)}{2\left(x+\beta_{0}\right)} \\
\mu_{3}=\frac{\gamma+\delta+2}{2\left(x+\beta_{0}\right)}, & \mu_{4}=0
\end{array}
$$

Then, from the relations in (38), we find

$$
\begin{array}{ll}
s(x, t)=\left(x+\beta_{0}\right)^{\delta}(t+\varepsilon), & q(x, y, t)=y\left(x+\beta_{0}\right)^{\frac{\gamma+\delta}{2}}, \\
\beta(x, t)=\left(x+\beta_{0}\right)^{\frac{\gamma+\delta}{2}+1}, & \alpha(x, y, t)=0 .
\end{array}
$$

The corresponding two-dimensional reduction has the form:

$$
\begin{aligned}
w_{q q q}+\left(1+\frac{\gamma+\delta}{2}\right) w w_{q q} & -\delta s\left(w_{q} w_{s q}+w_{s} w_{q q}\right)-\alpha_{0} w_{s q} \\
& -(\gamma+\delta+1) w_{q}^{2}+G(s)=0
\end{aligned}
$$

2. $\quad r(x)=\alpha_{0} \exp \left(\beta_{0} x\right), F(x, t)=\exp (2 \gamma x) G\left[(t+\varepsilon) \exp \left(-x\left(\beta_{0}-\gamma\right)\right)\right]$.

In this case, we have

$$
\mu_{1}=\left(\gamma-\beta_{0}\right)(t+\varepsilon), \quad \mu_{2}=\frac{\gamma y}{2}, \quad \mu_{3}=\frac{\gamma}{2}, \quad \mu_{4}=0
$$

Then, from the relations in (38), we find

$$
\begin{aligned}
& \left.s(x, t)=(t+\varepsilon) \exp \left[\left(\gamma-\beta_{0}\right) x\right)\right], \quad q(x, y, t)=y \exp \left(\frac{\gamma x}{2}\right), \\
& \beta(x, t)=\exp \left(\frac{\gamma x}{2}\right), \quad \alpha(x, y, t)=0 .
\end{aligned}
$$

The corresponding two-dimensional reduction has the form:

$$
\begin{aligned}
w_{q q q}-\left(\gamma-\beta_{0}\right) s w_{q} w_{s q} & -\alpha_{0} w_{s q}+\left(\gamma-\beta_{0}\right) s w_{s} w_{q q} \\
& +\frac{\gamma}{2} w w_{q q}-\gamma w_{q}^{2}+G(s)=0 .
\end{aligned}
$$

3. $r(x)=\alpha_{0} \exp \left(\beta_{0} x\right)+\gamma$,

$$
F(x, t)=-\frac{\gamma\left(\alpha_{0} \exp \left(\beta_{0} x\right)+\gamma\right)}{\beta_{0}(t+\varepsilon)^{2}}+G\left[(t+\varepsilon) \exp \left(-\beta_{0} x\right)\right] .
$$

In this case, we have

$$
\mu_{1}=-\beta_{0}(t+\varepsilon), \quad \mu_{2}=0, \quad \mu_{3}=0, \quad \mu_{4}=-\frac{\gamma y}{t+\varepsilon}
$$

Then, from the relations in (38), we find

$$
\begin{aligned}
& s(x, t)=(t+\varepsilon) \exp \left(-\beta_{0} x\right), \quad q(x, y, t)=y, \\
& \beta(x, t)=1, \quad \alpha(x, y, t)=\frac{\gamma y}{\beta_{0}(t+\varepsilon)} .
\end{aligned}
$$

The corresponding two-dimensional reduction has the form:

$$
w_{q q q}-\alpha_{0} w_{s q}+\beta_{0} s\left(w_{q} w_{s q}-w_{s} w_{q q}\right)+G(s)=0 .
$$

4. $r(x)=\alpha_{0}\left(x+\beta_{0}\right)^{-4 / 3}+\gamma\left(x+\beta_{0}\right)^{-2 / 3}$,

$$
F(x, t)=\frac{-3 \alpha_{0} \gamma\left(x+\beta_{0}\right)^{\delta-1 / 3}}{(3 \delta+2)(t+\varepsilon)^{2}}-\frac{9 \alpha_{0}^{2}(\delta+1)}{(3 \delta+2)^{2}\left(x+\beta_{0}\right)^{5 / 3}(t+\varepsilon)^{2}}+\frac{G\left[\left(x+\beta_{0}\right)^{-\delta-2 / 3}\right]}{\left(x+\beta_{0}\right)^{1 / 3}} .
$$

In this case, we have

$$
\begin{aligned}
& \mu_{1}=-\frac{(t+\varepsilon)(3 \delta+2)}{3\left(x+\beta_{0}\right)}, \quad \mu_{2}=-\frac{y}{3\left(x+\beta_{0}\right)}, \\
& \mu_{3}=\frac{2}{3\left(x+\beta_{0}\right)}, \quad \mu_{4}=-\frac{\alpha y(3 \delta+4)}{(3 \delta+2)(t+\varepsilon)\left(x+\beta_{0}\right)^{4 / 3}} .
\end{aligned}
$$

Then, from the relations in (38), we find

$$
\begin{aligned}
& s(x, t)=(t+\varepsilon)\left(x+\beta_{0}\right)^{-\delta-2 / 3}, \quad q(x, y, t)=\frac{y}{\left(x+\beta_{0}\right)^{1 / 3}} \\
& \beta(x, t)=\left(x+\beta_{0}\right)^{2 / 3}, \quad \alpha(x, y, t)=\frac{3 \alpha y}{(3 \delta+2)(t+\varepsilon)\left(x+\beta_{0}\right)^{1 / 3}} .
\end{aligned}
$$

The corresponding two-dimensional reduction has the form:

$$
\begin{aligned}
3 w_{q q q}-3 \gamma(3 \delta+2) w_{s q} & +(3 \delta+2) s\left(w_{q} w_{s q}-w_{s} w_{q q}\right) \\
& +2 w w_{q q}-w_{q}^{2}+3 G(s)=0 .
\end{aligned}
$$

5. $\quad r(x)=\alpha_{0}\left(x+\beta_{0}\right)^{-4 / 3}$,

$$
F(x, t)=G\left[\left(x+\beta_{0}\right)^{1 / 3} \exp \left(\int d t / a(t)\right)\right] \exp \left[\int d t / a(t)\right]+\frac{3 \alpha_{0}^{2}\left(a^{\prime}(t)-1\right)}{\left(x+\beta_{0}\right)^{5 / 3} a(t)^{2}} .
$$

In this case, we have

$$
\begin{array}{ll}
\mu_{1}=\frac{a(t)}{3\left(x+\beta_{0}\right)}, & \mu_{2}=-\frac{y}{3\left(x+\beta_{0}\right)}, \\
\mu_{3}=\frac{2}{3\left(x+\beta_{0}\right)}, & \mu_{4}=\frac{\alpha_{0} y\left(2-a^{\prime}(t)\right)}{a(t)\left(x+\beta_{0}\right)^{4 / 3}} .
\end{array}
$$

Then, from the relations in (38), we find

$$
\begin{aligned}
& s(x, t)=\left(x+\beta_{0}\right)^{1 / 3} \exp \left(\int \frac{d t}{a(t)}\right), \quad q(x, y, t)=\frac{y}{\left(x+\beta_{0}\right)^{1 / 3}} \\
& \beta(x, t)=\left(x+\beta_{0}\right)^{2 / 3}, \quad \alpha(x, y, t)=-\frac{3 \alpha_{0} y}{a(t)\left(x+\beta_{0}\right)^{1 / 3}} .
\end{aligned}
$$

The corresponding two-dimensional reduction has the form:

$$
3 w_{q q q}+s\left(w_{q q} w_{s}-w_{q} w_{s q}\right)+2 w w_{q q}-w_{q}^{2}+3 G(s)=0 .
$$

### 4.3. Reductions to ODEs (One-Dimensional Reductions)

Let us find all one-dimensional reductions of Equation (28), i.e., its reduction to an ODE. We are looking for such one-dimensional reductions in the following form:

$$
\begin{equation*}
u=U(x, y, t, w(z(x, y, t)) . \tag{40}
\end{equation*}
$$

Substituting Expression (40) into Equation (28), we obtain the following:

$$
\begin{equation*}
-U_{w} z_{y}^{3} w^{\prime \prime \prime}-3 U_{w w} z_{y}^{3} w^{\prime} w^{\prime \prime}+\cdots=0 \tag{41}
\end{equation*}
$$

Here, for simplicity, only two terms are written out. Similarly to the above, using the condition that the normalized Ratio (41) is an ODE, we obtain a linear form of one-dimensional reduction

$$
\begin{equation*}
u=\beta(x, y, t) w(z(x, y, t))+\alpha(x, y, t) . \tag{42}
\end{equation*}
$$

Remark 9. When deriving Expression (42), it was assumed that $z_{y} \neq 0$. The case $z_{y}=0$ corresponds to a degenerate reduction. This case is not of particular interest, and its detailed consideration is not given here.

Substituting Expression (42) into Equation (28), we obtain the following equation:

$$
-\beta z_{y}^{3} w^{\prime \prime \prime}+\beta z_{y}\left(\beta_{y} z_{x}-\beta_{x} z_{y}\right) w w^{\prime \prime}+w^{\prime 2}(z) \beta\left(-z_{x} \beta z_{y y}\right.
$$

$$
\begin{align*}
& \left.+z_{y}\left(\beta_{x} z_{y}-\beta_{y} z_{x}+\beta z_{x y}\right)\right)+w w^{\prime}\left(-\beta_{x}\left(2 \beta_{y} z_{y}+\beta z_{y y}\right)-\beta z_{x} \beta_{y y}\right. \\
& \left.+\beta_{y}\left(\beta_{x} z_{y}+\beta_{y} z_{x}+\beta z_{x y}\right)+\beta z_{y} \beta_{x y}\right)+w^{2}\left(\beta_{y} \beta_{x y}-\beta_{x} \beta_{y y}\right) \\
& +w^{\prime \prime}\left(-3 \beta z_{y} z_{y y}+r(x) \beta z_{y} z_{t}-\alpha_{x} \beta z_{y}^{2}+\alpha_{y} \beta z_{y} z_{x}-3 \beta_{y} z_{y}^{2}\right)  \tag{43}\\
& +w^{\prime}\left(-\alpha_{x}\left(2 \beta_{y} z_{y}+\beta z_{y y}\right)-\beta z_{x} \alpha_{y y}+\alpha_{y}\left(\beta_{x} z_{y}+\beta_{y} z_{x}+\beta z_{x y}\right)\right. \\
& \left.-3 \beta_{y} z_{y y}+r(x)\left(\beta_{t} z_{y}+\beta_{y} z_{t}+\beta z_{t y}\right)-3 \beta_{y y} z_{y}+\beta z_{y} \alpha_{x y}-\beta z_{y y y}\right) \\
& +w\left(-\alpha_{x} \beta_{y y}+r(x) \beta_{t y}-\beta_{x} \alpha_{y y}-\beta_{y y y}+\alpha_{y} \beta_{x y}+\beta_{y} \alpha_{x y}\right) \\
& +r(x) \alpha_{t y}-\alpha_{y y y}+\alpha_{y} \alpha_{x y}-\alpha_{x} \alpha_{y y}-F(x, t)=0
\end{align*}
$$

From the condition that Equation (43) is an ODE, we obtain the following overdetermined system of differential equations:

$$
\begin{align*}
& \frac{\beta_{y} z_{x}-\beta_{x} z_{y}}{z_{y}^{2}}=\Gamma_{1}(z) \\
& \frac{-\beta_{x} z_{y}^{2}+z_{y} \beta_{y} z_{x}-z_{y} \beta z_{x y}+z_{x} \beta z_{y y}}{z_{y}^{3}}=\Gamma_{2}(z), \\
& \frac{-\beta_{y} \beta_{x} z_{y}+\beta_{y}^{2} z_{x}+\beta_{y} \beta z_{x y}+\beta z_{y} \beta_{x y}-\beta_{x} \beta z_{y y}-\beta z_{x} \beta_{y y}}{\beta z_{y}^{3}}=\Gamma_{3}(z) \\
& \frac{\beta_{y} \beta_{x y}-\beta_{x} \beta_{y y}}{\beta z_{y}^{3}}=\Gamma_{4}(z), \\
& \frac{r(x) \beta z_{t} z_{y}+\alpha_{y} \beta z_{y} z_{x}-\alpha_{x} \beta z_{y}^{2}-3 \beta_{y} z_{y}^{2}-3 \beta z_{y} z_{y y}}{\beta z_{y}^{3}}=\Gamma_{5}(z),  \tag{44}\\
& \frac{r(x)\left(\beta_{y} z_{t}+\beta_{t} z_{y}+\beta z_{t y}\right)+\alpha_{y}\left(\beta_{x} z_{y}+\beta_{y} z_{x}+\beta z_{x y}\right)+\beta z_{y} \alpha_{x y}}{\beta z_{y}^{3}} \\
& +\frac{-\alpha_{x}\left(2 \beta_{y} z_{y}+\beta z_{y y}\right)-\beta z_{x} \alpha_{y y}-3 \beta_{y y} z_{y}-3 \beta_{y} z_{y y}-\beta z_{y y y}}{\beta z_{y}^{3}}=\Gamma_{6}(z), \\
& \frac{r(x) \beta_{t y}+\alpha_{y} \beta \beta_{x y}+\beta_{y} \alpha_{x y}-\alpha_{x} \beta_{y y}-\beta_{x} \alpha_{y y}-\beta_{y y y}}{\beta z_{y}^{3}}=\Gamma_{7}(z), \\
& \frac{r(x) \alpha_{t y}+\alpha_{y} \alpha_{x y-\alpha_{x} \alpha_{y y}-\alpha_{y y y}-F(x, t)}^{\beta z_{y}^{3}}=\Gamma_{8}(z) .}{}
\end{align*}
$$

Similarly to the case of two-dimensional reduction, the system of Equation (44) admits the following transformations:

$$
\begin{equation*}
z \rightarrow G_{1}(z), \quad \beta \rightarrow \beta G_{2}(z), \quad \alpha \rightarrow \alpha+\beta G_{3}(z) \tag{45}
\end{equation*}
$$

where $G_{1}(z), G_{2}(z)$ and $G_{3}(z)$ are arbitrary functions. These transformations are associated with arbitrariness in finding a reduction of Form (42). Consider auxiliary functions $\mu_{1}=\mu_{1}(x, y, t), \mu_{2}=\mu_{2}(x, y, t), \mu_{3}=\mu_{3}(x, y, t), \mu_{4}=\mu_{4}(x, y, t), \mu_{5}=\mu_{5}(x, y, t)$ and $\mu_{6}=\mu_{6}(x, y, t)$, determined from the following relations:

$$
\begin{align*}
& z_{x}-z_{y} \mu_{1}=0, \quad z_{t}-z_{y} \mu_{2}=0 \\
& \beta_{x}-\beta \mu_{3}-\beta_{y} \mu_{1}=0, \quad \beta_{t}-\beta \mu_{4}-\beta_{y} \mu_{2}=0  \tag{46}\\
& \alpha_{x}-\mu_{5}-\alpha_{y} \mu_{1}-\alpha \mu_{3}=0, \quad \alpha_{t}-\mu_{6}-\alpha_{y} \mu_{2}-\alpha \mu_{4}=0 .
\end{align*}
$$

It can be shown that the introduced auxiliary functions are invariants of the transformations in (45).

Each of the equations of the system of Equation (44) can be rewritten as two equations $\phi_{x} z_{y}-\phi_{y} z_{x}=0, \phi_{t} z_{y}-\phi_{y} z_{t}=0$, where $\phi(x, y, t)$ is the left-hand side of the corresponding equation. From here, we can obtain a system of sixteen equations, which is not given
here due to its bulkiness. Expressing the derivatives of $z_{x}, \alpha_{x}, \beta_{x}, z_{t}, \beta_{t}, \alpha_{t} \ldots$ via the derivatives $z_{y}, \beta_{y}, \alpha_{y} \ldots$ using the relations in (46), it is possible to rewrite the resulting system only in terms of the invariants $\mu_{1} \ldots, \mu_{6}$. To the resulting sixteen equations for $\mu_{1} \ldots, \mu_{6}$, three more compatibility conditions should be added, obtained from the relations $z_{x t}=z_{t x}, \beta_{x t}=\beta_{t x}$ and $\alpha_{x t}=\alpha_{t x}$. Thus, the following overdetermined system of equations consisting of nineteen equations for six unknown functions is subject to solution:

$$
\begin{align*}
& \mu_{1 y} \mu_{3}+\mu_{1} \mu_{3 y}-\mu_{3 x}-\mu_{3}^{2}=0, \quad \mu_{3} \mu_{2 y}-\mu_{3 t}-\mu_{3} \mu_{4}+\mu_{2} \mu_{3 y}=0, \\
& \mu_{1 y}^{2}-\mu_{1 x y}+\mu_{1} \mu_{1 y y}-\mu_{1 y} \mu_{3}=0, \quad \mu_{1 y} \mu_{4}-\mu_{1 y} \mu_{2 y}+\mu_{1 t y}-\mu_{2} \mu_{1 y y}=0, \\
& \mu_{3 y}\left(2 \mu_{1 y}-\mu_{3}\right)=0, \quad \mu_{4 y}\left(2 \mu_{1 y}-\mu_{3}\right)=0, \\
& \mu_{3 y}^{2}-\mu_{3} \mu_{3 y y}=0, \quad \mu_{3 y} \mu_{4 y}-\mu_{3} \mu_{4 y y}=0, \\
& 2 r(x) \mu_{2} \mu_{1 y}-r^{\prime}(x) \mu_{2}-r(x) \mu_{1 t}+\mu_{3} \mu_{5}-\mu_{5} \mu_{1 y}-\mu_{1} \mu_{5 y}+3 \mu_{3 y}+\mu_{5 x}+3 \mu_{1 y y}=0, \\
& 2 r(x) \mu_{2} \mu_{2 y}-r(x) \mu_{2 t}-\mu_{2 y} \mu_{5}-\mu_{2} \mu_{5 y}+3 \mu_{4}+3 \mu_{2 y y}+\mu_{5 t}+\mu_{3} \mu_{6}=0, \\
& 2 r(x) \mu_{1 y} \mu_{4}-r(x) \mu_{2} \mu_{3 y}+2 r(x) \mu_{1 y} \mu_{2 y}-r^{\prime}(x) \mu_{2 y}-r(x) \mu_{1 t y}-2 \mu_{3} \mu_{5 y}+\mu_{1 y y} \mu_{5} \\
& \quad+\mu_{1} \mu_{5 y y}+\mu_{3 y} \mu_{5}-r(x) \mu_{3 t}-r^{\prime}(x) \mu_{4}+\mu_{1 y y y}-\mu_{5 x y}+3 \mu_{3 y y}=0, \\
& r(x) \mu_{2} \mu_{4 y}-2 r(x) \mu_{2 y} \mu_{4}-2 \mu_{2 y} \mu_{5 y}-\mu_{2} \mu_{5 y y}-\mu_{2 y y} \mu_{5}+2 \mu_{1 y} \mu_{6 y}+2 \mu_{3} \mu_{6 y}+\mu_{3 y} \mu_{6} \\
& -2 \mu_{4 y} \mu_{5}-\mu_{2 y y y}-3 \mu_{4 y y}+\mu_{5 t y}+r(x) \mu_{2 t y}+r(x) \mu_{4 t}-2 r(x) \mu_{2 y}^{2}=0,  \tag{47}\\
& \quad-r(x) \mu_{3 y} \mu_{4}+2 r(x) \mu_{1 y} \mu_{4 y}=0,
\end{align*} \begin{array}{r}
\begin{array}{r}
\mu_{3} \mu_{5 y y}-r(x) \mu_{3 t y}-r^{\prime}(x) \mu_{4 y}+\mu_{3 y y y}+\mu_{3 y y} \mu_{5}-2 \mu_{3 y} \mu_{5 y} \\
r(x) \mu_{4} \mu_{4 y}-2 r(x) \mu_{2 y} \mu_{4 y}+r(x) \mu_{4 t y}-\mu_{3} \mu_{6 y y}-\mu_{4 y y y} \\
\quad+\mu_{3 y} \mu_{6 y}-\mu_{4 y y} \mu_{5}+\mu_{4 y} \mu_{5 y}=0,
\end{array} \\
\begin{array}{r}
2 r(x) \mu_{1 y} \mu_{6 y}-r(x) \mu_{3 y} \mu_{6}+\mu_{5 y y y}-\mu_{5 y}^{2}+F_{x}(x, t)-r(x) \mu_{5 t y}+\mu_{5 y y} \mu_{5} \\
\quad-r^{\prime}(x) \mu_{6 y}-3 F(x, t) \mu_{1 y}-F(x, t) \mu_{3}=0,
\end{array} \\
r(x) \mu_{4 y} \mu_{6}-2 r(x) \mu_{2 y} \mu_{6 y}-F_{t}(x, t)-\mu_{6 y y y}+r(x) \mu_{6 t y}+3 F(x, t) \mu_{2 y} \\
\quad-\mu_{6 y y} \mu_{5}+F(x, t) \mu_{4}+\mu_{6 y} \mu_{5 y}=0,
\end{array}
$$

The overdetermined system of Equation (47) is solved explicitly. To solve it, it is enough to consider two cases: (1) $\mu_{3}=\mu_{3}(x, t), \mu_{3} \neq 0$; (2) $\mu_{3}=0$ (it can be shown that $\mu_{3}=\mu_{3}(x, t)$ and $\left.\mu_{4}=\mu_{4}(x, t)\right)$. The system of Equation (47) has a solution only for certain types of functions $r(x)$ and $F(x, t)$. Let us write out these solutions.

1. $\quad r(x)=\alpha_{0}\left(x+\beta_{0}\right), F(x, t)=\gamma(x+a t+b)+\alpha(\alpha x+\beta)$.

In this case, we have

$$
\begin{aligned}
& \mu_{1}=0, \quad \mu_{2}=0 \\
& \mu_{3}=\frac{1}{x+a t+b}, \quad \mu_{4}=\frac{a}{x+a t+b} \\
& \mu_{5}=-\frac{a y(\alpha(a t+b)-\beta}{x+a t+b}, \quad \mu_{6}=\frac{a^{2} y(\alpha * x+\beta)}{x+a t+b} .
\end{aligned}
$$

Then, from the relations in (46), we find

$$
\begin{aligned}
& z(x, y, t)=y, \quad \beta(x, t)=(x+a t+b), \\
& \alpha(x, y, t)=-a y(\alpha x+\beta) .
\end{aligned}
$$

The corresponding one-dimensional reduction has the form:

$$
a \alpha\left(w^{\prime}-z w^{\prime \prime}\right)-w^{\prime 2}+w^{\prime \prime \prime}+w w^{\prime \prime}+\gamma=0 .
$$

2. $\quad r(x)$ is an arbitrary function, $F(x, t)=\alpha(x+\beta)^{\gamma-3}, \gamma \neq 0$.

In this case, we have

$$
\begin{array}{ll}
\mu_{1}=\frac{y(\gamma-4)}{4(x+\beta)}, & \mu_{2}=0 \\
\mu_{3}=\frac{\gamma}{4(x+\beta)}, & \mu_{4}=0 \\
\mu_{5}=0, \quad \mu_{6}=0 . &
\end{array}
$$

Then, from the relations in (46), we find

$$
\begin{aligned}
& z(x, y, t)=y(x+\beta)^{\gamma / 4-1}, \quad \beta(x, t)=(x+\beta)^{\gamma / 4} \\
& \alpha(x, y, t)=0 .
\end{aligned}
$$

The corresponding one-dimensional reduction has the form:

$$
4 w^{\prime 2}+4 w^{\prime \prime \prime}+4 \alpha+\gamma w w^{\prime \prime}-2 \gamma w^{\prime 2}=0
$$

3. $\quad r(x)=\alpha_{0}\left(x+\beta_{0}\right)^{\gamma}, F(x, t)=\delta(x+\beta)^{2 \gamma+1} / t^{2}$.

In this case, we have

$$
\begin{array}{ll}
\mu_{1}=\frac{\gamma y}{2\left(x+\beta_{0}\right)}, & \mu_{2}=-\frac{y}{2 t}, \\
\mu_{3}=\frac{\gamma+2}{2\left(x+\beta_{0}\right)}, & \mu_{4}=-\frac{1}{2 t}, \\
\mu_{5}=0, \quad \mu_{6}=0 . &
\end{array}
$$

Then, from the relations in (46), we find

$$
\begin{aligned}
& z(x, y, t)=y^{2}\left(x+\beta_{0}\right)^{\gamma} / t, \quad \beta(x, y, t)=\left(x+\beta_{0}\right) / y \\
& \alpha(x, y, t)=0 .
\end{aligned}
$$

The corresponding one-dimensional reduction has the form:

$$
\begin{aligned}
z(2+3 \gamma) w w^{\prime}+2 z^{2}(2+\gamma) w w^{\prime \prime} & -4 z^{2}(\gamma+1) w^{\prime 2}+z w^{\prime}\left(\alpha_{0} z+6\right) \\
& +2 \alpha_{0} z^{3} w^{\prime \prime}-6 w+w^{2}+8 w^{\prime \prime \prime} z^{3}=0
\end{aligned}
$$

4. $\quad r(x)=\alpha_{0} \exp \left(\beta_{0} x\right), F(x, t)=\gamma(x+\delta)^{-3+\varepsilon}$.

In this case, we obtain

$$
\begin{array}{ll}
\mu_{1}=\frac{(\varepsilon-4) y}{4(x+\delta)}, & \mu_{2}=0 \\
\mu_{3}=\frac{\varepsilon}{4(x+\delta)}, & \mu_{4}=0 \\
\mu_{5}=0, \quad \mu_{6}=0
\end{array}
$$

Then, from the relations in (46), we find

$$
\begin{aligned}
& z(x, y, t)=y(x+\delta)^{\gamma / 4-1}, \quad \beta(x, y, t)=(x+\delta)^{\varepsilon / 4}, \\
& \alpha(x, y, t)=0 .
\end{aligned}
$$

The corresponding one-dimensional reduction has the form:

$$
2(2-\varepsilon) w^{\prime 2}+\varepsilon w w^{\prime \prime}+4 w^{\prime \prime \prime}+4 \gamma=0
$$

5. $\quad r(x)=\alpha_{0}\left(x+\beta_{0}\right)^{-4 / 3}+\gamma\left(x+\beta_{0}\right)^{\delta}$,

$$
\begin{gathered}
F(x, t)=t^{-2}\left(x+\beta_{0}\right)^{2 \delta+1}\left[\varepsilon-3 \alpha_{0}(2+3 \delta)^{-2}\left(3 \alpha_{0}(\delta+1)\left(x+\beta_{0}\right)^{-8 / 3-2 \delta}\right.\right. \\
\left.\left.+\gamma(2+3 \delta)\left(x+\beta_{0}\right)^{-4 / 3-\delta}\right)\right] .
\end{gathered}
$$

In this case, we obtain

$$
\begin{aligned}
& \mu_{1}=-\frac{\delta y}{2\left(x+\beta_{0}\right)}, \quad \mu_{2}=-\frac{y}{2 t}, \\
& \mu_{3}=\frac{\delta+2}{2\left(x+\beta_{0}\right)}, \quad \mu_{4}=-\frac{1}{2 t}, \\
& \mu_{5}=-\frac{\alpha(4+3 \delta) y}{(2+3 \delta) t\left(x+\beta_{0}\right)^{4 / 3}}, \quad \mu_{6}=0 .
\end{aligned}
$$

Then, from the relations in (46), we find

$$
\begin{aligned}
& z(x, y, t)=\frac{t\left(x+\beta_{0}\right)^{-\delta}}{y^{2}}, \quad \beta(x, y, t)=\frac{\left(x+\beta_{0}\right)}{y} \\
& \alpha(x, y, t)=\frac{3 \alpha_{0} y}{\left(x+\beta_{0}\right)^{1 / 3} t(2+3 \delta)}
\end{aligned}
$$

The corresponding one-dimensional reduction has the form:

$$
\begin{aligned}
2 z^{4}(\delta+2) w w^{\prime \prime} & +z^{3}(\delta+6) w w^{\prime}-4 z^{4}(\delta+1) w^{\prime 2}+3 z^{2}(\gamma-18 z) w^{\prime} \\
& +2 z^{3}(\gamma-24 z) w^{\prime \prime}+z^{2} w^{2}-8 z^{5} w^{\prime \prime \prime}-6 w z^{2}+\varepsilon=0
\end{aligned}
$$

## 5. Conclusions

In this work, a method for constructing reductions of PDEs with two independent variables was considered. The method is based on the idea of invariance. An equation describing a steady-state laminar flat boundary layer with a pressure gradient was considered, and all reductions of this equation were obtained.

Further development of this method for finding reductions, based on the idea of invariance, was proposed for a PDE with three independent variables. The proposed method is a modification of the Clarkson-Kruskal direct method and expands the possibilities for its application. An equation describing a laminar unsteady axisymmetric boundary layer with a pressure gradient was analyzed. All reductions of this equation to ODEs (one-dimensional reductions) and simpler PDEs (two-dimensional reductions) were obtained.

Author Contributions: Conceptualization and methodology, A.V.A.; formal analysis and investigation, A.V.A. and A.A.K.; construction of symmetries and reductions, A.V.A. and A.A.K.; writingoriginal draft preparation, A.V.A. and A.A.K.; writing-review and editing, A.V.A.; supervision, A.V.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.
Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.
Data Availability Statement: Not applicable.
Conflicts of Interest: The authors declare no conflict of interest.

## References

1. Sedov, L.I. Similarity and Dimensional Methods in Mechanics, 10th ed.; CRC Press: Boca Raton, FL, USA, 1993. Ovsiannikov, L.V. Group Analysis of Differential Equations; Academic Press: Boston, MA, USA, 1982.
Barenblatt, G.I. Similarity, Self-Similarity, and Intermediate Asymptotics; Consultants Bureau: New York, NY, USA, 1979. Burde, G.I. A class of solutions of the boundary layer equations. Fluid Dyn. 1990, 25, 201-207.
Clarkson, P.A.; Kruskal, M.D. New similarity reductions of the Boussinesq equation. J. Math. Phys. 1989, 30, $2201-2213$.
Aksenov, A.V.; Kozyrev, A.A. Reductions of Stationary Boundary Layer Equation. Ufa Math. J. 2012, 4, 3-12.
2. Aksenov, A.V.; Kozyrev, A.A. Reductions of the Stationary Boundary Layer Equation with a Pressure Gradient. Dokl. Math. 2013, 87, 236-239.
3. Schlichting, H. Boundary Layer Theory, 7th ed.; McGraw-Hill: New York, NY, USA, 1975.
4. Laminar Boundary Layers; Rosenhead, L., Ed.; Oxford University Press: Oxford, UK, 1963.
5. Loitsyanskiy, L.G. Laminar Boundary Layers; Fizmatlit: Moscow, Russia, 1962. (In Russian)
6. Loitsyanskiy, L.G. Mechanics of Liquids and Gases; Begell House: New York, NY, USA, 1995.
7. Burde, G.I. New similarity reductions of the steady-state boundary layer equations. J. Phys. A Math. Gen. 1996, 29, 1665-1683.
8. Bluman, G.W.; Cole, J.D. The general similarity solution of the heat equation. J. Math. Mech. 1969, 18, 1025-1042.
9. Burde, G.I. The Construction of Special Explicit Solutions of the Boundary-Layer Equations: Steady Flows. Quart. J. Mech. Appl. Math. 1994, 47, 247-260.
10. Ignatovich, N.V. Partially invariant solutions, that are not reducible to invariant ones, of the equations of a steady boundary layer. Math. Notes 1993, 53, 98-100.
11. Pavlovskii, Y.N. Investigation of some invariant solutions to the boundary layer equations. Zhurn. Vychisl. Mat. i Mat. Fiziki 1961, 1, 280-294. (In Russian)
12. Polyanin, A.D. Exact solutions and transformations of the equations of a stationary laminar boundary layer. Theor. Found. Chem. Eng. 2001, 35, 319-328.
13. Polyanin, A.D.; Kutepov, A.M.; Vyazmin, A.V.; Kazenin, D.A. Hydrodynamics, Mass and Heat Transfer in Chemical Engineering; Taylor \& Francis: London, UK, 2002.
14. Polyanin, A.D.; Zaitsev, V.F. Handbook of Nonlinear Partial Differential Equations, 2nd ed.; CRC Press: Boca Raton, FL, USA, 2012.
15. Saccomandi, G. A remarkable class of non-classical symmetries of the steady two-dimensional boundary-layer equations. J. Phys. A Math. Gen. 2004, 37, 7005-7017.
16. Burde, G.I. The construction of special explicit solutions of the boundary-layer equations: Unsteady flows. Quart. J. Mech. Appl. Math. 1995, 48, 611-633.
17. Ludlow, D.K.; Clarkson, P.A.; Bassom, A.P. New similarity solutions of the unsteady incompressible boundary-layer equations. Quart. J. Mech. Appl. Math. 2000, 53, 175-206.
18. Ma, P.K.H.; Hui, W.H. Similarity solutions of the two-dimensional unsteady boundary layer equations. J. Fluid Mech. 1990, 216, 537-559.
19. Polyanin, A.D.; Zhurov, A.I. Separation of Variables and Exact Solutions to Nonlinear PDEs; CRC Press: Boca Raton, FL, USA, 2021.
20. Polyanin, A.D. Transformations and exact solutions containing arbitrary functions for boundary-layer equations. Dokl. Phys. 2001, 46, 526-531.
21. Polyanin, A.D.; Zaitsev, V.F. Equations of an unsteady-state laminar boundary layer: general transformations and exact solutions. Theor. Found. Chem. Eng. 2001, 35, 529-539.
22. Vereshchagina, L.I. Group fibering of the spatial unsteady boundary layer equations. Vestnik LGU 1973, 13, 82-86. (In Russian)
23. Xu, X. Algebraic Approaches to Partial Differential Equations; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2013.
24. Pavlov, K.B. Boundary-layer theory in non-Newtonian nonlinearly viscous media. Fluid Dyn. 1978, 13, 360-366.
25. Shulman, Z.P.; Berkovskii, B.M. Boundary Layer in Non-Newtonian Fluids; Nauka i Tehnika: Minsk, Belarus, 1966. (In Russian)
26. Zaitsev, V.F.; Polyanin, A.D. Exact solutions of the boundary layer equations for power law fluids. Fluid Dyn. 1989, 24, 686-690.
27. Zhizhin, G.V. Laminar boundary layer of non-Newtonian fluid (qualitative investigation). J. Appl. Mech. Techn. Phys. 1987, 3, 383-391.
28. Polyanin, A.D.; Zhurov, A.I. Unsteady axisymmetric boundary-layer equations: transformations, properties, exact solutions, order reduction and solution method. Int. J. Non-Linear Mech. 2015, 74, 40-50.
29. Polyanin, A.D.; Zhurov, A.I. Direct functional separation of variables and new exact solutions to axisymmetric unsteady boundary-layer equations. Commun. Nonlinear Sci. Numer. Simul. 2016, 31, 11-20.
