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Decay property of Timoshenko system in thermoelasticity

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We investigate the decay property of a Timoshenko system of thermoelasticity in the whole space for both Fourier and Cattaneo laws of heat conduction. We point out that although the paradox of infinite propagation speed inherent in the Fourier law is removed by changing to the Cattaneo law, the latter always leads to a solution with the decay property of the regularity-loss type. The main tool used to prove our results is the energy method in the Fourier space together with some integral estimates. We derive L^2 decay estimates of solutions and observe that for the Fourier law the decay structure of solutions is of the regularity-loss type if the wave speeds of the first and the second equations in the system are different. For the Cattaneo law, decay property of the regularity-loss type occurs no matter what the wave speeds are. In addition, by restricting the initial data to $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with a suitably large s and $\gamma \in [0, 1]$, we can derive faster decay estimates with the decay rate improvement by a factor of $t^{-\gamma/2}$. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

Our goal is to investigate the effect of heat conduction on the stability of solutions of Timoshenko systems describing coupled dynamics of elastic wave propagation and thermal dissipation in vibrating beams. The thermal dissipation is described by either Fourier or Cattaneo laws of heat conduction.

We first consider the following coupled set of two wave equations of Timoshenko theory of a vibrating beam [1] with and additional effect of heat conduction according to the Fourier law,

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \lambda\psi_t + \beta\theta_x = 0, \\ \theta_t - \theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \tag{1}$$

where $t \in (0, \infty)$ denotes the time variable and $x \in \mathbb{R}$ is the space variable, the functions φ and ψ denote the displacements of the elastic material, the function θ is the temperature difference, and a, λ, and β are certain positive constants depending on the material elastic and thermal properties.

Initial conditions are of the following form:

$$\begin{cases} \varphi(., 0) = \varphi_0(x), \quad \varphi_t(., 0) = \varphi_1(x), \quad \psi(., 0) = \psi_0(x), \\ \psi_t(., 0) = \psi_1(x), \quad \theta(., 0) = \theta_0(x). \end{cases} \tag{2}$$

The original model of Timoshenko [1] consists of a coupled system of two wave equations describing the transverse vibration of a beam, and it ignores damping effects of any nature. Specifically, Timoshenko derived the following system

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0, L) \times (0, +\infty), \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi)), & \text{in } (0, L) \times (0, +\infty), \end{cases} \tag{3}$$

where t is time and x is the coordinate along the beam length. The function $\varphi = \varphi(t, x)$ is the transverse displacement of the beam from an equilibrium state, and $\psi = \psi(t, x)$ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I, and K are, respectively,

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the density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. For a derivation of Timoshenko's system, we refer the reader to [2].

The subject of stability of Timoshenko-type systems (in bounded domains) has received much attention in the last years, and quite a number of results concerning uniform and asymptotic decay of energy have been established. See for instance [3–13] and references therein.

An important problem in Timoshenko systems is to find a minimum dissipation by which their solutions decay uniformly to zero in time. Several types of dissipative mechanisms have previously been introduced: of frictional type, of viscoelastic type, and of thermal nature. We recall here relevant results on the Timoshenko systems with thermal dissipation. The interested reader is referred to [3, 6, 7, 10, 11, 13] for the Timoshenko systems with frictional damping and to [4, 5, 8, 12] for Timoshenko systems with viscoelastic damping.

The proof of the stability results of the Timoshenko systems in a bounded domain is based on the Poincaré inequality and the type of the boundary conditions. The problem in the whole space $\mathbb{R}$ has been considered in recent papers by Kawashima and his collaborators [14, 15] and by Racke and Said-Houari [16]. We give more details on these papers in the succeeding discussions.

In [14], Ide *et al*. investigated the problem

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - a^2\psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \lambda\psi_t(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{4}$$

where a and λ are positive constants and other variables have the same meaning as before. They proved that if $a = 1$, then solutions of (4) decay as

$$\left\| \partial_x^k U(t) \right\|_2 \leq C(1+t)^{-1/4-k/2} \|U_0\|_1 + Ce^{-ct} \left\| \partial_x^k U_0 \right\|_2, \tag{5}$$

where $U = (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t)^T$. If on the other hand, $a \neq 1$, then decay property of system (4) is of regularity-loss type and solutions decay as

$$\left\| \partial_x^k U(t) \right\|_2 \leq C(1+t)^{-1/4-k/2} \|U_0\|_1 + C(1+t)^{-l/2} \left\| \partial_x^{k+l} U_0 \right\|_2. \tag{6}$$

The parameters k and l in (5) and (6) are non-negative integers, and C and c are positive constants. The work in [14] was followed by [15] in which the above decay results have been generalized to include nonlinear effects as in

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - \sigma(\psi_x)_x(t,x) - (\varphi_x - \psi)(t,x) + \lambda\psi_t(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{7}$$

where $\sigma(\eta) > 0$ is a smooth function of η. The existence of global solutions and the asymptotic decay of these solutions under the smallness condition on the initial data in $H^s \cap L^1$ with suitably large s (in fact, for $s \geq 6$) was proved in that work. In both papers [14, 15], the authors have found the diffusion phenomenon in systems (6) and (7), meaning that as time tends to infinity, the solutions approach the diffusion wave expressed in terms of superposition of heat kernels.

Racke and Said-Houari [16] analyzed the semilinear problem

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - a^2\psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \lambda\psi_t(t,x) = f(\psi(t,x)), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{8}$$

where $f(\psi(t,x)) = |\psi(t,x)|^p$ with $p > 1$. For the linear case (i.e., $f = 0$), they improved the decay results obtained in [14] so that for initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with a suitably large s and $\gamma \in [0,1]$, solutions decay faster than those given in [14]. Further, they analyzed the asymptotic behavior of the semilinear problem (8) with the power type nonlinearity $|u|^p$ with $p > 12$. Using the decay estimates for the linear problem combined with the weighted energy method of Todorova and Yordanov [17] with the special weight given in [18], the authors proved the small-data global existence and obtained optimal decay estimates for the semilinear problem.

To the best of our knowledge, [19] is the first paper treating the Timoshenko system with thermal dissipation. Specifically, the following problem was analyzed:

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0,L) \times (0,+\infty), \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } (0,L) \times (0,+\infty). \end{cases} \tag{9}$$

Under appropriate conditions on σ, ρ_i, b, k, and γ, they proved several exponential decay results for the linearized system and found a non-exponential stability result for the case of different wave speeds (of the linearized system).

In system (9), the heat conduction is given by the classical Fourier law, which assumes the heat flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q + \kappa\nabla\theta = 0, \tag{10}$$

where $\kappa > 0$ is the thermal conductivity. The last equation in (9) corresponds to the classical heat equation if the coupling term proportional to γ is ignored.

As an alternative to the Fourier law, we can consider the Cattaneo law of heat conduction that differs from the Fourier law by the presence of q_t term as in

$$\tau_0 q_t + q + \kappa \nabla\theta = 0, \qquad (\tau_0 > 0, \text{ relatively small})^{\ddagger}. \tag{11}$$

One can express q in terms of θ directly from this equation, but it becomes now a *nonlocal-in-time* relationship,

$$q(t) = -\frac{\kappa}{\tau_0} \int_{-\infty}^{t} \nabla\theta(s)\, e^{(s-t)/\tau_0} ds. \tag{12}$$

As a result, we obtain what is called the system of thermoelasticity of *second sound* wherein the heat transport equation is of hyperbolic type and therefore predicts a finite speed of heat propagation. See [21–23] and references therein for more information on this theory. Clearly, if $\tau_0 = 0$, the Cattaneo law (11) reduces to the classical Fourier law (10) and the hyperbolic system (13) reduces to the classical hyperbolic–parabolic system (9).

Messaoudi *et al.* [24] studied the following Timoshenko system with second sound:

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases} \tag{13}$$

with $(x,t) \in (0,L) \times (0,\infty)$ and the nonlinear function σ is assumed to be sufficiently smooth and to satisfy

$$\sigma_{\varphi_x}(0,0) = \sigma_{\psi}(0,0) = k$$

and

$$\sigma_{\varphi_x\varphi_x}(0,0) = \sigma_{\varphi_x\psi}(0,0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

Green and Naghdi [25, 26] proposed a new law of heat conduction:

$$q + \kappa^* p_x + \tilde{\kappa}\nabla\theta = 0, \quad (\tilde{\kappa} > \kappa^* > 0,\ p_t = \theta),$$

introducing what is called *thermoelasticity of type III*.

The Timoshenko systems in thermoelasticity of type III has recently been studied by Messaoudi and Said-Houari [27, 28]. In [27], the authors investigated the asymptotic behavior of the problem

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, \end{cases} \tag{14}$$

in $(0,\infty) \times (0,1)$ and proved an exponential decay result similar to the one in [19]. The same problem (14) with an additional damping of history type of the form

$$\int_0^{\infty} g(s)\psi_{xx}(x, t-s) ds \tag{15}$$

acting in the second equation has been analyzed in [28]. The authors of [28] proved exponential and polynomial stability results for equal as well as non-equal wave speeds under conditions on the relaxation function g weaker than those in [4, 12].

The second problem we investigate in the present work is the following Timoshenko system in thermoelasticity of second sound

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \beta\theta_x + \lambda\psi_t = 0, \\ \theta_t + \kappa q_x + \beta\psi_{tx} = 0, \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0. \end{cases} \tag{16}$$

where $t \in (0,\infty)$ and $x \in \mathbb{R}$ and $\gamma, \tau_0, \delta, \kappa, \lambda$, and β are positive constants and the following initial conditions are assumed:

$$\begin{cases} \varphi(.,0) = \varphi_0(x), \quad \varphi_t(.,0) = \varphi_1(x), \quad \psi(.,0) = \psi_0(x), \\ \psi_t(.,0) = \psi_1(x), \quad \theta(.,0) = \theta_0(x), \quad q(.,0) = q_0(x). \end{cases} \tag{17}$$

‡ *The constant τ_0 represents the time lag needed to establish the steady state of the heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element. See the survey paper* [20] *for more details.*

As mentioned earlier, the Fourier law of heat conduction results in an infinite speed of propagation of a signal that is physically unsatisfactory. To fix the problem, authors have proposed a number of modifications of the basic assumption on the relation between the heat flux and the temperature, such as the Cattaneo law, the Gurtin and Pipkin theory, the Jeffreys law, and the Green and Naghdi theory. The common feature of these modified theories is that they all lead to a hyperbolic differential equation and the speed of propagation becomes finite. The reader can learn more about these modified models from [29, 30].

The Cattaneo law (11) was proposed by Cattaneo in his celebrated paper [31]. It is perhaps the most obvious and simplest generalization of the Fourier law that gives rise to a finite speed of propagation of heat. Indeed, from the energy balance law,

$$\rho_3\theta_t + \varrho \mathrm{div} q = 0 \tag{18}$$

and (11), we obtain the telegraph equation

$$\rho_3\theta_{tt} - \frac{\varrho\kappa}{\tau_0}\Delta\theta + \frac{\rho_3}{\tau_0}\theta_t = 0, \tag{19}$$

which is a hyperbolic equation and predicts a finite signal speed equals to $(\varrho\kappa/(\rho_3\tau_0))^{1/2}$.

The Cattaneo equation (11) can also be expressed as an integral over the history of the temperature gradient as shown in (12). A more general form of Equation (12) has been given by Gurtin and Pipkin [32]:

$$q(t) = -\int_{-\infty}^{t} a(t-s)\,\nabla\theta(s)\,\mathrm{d}s, \tag{20}$$

where $a(s)$ is the heat flux relaxation function.

Many different constitutive models arise from different choices of $a(s)$. Equation (12) can easily be recovered from (20) by assuming

$$a(s) = \frac{\kappa}{\tau_0}e^{-s/\tau_0}.$$

If we assume that $\nabla\theta$ is constant for all time and let $\kappa = \int_0^\infty a(s)\mathrm{d}s$, then Equation (20) reduces to the classical Fourier law (10). Also, the heat flux law of Jeffreys type

$$q(t) = -\kappa_1\nabla\theta(t) - \frac{\kappa_2}{\tau_0}\int_{-\infty}^{t}\nabla\theta(s)\,e^{(s-t)/\tau_0}\mathrm{d}s,$$

can be seen by letting

$$a(s) = \kappa_1\delta(s) + \frac{\kappa_2}{\tau_0}e^{-s/\tau_0}$$

in (20), where κ_1 and κ_2 are two positive constants and δ is the Dirac function. See [30] for more details.

We should note here that the dissipative effects introduced by the Cattaneo law are usually weaker than those induced by the Fourier law. Yet, replacing the Fourier law by the Cattaneo law is not always beneficial. Recently, Fernández Sare and Racke [33] have shown that it is possible to destroy the stability property, when the heat flux obeys the Cattaneo law. Specifically, they showed that in a bounded domain, system (16) (with $\lambda = 0$) is no longer exponentially stable even when $a = 1$, whereas for $a = 1$, system (1) is exponentially stable. Moreover, they proved that even an additional damping of the form (15) acting on the second equation in (16) is not strong enough to stabilize the solutions exponentially.

We now summarize the main findings of the present work:

- We have extended the decay results of [14] to the Timoshenko system in thermoelasticity.
- We have found optimal decay estimates for weighted initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with a suitably large s and $\gamma \in [0, 1]$. These are obtained for the classical thermoelasticity as well as for the second sound model.
- We contrast the decay results with the Fourier law with the one obtained for the Cattaneo law.

The plan of the paper is as follows. In Section 2, we prove pointwise estimates by transforming our system (1)–(2) to a first-order linear system in the Fourier space and making use of the energy method in the Fourier space. In addition, we derive appropriate decay estimates. Section 3 is devoted to showing the improved decay estimates for $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$, $\gamma \in [0.1]$. In Section 4, we treat the second sound model and obtain optimal decay estimates. Finally, in Section 5, we draw conclusions from the analysis.

Before closing this section, we explain some notations. Throughout this paper, $\|.\|_q$ and $\|.\|_{H^l}$ stand for the $L^q(\mathbb{R})$-norm $(1 \le q \le \infty)$ and the $H^l(\mathbb{R})$-norm, respectively. Also, for $\gamma \in [0, +\infty)$, we define the weighted function space $L^{1,\gamma}(\mathbb{R})$ as follows: $u \in L^{1,\gamma}(\mathbb{R})$ iff $u \in L^1(\mathbb{R})$ and

$$\|u\|_{1,\gamma} = \int_{\mathbb{R}}(1+|x|)^\gamma|u(x)|\mathrm{d}x < +\infty.$$

We also denote by $\hat{f} = \mathcal{F}(f)$ the Fourier transform of f with inverse $\mathcal{F}^{-1}$,

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)\,e^{-i\xi x}\mathrm{d}x.$$

2. Decay estimates

In this section, we prove certain decay estimates of the total energy of system (1)–(2). We employ the technique of converting the original set of second-order equations to a first-order system with a subsequent Fourier transform in the spatial variable and estimation of solutions of the transformed system. The technique is well-known [34] and in the context of Timoshenko systems has previously been used by Kawashima and co-workers [14]. Let us first write system (1)–(2) as a first-order (in time) system of the form

$$\begin{cases} U_t + AU_x + LU = BU_{xx}, \\ U(x,0) = U_0, \end{cases} \tag{21}$$

where A, L, and B are matrices and U is a solution vector identified in the succeeding equations. To obtain system (21), we introduce the following variables:

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t.$$

Consequently, system (1) can be rewritten into the following first-order system of hyperbolic–parabolic type

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \lambda y + \beta\theta_x = 0, \\ \theta_t - \theta_{xx} + \beta y_x = 0, \end{cases} \tag{22}$$

and the initial condition (2) takes the form

$$(v,u,z,y,\theta)(x,0) = (v_0,u_0,z_0,y_0,\theta_0) \tag{23}$$

where

$$v_0 = \varphi_{0,x} - \psi_0, \quad u_0 = \psi_1, \quad z_0 = a\psi_{0,x}, \quad y_0 = \psi_1.$$

System (22)–(23) is equivalent to system (21) with

$$U = \begin{pmatrix} v \\ u \\ z \\ y \\ \theta \end{pmatrix}, \ A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -a & 0 & \beta \\ 0 & 0 & 0 & \beta & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $U_0 = (v_0,u_0,z_0,y_0,\theta_0)^T$. We see that A is real and symmetric and L is non-negative definite, but not symmetric. As a result of the latter, the general theory introduced in [35] is not applicable here. Consequently, the decay estimates of system (21) have to be performed by direct computations using the energy method in the Fourier space together with some integral estimates.

By taking the Fourier transform of (21), we obtain the following Cauchy problem for a first-order system,

$$\begin{cases} \hat{U}_t + i\xi A\hat{U} + L\hat{U} = -\xi^2 B\hat{U}, \\ \hat{U}(\xi,0) = \hat{U}_0(\xi), \end{cases} \tag{24}$$

which can be written out as follows,

$$\begin{cases} \hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \\ \hat{u}_t - i\xi\hat{v} = 0, \\ \hat{z}_t - ai\xi\hat{y} = 0, \\ \hat{y}_t - ai\xi\hat{z} - \hat{v} + \lambda\hat{y} + \beta i\xi\hat{\theta} = 0, \\ \hat{\theta}_t + \xi^2\hat{\theta} + \beta i\xi\hat{y} = 0. \end{cases} \tag{25}$$

Let us now define the following energy functional

$$\mathscr{E}(t) = \frac{1}{2}\left(|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2\right) = \frac{1}{2}\left|\hat{U}(\xi,t)\right|^2. \tag{26}$$

A starting point is, as usual, the dissipativity inequality, which states that the energy $\mathscr{E}(t)$ of the entire system (25) is a non-increasing function, that is, we have the following.

Lemma 2.1
Let $(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta})$ be a solution of (25), then the energy $\mathscr{E}(t)$ is a non-increasing function and satisfies for all $t \geq 0$

$$\begin{aligned}\frac{d\mathscr{E}(t)}{dt} &= -\left(\lambda\left|\hat{y}\right|^2 + \xi^2|\hat{\theta}|^2\right)\\ &\leq -\lambda\left|\hat{y}\right|^2 - \frac{1}{2}\xi^2|\hat{\theta}|^2 - \frac{\xi^2}{2\left(1+\xi^2\right)}|\hat{\theta}|^2.\end{aligned} \tag{27}$$

Proof
Multiplying the first equation in (25) by $\bar{\hat{v}}$, the second equation by $\bar{\hat{u}}$, the third equation by $\bar{\hat{z}}$, the fourth equation by $\bar{\hat{y}}$, and the fifth equation $\bar{\hat{\theta}}$, adding these equalities and taking the real part, (27) follows. □

The following lemma will play a key role in the proof of our main result, and it is similar to proposition 2.1 in [14].

Lemma 2.2
Let $\hat{U}(\xi, t)$ be the solution of (24). Then, for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following pointwise estimates:

$$\left|\hat{U}(\xi,t)\right|^2 \leq \begin{cases} Ce^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2, & \text{if } a=1\\ Ce^{-c\rho_2(\xi)t}\left|\hat{U}(\xi,0)\right|^2, & \text{if } a\neq 1\end{cases} \tag{28}$$

where $\rho_1(\xi) = \xi^2/\left(1+\xi^2\right), \rho_2(\xi) = \xi^2/\left(1+\xi^2\right)^2$ and C and c are two positive constants.

Proof
Our main tool to prove Lemma 2.2 is the energy method in the Fourier space, and we follow the method as employed in [14].
Throughout the paper, we make repeated use of Young's inequality

$$|ab| \leq \epsilon a^2 + C(\epsilon)b^2.$$

Constants $C(\epsilon)$ here and in the sequel denote possibly different values in different places, but are in principle easy to determine.
First, multiplying the first equation in (25) by $i\xi\bar{\hat{u}}$ and the second equation by $i\xi\bar{\hat{v}}$, we obtain

$$i\xi\hat{v}_t\bar{\hat{u}} + \xi^2\left|\hat{u}\right|^2 + i\xi\hat{y}\bar{\hat{u}} = 0$$

and

$$i\xi\hat{u}_t\bar{\hat{v}} + \xi^2\left|\hat{v}\right|^2 = 0,$$

respectively. By subtracting the last two equalities, taking the real part of the result and using the fact that

$$\text{Re}\left\{i\xi\left(\bar{\hat{u}}\hat{v}_t - \hat{u}_t\bar{\hat{v}}\right)\right\} = \left\{\text{Re}\left(i\xi\hat{v}\bar{\hat{u}}\right)\right\}_t,$$

we obtain

$$\left\{\text{Re}\left(i\xi\hat{v}\bar{\hat{u}}\right)\right\}_t + \xi^2\left(\left|\hat{u}\right|^2 - \left|\hat{v}\right|^2\right) + \text{Re}\left(i\xi\hat{y}\bar{\hat{u}}\right) = 0. \tag{29}$$

Similarly, multiplying the fourth equation by $i\xi\bar{\hat{z}}$ and the third equation in (25) by $i\xi\bar{\hat{y}}$, we obtain by the same method as above

$$\left\{\text{Re}\left(i\xi\hat{y}\bar{\hat{z}}\right)\right\}_t + a\xi^2\left(\left|\hat{z}\right|^2 - \left|\hat{y}\right|^2\right) - \text{Re}\left(i\xi\bar{\hat{z}}\left(\hat{v} - \lambda\hat{y} - i\beta\xi\hat{\theta}\right)\right) = 0. \tag{30}$$

Now, we add the equalities (29) and (30) to obtain

$$\begin{aligned}&\xi\frac{d\mathscr{F}(t)}{dt} + \xi^2\left(\left|\hat{u}\right|^2 + a\left|\hat{z}\right|^2\right) - \xi^2\left(\left|\hat{v}\right|^2 + a\left|\hat{y}\right|^2\right)\\ &= -\text{Re}\left(i\xi\hat{y}\bar{\hat{u}}\right) + \text{Re}\left(i\xi\bar{\hat{z}}\left(\hat{v} - \lambda\hat{y} - \beta i\xi\hat{\theta}\right)\right)\end{aligned} \tag{31}$$

where

$$\mathscr{F}(t) = \text{Re}\left(i\hat{v}\bar{\hat{u}} + i\hat{y}\bar{\hat{z}}\right). \tag{32}$$

Using Young's inequality, we obtain for any $\epsilon > 0$

$$-\text{Re}\left(i\xi\hat{y}\bar{\hat{u}}\right) + \text{Re}\left(i\xi\bar{\hat{z}}\left(\hat{v} - \lambda\hat{y}\right)\right) \leq \epsilon\xi^2\left(\left|\hat{u}\right|^2 + a\left|\hat{z}\right|^2\right) + C(\epsilon)\left(\left|\hat{v}\right|^2 + \left|\hat{y}\right|^2\right), \tag{33}$$

and

$$\mathrm{Re}\left(\beta\xi^2\bar{\hat{z}}\hat{\theta}\right) \le \xi^2\epsilon\left|\hat{z}\right|^2 + C(\epsilon)\,\xi^2|\hat{\theta}|^2. \tag{34}$$

Next, plugging the inequalities (33)–(34) into (31), we obtain

$$\begin{aligned}&\xi\frac{\mathrm{d}\mathscr{F}(t)}{\mathrm{d}t} + (1-\epsilon)\,\xi^2\left|\hat{u}\right|^2 + (a-\epsilon(1+a))\,\xi^2\left|\hat{z}\right|^2 - \xi^2\left(\left|\hat{v}\right|^2 + a\left|\hat{y}\right|^2\right)\\ &\le C(\epsilon)\left(1+\xi^2\right)\left(\left|\hat{v}\right|^2 + \left|\hat{y}\right|^2\right) + C(\epsilon)\,\xi^2|\hat{\theta}|^2.\end{aligned}$$

Consequently, the above estimate leads to

$$\begin{aligned}&\frac{\xi}{1+\xi^2}\frac{\mathrm{d}\mathscr{F}(t)}{\mathrm{d}t} + (1-\epsilon)\frac{\xi^2}{1+\xi^2}\left|\hat{u}\right|^2 + (a-\epsilon(1+a))\frac{\xi^2}{1+\xi^2}\left|\hat{z}\right|^2\\ &\le C(\epsilon)\left(\left|\hat{v}\right|^2 + \left|\hat{y}\right|^2\right) + C(\epsilon)\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2.\end{aligned} \tag{35}$$

Multiplying the first equation in (25) by $-\bar{\hat{y}}$ and the fourth equation by $-\bar{\hat{v}}$, we obtain

$$-\hat{v}_t\bar{\hat{y}} + i\xi\hat{u}\bar{\hat{y}} - \left|\hat{y}\right|^2 = 0, \tag{36}$$

and

$$-\hat{y}_t\bar{\hat{v}} + ai\xi\hat{z}\bar{\hat{v}} + \left|\hat{v}\right|^2 - \lambda\hat{y}\bar{\hat{v}} - \beta i\xi\hat{\theta}\bar{\hat{v}} = 0, \tag{37}$$

respectively.

Adding Equations (36) and (37), and taking the real part, we find

$$-\left(\mathrm{Re}\hat{v}\bar{\hat{y}}\right)_t + \mathrm{Re}\left(i\xi\hat{u}\bar{\hat{y}}\right) - \left|\hat{y}\right|^2 + \mathrm{Re}\left(ai\xi\hat{z}\bar{\hat{v}}\right) + \left|\hat{v}\right|^2 - \mathrm{Re}\left(\lambda\hat{y}\bar{\hat{v}} + \beta i\xi\hat{\theta}\bar{\hat{v}}\right) = 0. \tag{38}$$

Similarly, multiplying the second equation in (25) by $-a\bar{\hat{z}}$ and the third equation by $-a\bar{\hat{u}}$, we find

$$-a\bar{\hat{z}}\hat{u}_t + ai\xi\bar{\hat{z}}\hat{v} = 0 \tag{39}$$

and

$$-a\bar{\hat{u}}\hat{z}_t + i\xi a^2\bar{\hat{u}}\hat{y} = 0, \tag{40}$$

respectively.

Adding Equations (39) and (40) and taking the real part, we obtain

$$-\left(\mathrm{Re}a\bar{\hat{z}}\hat{u}\right)_t + \mathrm{Re}\left(ai\xi\bar{\hat{z}}\hat{v}\right) + \mathrm{Re}\left(i\xi a^2\bar{\hat{u}}\hat{y}\right) = 0. \tag{41}$$

Addition of Equations (38) and (41), results in

$$\frac{\mathrm{d}\mathscr{K}(t)}{\mathrm{d}t} - \left|\hat{y}\right|^2 + \left|\hat{v}\right|^2 = \mathrm{Re}\left(\lambda\hat{y}\bar{\hat{v}} + \beta i\xi\hat{\theta}\bar{\hat{v}}\right) + \left(1-a^2\right)\mathrm{Re}\left(i\xi\bar{\hat{u}}\hat{y}\right) \tag{42}$$

where

$$\mathscr{K}(t) = -\mathrm{Re}\left(\hat{v}\bar{\hat{y}} + a\bar{\hat{z}}\hat{u}\right). \tag{43}$$

Applying Young's inequality to the first term in the right-hand side of (42), we obtain

$$\mathrm{Re}\left(\lambda\hat{y}\bar{\hat{v}} + \beta i\xi\hat{\theta}\bar{\hat{v}}\right) \le \epsilon\left|\hat{v}\right|^2 + C(\epsilon)\left|\hat{y}\right|^2 + C(\epsilon)\,\xi^2|\hat{\theta}|^2. \tag{44}$$

Consequently, (42) takes the form

$$\begin{aligned}\frac{\mathrm{d}\mathscr{K}(t)}{\mathrm{d}t} - \left|\hat{y}\right|^2 + \left|\hat{v}\right|^2 \quad &\le \quad \epsilon\left|\hat{v}\right|^2 + C(\epsilon)\left|\hat{y}\right|^2 + C(\epsilon)\,\xi^2|\hat{\theta}|^2\\ &\quad + \left|1-a^2\right|\left|\xi\right|\left|\hat{u}\right|\left|\hat{y}\right|.\end{aligned} \tag{45}$$

Now, we have to distinguish two different cases:

Case 1
$a = 1$.

Let us define the Lyapunov functional $\mathscr{L}$ as

$$\mathscr{L}(t) = N\mathscr{E}(t) + \alpha_1 \frac{\xi}{1+\xi^2}\mathscr{F}(t) + \alpha_2 \mathscr{K}(t), \tag{46}$$

where α_1, α_2, and N are positive constants to be fixed later. From (27), (35), and (45), we obtain

$$\begin{aligned}&\frac{\mathrm{d}\mathscr{L}(t)}{\mathrm{d}t} + \alpha_1 (1-\epsilon) \frac{\xi^2}{1+\xi^2} \left|\hat{u}\right|^2 + \alpha_1 (1-2\epsilon) \frac{\xi^2}{1+\xi^2} \left|\hat{z}\right|^2 \\ &(N\lambda - (\alpha_2+\alpha_1) C(\epsilon)) \left|\hat{y}\right|^2 + (\alpha_2 (1-\epsilon) - \alpha_1 C(\epsilon)) \left|\hat{v}\right|^2 + \left(\frac{N}{2} - \alpha_2 C(\epsilon)\right) \xi^2 |\hat{\theta}|^2 \\ &+ \left(\frac{N}{2} - \alpha_1 C(\epsilon)\right) \frac{\xi^2}{1+\xi^2} |\hat{\theta}|^2 \le 0.\end{aligned} \tag{47}$$

An important step now is to choose the constants ϵ, α_1, α_2, and N so that all the coefficients in (47) are positive. To do so, let us first fix ϵ small enough such that

$$\epsilon < \frac{1}{2}. \tag{48}$$

Next, pick α_1 and α_2 such that

$$\alpha_2 (1-\epsilon) - \alpha_1 C(\epsilon) > 0.$$

Finally, we take N large enough such that

$$N > \max\left\{\frac{(\alpha_2+\alpha_1) C(\epsilon)}{\lambda}, 2\alpha_1 C(\epsilon), 2\alpha_2 C(\epsilon)\right\}.$$

With these choices, (47) takes the form

$$\frac{\mathrm{d}\mathscr{L}(t)}{\mathrm{d}t} + c\mathscr{W}(t) \le 0, \tag{49}$$

where

$$\mathscr{W}(t) = \left|\hat{y}\right|^2 + \left|\hat{v}\right|^2 + \xi^2|\hat{\theta}|^2 + \frac{\xi^2}{1+\xi^2}\left(\left|\hat{z}\right|^2 + \left|\hat{u}\right|^2 + |\hat{\theta}|^2\right)$$

and c is a positive constant.

On the other hand, we have the following simple but key result.

Lemma 2.3
For N large enough there exist three positive constants β_1, β_2, and β_3 such that for all $t \ge 0$, we have

$$\beta_1\mathscr{E}(t) \le \mathscr{L}(t) \le \beta_2\mathscr{E}(t) \quad \text{and} \quad \mathscr{W}(t) \ge \beta_3\rho_1(\xi)\mathscr{E}(t), \tag{50}$$

where $\rho_1(\xi) = \xi^2/\left(1+\xi^2\right)$.

Consequently, from (49) and (50), we can find $\eta > 0$ such that

$$\mathscr{E}(t) = \left|\hat{U}(\xi,t)\right|^2 \le e^{-\eta\rho_1(\xi)}\mathscr{E}(0).$$

Thus, the first inequality in (28) is proved.

Proof of Lemma 2.3
The second estimate in (50) is easily checked. To prove the first estimate, from (46), we have

$$|\mathscr{L}(t) - N\mathscr{E}(t)| = \left|\alpha_1 \frac{\xi}{1+\xi^2}\mathscr{F}(t) + \alpha_2\mathscr{K}(t)\right|. \tag{51}$$

For all $\xi \in \mathbb{R}$, we have $\xi/\left(1+\xi^2\right) < 1$, and exploiting (32) and (43), the right-hand side of (51) can be estimated as

$$\left|\alpha_1 \frac{\xi}{1+\xi^2}\mathscr{F}(t) + \alpha_2\mathscr{K}(t)\right| \le \hat{C}\mathscr{E}(t). \tag{52}$$

From (51) and (52), we obtain

$$(N-\hat{C})\mathscr{E}(t) \leq \mathscr{L}(t) \leq (N+\hat{C})\mathscr{E}(t). \tag{53}$$

This last inequality gives the first estimate in (50) provided that N is sufficiently large. This completes the proof. □

Case 2
$a \neq 1$.

In this case, we estimate the last term in (45) as follows:

$$\left|1-a^2\right| |\xi| |\hat{u}| |\hat{y}| \leq \epsilon \frac{\xi^2}{1+\xi^2} |\hat{u}|^2 + C(\epsilon)\left(1+\xi^2\right)|\hat{y}|^2. \tag{54}$$

Define the Lyapunov functional $\mathscr{V}$ as follows:

$$\mathscr{V}(t) = M\mathscr{E}(t) + \frac{\eta_1}{1+\xi^2}\left(\frac{\xi}{1+\xi^2}\right)\mathscr{F}(t) + \frac{\eta_2}{1+\xi^2}\mathscr{K}(t) \tag{55}$$

where η_1, η_2, and M are positive constants to be chosen later.
Inequality (27) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}\mathscr{E}(t)}{\mathrm{d}t} &\leq -\lambda|\hat{y}|^2 - \frac{1}{2}\xi^2|\hat{\theta}|^2 - \frac{\xi^2}{2\left(1+\xi^2\right)}|\hat{\theta}|^2 \\ &\leq -\lambda|\hat{y}|^2 - \frac{1}{2}\frac{\xi^2}{\left(1+\xi^2\right)^2}|\hat{\theta}|^2 - \frac{\xi^2}{2\left(1+\xi^2\right)}|\hat{\theta}|^2.\end{aligned} \tag{56}$$

Therefore, from (35), (54), (55), and (56), we obtain

$$\begin{aligned}&\frac{\mathrm{d}\mathscr{V}(t)}{\mathrm{d}t} + (1-2\epsilon)\frac{\eta_1\xi^2}{\left(1+\xi^2\right)^2}|\hat{u}|^2 + (a-\epsilon(1+a))\frac{\eta_1\xi^2}{\left(1+\xi^2\right)^2}|\hat{z}|^2 \\ &+ (\lambda M - 2\eta_2 - (\eta_1+\eta_2)C(\epsilon))|\hat{y}|^2 + (\eta_2(1-\epsilon) - \eta_1 C(\epsilon))\frac{1}{1+\xi^2}|\hat{v}|^2 \\ &+ \left(\frac{M}{2} - \eta_1 C(\epsilon)\right)\frac{\xi^2}{\left(1+\xi^2\right)^2}|\hat{\theta}|^2 + \left(\frac{M}{2} - \eta_2 C(\epsilon)\right)\frac{\xi^2}{\left(1+\xi^2\right)}|\hat{\theta}|^2 \leq 0,\end{aligned} \tag{57}$$

where we have used the fact that $1/\left(1+\xi^2\right) \leq 1$.
Now, we fix our constants in (57). First, we choose ϵ as in (48) and η_1, η_2 such that

$$\eta_2(1-\epsilon) - \eta_1 C(\epsilon) > 0.$$

Once the above constants are fixed, we choose M large enough such that

$$M > \max\left(2\eta_1 C(\epsilon), 2\eta_2 C(\epsilon), \frac{2\eta_2 + (\eta_1+\eta_2)C(\epsilon)}{\lambda}\right).$$

Consequently, the estimate (57) takes the form

$$\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}t} + \tilde{c}\tilde{\mathscr{W}} \leq 0, \tag{58}$$

where $\tilde{c}$ is a positive constant and

$$\tilde{\mathscr{W}} = |\hat{y}|^2 + \frac{1}{1+\xi^2}|\hat{v}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2 + \frac{\xi^2}{\left(1+\xi^2\right)^2}\left(|\hat{z}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2\right).$$

It is clear that for M large enough, there exist three positive constants $\tilde{\beta}_1$, $\tilde{\beta}_2$, and $\tilde{\beta}_3$ such that for all $t \geq 0$, we have

$$\tilde{\beta}_1\mathscr{E}(t) \leq \mathscr{V}(t) \leq \tilde{\beta}_2\mathscr{E}(t) \quad \text{and} \quad \tilde{\mathscr{W}}(t) \geq \tilde{\beta}_3\rho_2(\xi)\mathscr{E}(t), \tag{59}$$

where $\rho_2(\xi) = \xi^2/\left(1+\xi^2\right)^2$. Consequently, the proof of the second inequality in (28) follows by the same method as in the case $a=1$. Thus, the proof of Lemma 2.2 is complete. □

Our first decay estimates can be stated as follows:

Theorem 2.1
Let s be a non-negative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution U of problem (21) satisfies the following decay estimates:

- When $a = 1$,

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2. \tag{60}$$

- When $a \neq 1$,

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\left\|\partial_x^{k+l} U_0\right\|_2, \tag{61}$$

where k and l are non-negative integers satisfying $k + l \leq s$ and C, c are two positive constants.

Remark 2.1
The decay estimate (61) indicates that for $a \neq 1$, the decay property of system (1)–(2) is of the regularity-loss type, which means that we have the optimal decay estimates of solutions under the additional regularity assumption on the initial data. This decay structure of regularity-loss type appears also for a hyperbolic–elliptic system related to a radiating gas [36, 37].

Proof of Theorem 2.1.
Using the Fourier transform, the proof of Theorem 2.1 is reduced to the analysis of the behavior of the spectral parameter ξ in the high-frequency and low-frequency regions. The proof is essentially based on the pointwise estimates in Lemma 2.2.

First, suppose that $a = 1$. Then, applying the Plancherel theorem and making use of the first inequality in (28), we obtain

$$\begin{aligned}
\left\|\partial_x^k U(t)\right\|_2^2 &= \int_{\mathbb{R}} |\xi|^{2k}\left|\hat{U}(\xi,t)\right|^2 \mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi \\
&= C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi + C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi \\
&= I_1 + I_2.
\end{aligned} \tag{62}$$

The integral here is split into two parts: the low-frequency part ($|\xi| \leq 1$) and the high-frequency part ($|\xi| \geq 1$). As $\rho_1(\xi) \geq \frac{1}{2}\xi^2$ for $|\xi| \leq 1$, then we have for the low-frequency part that

$$\begin{aligned}
I_1 &= C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi \\
&\leq C\left\|\hat{U}_0\right\|_{L^\infty}^2 \int_{|\xi|\leq 1} |\xi|^{2k} e^{-\frac{c}{2}\xi^2 t}\mathrm{d}\xi.
\end{aligned} \tag{63}$$

By using the inequality

$$\int_0^1 |\xi|^\sigma e^{-\frac{c}{2}\xi^2 t}\mathrm{d}\xi \leq C(1+t)^{-(\sigma+1)/2}, \tag{64}$$

we deduce from (63) that

$$I_1 \leq C(1+t)^{-1/2-k}\|U_0\|_{L^1}^2. \tag{65}$$

For the high-frequency part, I_2, on the other hand, we have $\rho_1(\xi) \geq \frac{1}{2}$, and therefore,

$$\begin{aligned}
I_2 &= C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi \\
&\leq Ce^{-\frac{c}{2}t}\int_{|\xi|\geq 1} |\xi|^{2k}\left|\hat{U}(\xi,0)\right|^2 \mathrm{d}\xi \\
&\leq Ce^{-ct}\left\|\partial_x^k U_0\right\|_2^2.
\end{aligned} \tag{66}$$

Combining (62), (65), and (66), we obtain the estimate (60).

Second, assume that $a \neq 1$. As above,

$$\begin{aligned} \left\| \partial_x^k U(t) \right\|_2^2 &= \int_{\mathbb{R}} |\xi|^{2k} \left| \hat{U}(\xi,t) \right|^2 \mathrm{d}\xi \\ &\leq C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_2(\xi)t} \left| \hat{U}(\xi,0) \right|^2 \mathrm{d}\xi \\ &= C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} \left| \hat{U}(\xi,0) \right|^2 \mathrm{d}\xi + C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} \left| \hat{U}(\xi,0) \right|^2 \mathrm{d}\xi \\ &= L_1 + L_2. \end{aligned} \tag{67}$$

Now, using the second estimate in (28) and the fact that $\rho_2(\xi) \geq \frac{1}{4}\xi^2$ for $|\xi| \leq 1$, we have by the same method as in the proof of the estimate of I_1,

$$L_1 \leq C(1+t)^{-1/2-k} \|U_0\|_{L^1}^2 . \tag{68}$$

To estimate the term L_2, we use the inequality $\rho_2(\xi) \geq c\xi^{-2}$ for $|\xi| \geq 1$ to obtain

$$\begin{aligned} L_2 &= C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} \left| \hat{U}(\xi,0) \right|^2 \mathrm{d}\xi \\ &\leq C \sup_{|\xi|\geq 1} \left(|\xi|^{-2l} e^{-c\xi^{-2}t} \right) \int_{|\xi|\geq 1} |\xi|^{2(k+l)} \left| \hat{U}(\xi,0) \right|^2 \mathrm{d}\xi \\ &\leq C(1+t)^{-l} \left\| \partial_x^{k+l} U_0 \right\|_2^2 . \end{aligned} \tag{69}$$

Inserting the estimates (68) and (69) into (67), (61) is obtained. This completes the proof of Theorem 2.1. □

3. Improved decay estimates

In this section, we improve the decay results obtained in Theorem 2.1, namely, by restricting the initial data to $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with a sufficiently large s and $\gamma \in [0,1]$, we can derive faster decay estimates than those given in Theorem 2.1. Indeed, by using the pointwise estimates derived in Lemma 2.2 and adapting the method introduced by Ikehata in [38], to treat the Fourier transform in the low-frequency region, we can improve the decay rate given in Theorem 2.1 by $t^{-\gamma/2}$, $\gamma \in [0,1]$, especially in the case of equal wave speeds, that is, $a = 1$.

Our main result in this section is as follows.

Theorem 3.1

Let $\gamma \in [0,1]$. Let s be a non-negative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$. Then, the solution U of problem (21) satisfies the following decay estimates:

- When $a = 1$, we have

$$\begin{aligned} \left\| \partial_x^k U(t) \right\|_2 \leq\ & C(1+t)^{-1/4-k/2-\gamma/2} \|U_0\|_{L^{1,\gamma}} + Ce^{-ct} \left\| \partial_x^k U_0 \right\|_2 \\ & + C(1+t)^{-1/4-k/2} \left| \int_{\mathbb{R}} U_0(x)\, \mathrm{d}x \right| . \end{aligned} \tag{70}$$

- When $a \neq 1$, we have

$$\begin{aligned} \left\| \partial_x^k U(t) \right\|_2 \leq\ & C(1+t)^{-1/4-k/2-\gamma/2} \|U_0\|_{L^{1,\gamma}} + C(1+t)^{-l/2} \left\| \partial_x^{k+l} U_0 \right\|_2 \\ & + C(1+t)^{-1/4-k/2} \left| \int_{\mathbb{R}} U_0(x)\, \mathrm{d}x \right| , \end{aligned} \tag{71}$$

where k and l are non-negative integers satisfying $k + l \leq s$ and C, c are two positive constants.

Remark 3.1

Theorem 3.1 shows that by taking the initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ such that $\int_{\mathbb{R}} U_0(x)\, \mathrm{d}x = 0$, the decay rates given in Theorem 2.1 can be improved by $t^{-\gamma/2}$, $\gamma \in [0,1]$.

Proof of Theorem 3.1.

The proof of Theorem 3.1 is reduced to the analysis of the behavior of the Fourier transform of the solution in the low-frequency range, $|\xi| < 1$. The high-frequency part has an exponential decay rate for $a = 1$. Indeed, assume first that $a = 1$ and recall the identity (62), that is,

$$\begin{aligned} \left\| \partial_x^k U(t) \right\|_2^2 &= \int_{\mathbb{R}} |\xi|^{2k} \left| \hat{U}(\xi,t) \right|^2 \mathrm{d}\xi \\ &= I_1 + I_2, \end{aligned} \tag{72}$$

where I_1 and I_2 are defined in (63) and (66), respectively. We have the following estimate for the low-frequency part, I_1.

Lemma 3.1
Let $\gamma \in [0,1]$. Then,

$$I_1 \leq C\,(1+t)^{-1/2-(k+\gamma)}\,\|U_0\|^2_{1,\gamma} + C\,(1+t)^{-1/2-k}\left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right|^2. \tag{73}$$

Proof
From (63), we have

$$I_1 = C\int_{|\xi|\leq 1} |\xi|^{2k}\,e^{-c\rho_1(\xi)t}\left|\hat{U}_0\,(\xi)\right|^2 \mathrm{d}\xi.$$

Using [38, Lemma 3.1], the term $\left|\hat{U}_0\,(\xi)\right|$ can be estimated as follows:

$$\begin{aligned}\left|\hat{U}_0\,(\xi)\right| &= \left|\int_{\mathbb{R}} e^{-ix\xi}U_0\,(x)\,\mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}} |\cos\,(x\xi)-1|\,|U_0\,(x)|\,\mathrm{d}x + \int_{\mathbb{R}} |\sin\,(x\xi)|\,|U_0\,(x)|\,\mathrm{d}x + \left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right|.\end{aligned}$$

Because

$$\begin{cases} K_\gamma = \sup_{\theta\neq 0}\dfrac{|1-\cos\theta|}{|\theta|^\gamma} < +\infty, \\ M_\gamma = \sup_{\theta\neq 0}\dfrac{\sin\theta}{|\theta|^\gamma} < +\infty, \end{cases}$$

for $0 \leq \gamma \leq 1$, then we deduce

$$\left|\hat{U}_0\,(\xi)\right| \leq C_\gamma\,|\xi|^\gamma\,\|U_0\|_{1,\gamma} + \left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right| \tag{74}$$

with $C_\gamma = K_\gamma + M_\gamma$.

Therefore, using (74), we obtain

$$I_1 \leq C\,\|U_0\|^2_{1,\gamma}\int_{|\xi|\leq 1} |\xi|^{2(k+\gamma)}\,e^{-c\rho_1(\xi)t}\mathrm{d}\xi + \left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right|^2\int_{|\xi|\leq 1}|\xi|^{2k}\,e^{-c\rho_1(\xi)t}\mathrm{d}\xi. \tag{75}$$

Because in the low-frequency part, $|\xi| \leq 1$, we have $\rho_1\,(\xi) = \xi^2/\left(1+|\xi|^2\right) \geq |\xi|^2/2$, then (75) becomes

$$I_1 \leq C\,\|U_0\|^2_{1,\gamma}\int_{|\xi|\leq 1} |\xi|^{2(k+\gamma)}\,e^{-c_1|\xi|^2 t}\mathrm{d}\xi + \left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right|^2\int_{|\xi|\leq 1}|\xi|^{2k}\,e^{-c_1|\xi|^2 t}\mathrm{d}\xi$$

where $c_1 = c/2$. By exploiting inequality (64), we arrive at

$$I_1 \leq C(1+t)^{-(k+\gamma)-1/2}\,\|U_0\|^2_{1,\gamma} + (1+t)^{-k-1/2}\left|\int_{\mathbb{R}} U_0\,(x)\,\mathrm{d}x\right|^2. \tag{76}$$

Consequently, the result of Lemma 3.1 holds from (74) and (76). This completes the proof of Lemma 3.1. □

To complete the proof of (70) in Theorem 3.1, it suffices to combine (66) and (73). By the same method, we can show that the estimate (71) also holds, but we omit the details. Thus, the proof of Theorem 3.1 is now complete. □

4. The second sound model

In this section, we consider the system (16)–(17) and our goal now is to study the asymptotic stability of the solutions.

By introducing the following variables,

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad w = \tau_0 q.$$

System (16) can be rewritten as the following first-order hyperbolic system

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \lambda y + \beta\theta_x = 0, \\ \theta_t + \dfrac{\kappa}{\tau_0} w_x + \beta y_x = 0, \\ w_t + \dfrac{\delta}{\tau_0} w + \kappa\theta_x = 0 \end{cases} \tag{77}$$

and the initial conditions (17) take the form

$$(v, u, z, y, \theta, w)\,(x, 0) = (v_0, u_0, z_0, y_0, \theta_0, w_0) \tag{78}$$

where

$$v_0 = \varphi_{0,x} - \psi_0, \quad u_0 = \psi_1, \quad z_0 = a\psi_{0,x}, \quad y_0 = \psi_1, \quad w_0 = \tau_0 q_0.$$

System (77)–(78) is equivalent to

$$\begin{cases} U_t + AU_x + LU = 0, \\ U\,(x, 0) = U_0. \end{cases} \tag{79}$$

where

$$U = \begin{pmatrix} v \\ u \\ z \\ y \\ \theta \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta & 0 & \kappa/\tau_0 \\ 0 & 0 & 0 & 0 & \kappa & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta/\tau_0 \end{pmatrix} \tag{80}$$

and $U_0 = (v_0, u_0, z_0, y_0, \theta_0, w_0)^T$. Notice that L is a non-negative definite matrix and that A is not in general symmetric.

System (79) can be seen as a particular case of a general hyperbolic system of balance laws. We point out that Shizuta and Kawashima [39] introduced the so-called algebraic condition (SK), namely

(SK) $\quad \forall \xi \in \mathbb{R} - \{0\}, \quad Ker(L) \cap \{\text{eigenvectors of } (\xi A)\} = \{0\},$

which is satisfied in many examples and sufficient to establish a general result of global existence for small perturbations of constant-equilibrium state. Our system (79) satisfies the condition (SK), but the general theory on the dissipative structure established in [39] is not applicable as the matrices A and L are not real symmetric. Consequently, to treat the global existence and asymptotic stability of (79), new ideas are required. See [15] for more details.

As a side note, we remark that Beauchard and Zuazua [40] have recently shown that the condition (SK) is equivalent to the classical Kalman rank condition in control theory for the pair (A, L), that is,

$$rk[L, \tilde{A}(i\xi)L, \dots, \tilde{A}(i\xi)^{N-1}L] = N,$$

where $\tilde{A}(i\xi) = i\xi A$ is an $N \times N$ matrix.

If we take the Fourier transform of (79), we obtain the following Cauchy problem for a first-order system

$$\begin{cases} \hat{U}_t + i\xi A\hat{U} + L\hat{U} = 0, \\ \hat{U}\,(\xi, 0) = \hat{U}_0. \end{cases} \tag{81}$$

Solving this equation, we find

$$\hat{U}(\xi, t) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0(\xi),$$

where

$$\hat{\Phi}\,(i\xi) = -\,(i\xi A + L)\,. \tag{82}$$

The solution of (79) is then given by

$$U\,(x, t) = e^{t\Phi}U_0\,(x)\,,$$

where

$$\left(e^{t\Phi}\omega\right)(x) = \mathcal{F}^{-1}\left[e^{t\hat{\Phi}(i\xi)}\hat{\omega}(\xi)\right](x).$$

In what follows, we will show that there exists a function $\rho(\xi)$ and a positive constant C such that

$$|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-\rho(\xi)t}.$$

Thus, the behavior of the solution of system (79) will depend in a crucial way on the behavior of the function $\rho(\xi)$ in high-frequency and low-frequency regions. As we have seen in Lemma 2.2, the function $\rho(\xi)$ may

- take the form of $\rho_1(\xi)$ and the behavior of the solution in this case will be completely different depending on the behavior of $\rho_1(\xi)$ near $\xi = 0$, as the high-frequency part decays exponentially.
- take the form of $\rho_2(\xi)$ and in this case the dissipation will be very weak in the high-frequency region and will result in a solution with decay property of regularity-loss type.

System (81) can be rewritten into the following form:

$$\begin{cases} \hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \\ \hat{u}_t - i\xi\hat{v} = 0, \\ \hat{z}_t - ai\xi\hat{y} = 0, \\ \hat{y}_t - ai\xi\hat{z} - \hat{v} + \lambda\hat{y} + \beta i\xi\hat{\theta} = 0, \\ \hat{\theta}_t + \dfrac{\kappa}{\tau_0}i\xi\hat{w} + \beta i\xi\hat{y} = 0, \\ \hat{w}_t + \dfrac{\delta}{\tau_0}\hat{w} + \kappa i\xi\hat{\theta} = 0. \end{cases} \tag{83}$$

Let us now define the following energy functional

$$\begin{aligned} \hat{\mathscr{E}}(t) &= \frac{1}{2}\left(\left|\hat{v}\right|^2 + \left|\hat{u}\right|^2 + \left|\hat{z}\right|^2 + \left|\hat{y}\right|^2 + |\hat{\theta}|^2 + \left|\hat{w}\right|^2\right) \\ &= \frac{1}{2}|\hat{U}|_2^2. \end{aligned} \tag{84}$$

Note that if $-\hat{\Phi}$ is a coercive matrix, then system (81) will be entirely dissipative, and it is easy in this case to prove some pointwise estimates for the Fourier image $\hat{U}(\xi, t)$. However, we have

$$\mathrm{Re}\left(\hat{U}^T\hat{\Phi}(i\xi)\hat{U}\right) = -\mathrm{Re}\left(\hat{V}^TD\hat{V}\right),$$

where

$$\hat{V} = (\hat{y}, \hat{w})^T \text{ and } D = \begin{pmatrix} \lambda & 0 \\ 0 & \delta/\tau_0 \end{pmatrix}.$$

Therefore, the above quadratic form on $\hat{V}$ does not provide any information on the vector $\hat{W} = (\hat{v}, \hat{u}, \hat{z}, \hat{\theta})^T$. In fact, this is the case for a large class of hyperbolic systems. In our case, we will show that the interaction of the dissipation of the components in $\hat{V}$ with the time dynamics may eventually dissipate the other components in $\hat{W}$.

The next lemma states that the energy $\hat{\mathscr{E}}(t)$ of the entire system (16)–(17) is a non-increasing function. More precisely, we have the following result.

Lemma 4.1
Let $(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{w})$ be a solution of (83). Then, the energy $\hat{\mathscr{E}}(t)$ is a non-increasing function and satisfies for all $t \ge 0$

$$\frac{\mathrm{d}\hat{\mathscr{E}}(t)}{\mathrm{d}t} = -\lambda\left|\hat{y}\right|^2 - \frac{\delta}{\tau_0}\left|\hat{w}\right|^2. \tag{85}$$

Proof
The proof of Lemma 4.1 can be performed along the same line as the proof of Lemma 2.1. It can be also proved in a more general context. Indeed, let us first remark that the matrix L defined in (80) can be written as the sum of a symmetric matrix L_1 and a skew-symmetric matrix L_2. On the other hand, multiplying the first equation in (79) by $\overline{\hat{U}}^T$, we obtain

$$\overline{\hat{U}}^T\hat{U}_t = \overline{\hat{U}}^T\hat{\Phi}(i\xi)\hat{U}.$$

We have also the identity

$$\hat{U}_t^T\overline{\hat{U}} = \hat{U}^T\hat{\Phi}^T(i\xi)\overline{\hat{U}}.$$

Adding the above two identities, taking the real part and using the fact that A and L_1 are symmetric, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\hat{U}(t)|_2^2 &= \overline{\hat{U}}^T\hat{\Phi}(i\xi)\hat{U}+\hat{U}^T\hat{\Phi}^T(i\xi)\overline{\hat{U}}\\ &= -\overline{\hat{U}}^T L_1\overline{\hat{U}} = -\lambda|\hat{y}|^2-\frac{\delta}{\tau_0}|\hat{w}|^2.\end{aligned}$$

This completes the proof of Lemma 4.1. □

The following lemma will play an essential role in the proof of our main result.

Lemma 4.2
Let $\hat{\Phi}(i\xi) = -(i\xi A + L)$ be the matrix defined in (82). Then, for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following estimate:

$$\left|e^{t\hat{\Phi}(i\xi)}\right| \leq Ce^{-c\rho_2(\xi)t} \tag{86}$$

where $\rho_2(\xi) = \xi^2/\left(1+\xi^2\right)^2$ and C, c are two positive constants.

Proof
Note that here we do not need to distinguish between $a = 1$ and $a \neq 1$ as in both cases the decay property of the solutions of (79) is of regularity loss type. To prove Lemma 4.2, we apply the energy method to the problem (83) formulated in the Fourier space and derive pointwise estimates of solutions to (83).

Following the same computations as in Lemma 2.2 (up to estimate (35)), we obtain for any $\epsilon > 0$,

$$\begin{aligned}&\xi\frac{\mathrm{d}\hat{\mathscr{F}}(t)}{\mathrm{d}t}+(1-\epsilon)\xi^2|\hat{u}|^2+(a-\epsilon(a+1))\xi^2|\hat{z}|^2\\ &\leq C(\epsilon)\left(1+\xi^2\right)\left(|\hat{v}|^2+|\hat{y}|^2\right)+C(\epsilon)\xi^2|\hat{\theta}|^2,\end{aligned} \tag{87}$$

where

$$\hat{\mathscr{F}}(t) = \mathrm{Re}\left(i\hat{v}\bar{\hat{u}}+i\hat{y}\bar{\hat{z}}\right). \tag{88}$$

Also, multiplying the fifth equation in (83) by $i\xi\bar{\hat{w}}$, and the sixth equation by $i\xi\bar{\hat{\theta}}$, we find by the same method

$$\xi\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{N}}(t)+\xi^2\left(\kappa|\hat{\theta}|^2-\frac{\kappa}{\tau_0}|\hat{w}|^2\right)-\xi^2\beta\mathrm{Re}\left(\bar{\hat{w}}\hat{y}\right)-\frac{\delta}{\tau_0}\mathrm{Re}\left(i\xi\bar{\hat{\theta}}\hat{w}\right)=0, \tag{89}$$

where

$$\hat{\mathscr{N}}(t) = -\mathrm{Re}\left(i\hat{w}\bar{\hat{\theta}}\right). \tag{90}$$

Young's inequality leads to

$$\xi^2\beta\mathrm{Re}\left(\bar{\hat{w}}\hat{y}\right) \leq \xi^2\epsilon|\hat{y}|^2+\xi^2C(\epsilon)|\hat{w}|^2. \tag{91}$$

Similarly,

$$\frac{\delta}{\tau_0}\mathrm{Re}\left(i\xi\bar{\hat{\theta}}\hat{w}\right) \leq \epsilon\xi^2|\hat{\theta}|^2+C(\epsilon)|\hat{w}|^2. \tag{92}$$

Next, inserting the inequalities (91) and (92) into (89), we obtain

$$\begin{aligned}&\xi\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{N}}(t)+\kappa\xi^2|\hat{\theta}|^2\\ &\leq \epsilon\xi^2|\hat{\theta}|^2+C(\epsilon)\left(1+\xi^2\right)\left(|\hat{w}|^2+|\hat{y}|^2\right).\end{aligned} \tag{93}$$

Also, repeating the computations in Lemma 2.2 (the estimates from (36) up to (44)) and making use of (54), we obtain for any $\epsilon, \tilde{\epsilon} > 0$

$$\frac{\mathrm{d}\hat{\mathscr{K}}(t)}{\mathrm{d}t}-|\hat{y}|^2+|\hat{v}|^2 \leq \epsilon|\hat{v}|^2+C(\epsilon,\tilde{\epsilon})\left(1+\xi^2\right)|\hat{y}|^2+C(\epsilon)\xi^2|\hat{\theta}|^2+\tilde{\epsilon}\frac{\xi^2}{1+\xi^2}|\hat{u}|^2, \tag{94}$$

where

$$\hat{\mathscr{K}}(t) = -\mathrm{Re}\left(\hat{v}\bar{\hat{y}}+a\bar{\hat{z}}\hat{u}\right). \tag{95}$$

Let us now define the Lyapunov functional $\hat{\mathscr{L}}$ as follows:

$$\hat{\mathscr{L}}(t) = \hat{M}\left(1+\xi^2\right)\hat{\mathscr{E}}(t) + \frac{\hat{\alpha}_1}{1+\xi^2}\xi\hat{\mathscr{F}}(t) + \xi\hat{\mathscr{N}}(t) + \hat{\alpha}_2\hat{\mathscr{K}}(t), \tag{96}$$

where $\hat{M}$, $\hat{\alpha}_1$, and $\hat{\alpha}_2$ are positive constants to be chosen later. Consequently, exploiting the inequalities (85), (87), (93), and (94), we obtain for all $t \geq 0$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{L}}(t) + \hat{\alpha}_1\{a-\epsilon(a+1)\}\frac{\xi^2}{1+\xi^2}|\hat{z}|^2 + \{\hat{\alpha}_1(1-\epsilon)-\hat{\alpha}_2\tilde{\epsilon}\}\frac{\xi^2}{1+\xi^2}|\hat{u}|^2 \\ &\quad + \{(\kappa-\epsilon)-C(\epsilon)(\hat{\alpha}_1+\hat{\alpha}_2)\}\xi^2|\hat{\theta}|^2 + \{\hat{\alpha}_2(1-\epsilon)-\hat{\alpha}_1C(\epsilon)\}|\hat{v}|^2 \\ &\quad + \left\{\hat{M}\lambda - C(\epsilon,\tilde{\epsilon},\hat{\alpha}_1,\hat{\alpha}_2)\right\}\left(1+\xi^2\right)|\hat{y}|^2 + \left\{\hat{M}\delta/\tau_0 - C(\epsilon)\right\}\left(1+\xi^2\right)|\hat{w}|^2 \leq 0, \end{aligned} \tag{97}$$

where we have used the fact that $\xi^2/\left(1+\xi^2\right) \leq \xi^2$. Our goal now is to choose the constants ϵ, $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{M}$ in (97) such that all the coefficients in the left-hand side in (97) become positive. Indeed, we fix first ϵ small enough such that

$$\epsilon < \min\left(\frac{a}{1+a}, \kappa\right). \tag{98}$$

After that, we pick $\hat{\alpha}_1$ and $\hat{\alpha}_2$ such that

$$\hat{\alpha}_2(1-\epsilon) - \hat{\alpha}_1C(\epsilon) > 0$$

and

$$(\kappa-\varepsilon) - C(\epsilon)(\hat{\alpha}_1+\hat{\alpha}_2) > 0.$$

Next, choose $\tilde{\epsilon}$ small enough such that

$$\tilde{\epsilon} < \frac{\hat{\alpha}_1(1-\varepsilon)}{\hat{\alpha}_2}.$$

Finally, we take $\hat{M}$ large enough such that

$$\hat{M} > \max\left\{\frac{C(\epsilon,\tilde{\epsilon},\hat{\alpha}_1,\hat{\alpha}_2)}{\lambda}, \frac{C(\epsilon)\tau_0}{\delta}\right\}.$$

Consequently, (97) takes the form

$$\frac{\mathrm{d}\hat{\mathscr{L}}(t)}{\mathrm{d}t} + c\hat{\mathscr{W}}(t) \leq 0, \tag{99}$$

where

$$\hat{\mathscr{W}}(t) = \left(1+\xi^2\right)\left(|\hat{y}|^2 + |\hat{w}|^2\right) + |\hat{v}|^2 + \frac{\xi^2}{1+\xi^2}\left(|\hat{z}|^2 + |\hat{u}|^2\right) + \xi^2|\hat{\theta}|^2$$

and c is a positive constant.

On the other hand, we have the following result.

Lemma 4.3

For $\hat{M}$ large enough, there exist three positive constants $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\beta}_3$ such that for all $t \geq 0$, we have

$$\hat{\beta}_1\left(1+\xi^2\right)\hat{\mathscr{E}}(t) \leq \hat{\mathscr{L}}(t) \leq \hat{\beta}_2\left(1+\xi^2\right)\hat{\mathscr{E}}(t) \quad \text{and} \quad \hat{\mathscr{W}}(t) \geq \hat{\beta}_3\rho_1(\xi)\hat{\mathscr{E}}(t), \tag{100}$$

where $\rho_1(\xi) = \xi^2/\left(1+\xi^2\right)$.

From (99) and (100) we can then find $\eta > 0$ such that

$$\hat{\mathscr{E}}(t) = \left|\hat{U}(\xi,t)\right|^2 \leq e^{-\eta\rho_2(\xi)}\hat{\mathscr{E}}(0).$$

This completes the proof of Lemma 4.2. □

Proof of Lemma 4.3
The second estimate is easily verified. To prove the first estimate, we make use of formula (96) to write

$$\left|\hat{\mathscr{L}}(t)-\hat{M}\left(1+\xi^2\right)\hat{\mathscr{E}}(t)\right|=|H(t)|,$$

where

$$H(t)=\frac{\hat{\alpha}_1}{1+\xi^2}\xi\hat{\mathscr{F}}(t)+\xi\hat{\mathscr{N}}(t)+\hat{\alpha}_2\hat{\mathscr{K}}(t).$$

Furthermore, recalling (88), (90), and (95) and using Young's inequality, we arrive at

$$\begin{aligned}|H(t)|\leq&\frac{\hat{\alpha}_1|\xi|}{2\left(1+\xi^2\right)}\left(|\hat{v}|^2+|\hat{z}|^2+|\hat{u}|^2+|\hat{y}|^2\right)+\frac{|\xi|}{2}\left(|\hat{w}|^2+|\hat{\theta}|^2\right)\\&+\frac{\hat{\alpha}_2}{2}\left(|\hat{v}|^2+|\hat{y}|^2+a|\hat{z}|^2+a|\hat{u}|^2\right)\\\leq&C\left(\hat{\alpha}_1,\hat{\alpha}_2\right)\left(1+\xi^2\right)\hat{\mathscr{E}}(t).\end{aligned}$$

Consequently, we deduce from above that

$$\left(\hat{M}-C\left(\hat{\alpha}_1,\hat{\alpha}_2\right)\right)\left(1+\xi^2\right)\hat{\mathscr{E}}(t)\leq\hat{\mathscr{L}}(t)\leq\left(\hat{M}+C\left(\hat{\alpha}_1,\hat{\alpha}_2\right)\right)\left(1+\xi^2\right)\hat{\mathscr{E}}(t).$$

Therefore, if $\hat{M}$ is large enough, then our result holds true. This finishes the proof of Lemma 4.3. □

Our main theorem in this section can be stated as follows.

Theorem 4.1
Let s be a non-negative integer and assume that $U_0\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$. Then, the solution $U=(\hat{v},\hat{u},\hat{z},\hat{y},\hat{\theta},\hat{w})'$ of problem (79) satisfies the following decay estimates:

$$\left\|\partial_x^kU(t)\right\|_2\leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1}+C(1+t)^{-l/2}\left\|\partial_x^{k+l}U_0\right\|_2,\tag{101}$$

where k and l are non-negative integers satisfying $k+l\leq s$ and C,c are positive constants.

The decay estimate (101) can be improved (under suitable restrictions on the initial data) by $t^{-\gamma/2}$, $\gamma\in[0,1]$ and we have the following theorem.

Theorem 4.2
Let $\gamma\in[0,1]$, let s be a non-negative integer, and assume that $U_0\in H^s(\mathbb{R})\cap L^{1,\gamma}(\mathbb{R})$. Then, the solution $U=(\hat{v},\hat{u},\hat{z},\hat{y},\hat{\theta},\hat{w})'$ of problem (79) satisfies the following decay estimates:

$$\begin{aligned}\left\|\partial_x^kU(t)\right\|_2\leq&C(1+t)^{-1/4-k/2-\gamma/2}\|U_0\|_{L^{1,\gamma}}+C(1+t)^{-l/2}\left\|\partial_x^{k+l}U_0\right\|_2\\&+C(1+t)^{-1/4-k/2}\left|\int_{\mathbb{R}}U_0(x)\,dx\right|,\end{aligned}\tag{102}$$

where k and l are non-negative integers satisfying $k+l\leq s$ and C and c are two positive constants.

We omit the proofs of Theorems 4.1 and 4.2 as they are similar to the proofs of Theorem 2.1 and 3.1, respectively.

5. Concluding remarks

The pure Timoshenko system ($(1)_1$ and $(1)_2$ with $\beta=0$) has the same decay rate as in (60) and (61) (see [14]). This means that the Fourier model of heat conduction preserves the decay rate of the system, whereas the decay property of the solution in the Cattaneo model is of regularity-loss type (even if the wave speeds of the first two equations in (16) are the same, i.e, $a=1$) in the sense that the decay rate of the pure Timoshenko system (without the heat conduction) can be preserved only under additional regularity on the solution. Recently, Fernández Sare and Racke [33] proved that even in bounded domain and with nice boundary conditions, the removal of the paradox of infinite propagation speed inherent in the Fourier law by changing to the Cattaneo law causes a loss of the exponential stability property even if the wave speeds are the same. In fact, even an additional damping term of history type of the form (15) acting in the second equation in (16) is not strong enough to restore the exponential stability of the system. As a consequence, we conclude that the behavior under the Fourier law is essentially different from the behavior under the Cattaneo law.

Before the paper by Fernández Sare and Racke, [33], the time asymptotic behavior of solutions under the coupling through the Cattaneo law was widely believed to be the same as in coupling with the Fourier law. For example, Racke in [22] has shown that the norm of the difference between the solution $U=(u,u_t,\theta,q)$ of the classical thermoelastic system

$$\begin{cases} u_{tt}-\alpha u_{xx}+\beta\theta_x=0,\\ \theta_t+q_x+\delta u_{tx}=0,\\ q+\kappa\theta_x=0,\end{cases}$$

and the solution $V=(u,u_t,\theta,q)$ of the second sound thermoelastic model

$$\begin{cases} u_{tt}-\alpha u_{xx}+\beta\theta_x=0,\\ \theta_t+q_x+\delta u_{tx}=0,\\ \tau_0 q_t+q+\kappa\theta_x=0,\end{cases}$$

can be estimated as $\|U(t,x)-V(t,x)\|_2\le C\tau_0^2$. This last estimate means that the difference will tend to zero as $\tau_0\to 0$. However, this may not be true for some other systems. For example, this statement is not true for the Timoshenko system or for the contact problem in thermoelasticity. See [41] for more details.

It has been observed in [42] for the classical thermoelasticity (Fourier law) that the thermal dissipation does not assure the uniform decay of the energy except for special cases such as unidimensional problem, suitable conditions on the domain or symmetric solutions. It has been proved in [19], that in bounded interval $[0,L]$, the dissipation given by the Fourier law is strong enough to stabilize system (1) (with $\delta=0$) exponentially, provided that the wave speeds of the first and the second equations are the same, that is, $a=1$. One may ask whether a similar result holds in an unbounded domain when $\delta=0$. This remains an open question. The difficulty here is in constructing the Lyapunov functional when the dissipative term proportional to $\hat{y}$ is removed from the system.

The improved decay rates as presented in the proof of Theorem 3.1 can be obtained also for a class of symmetric hyperbolic systems, symmetric hyperbolic–parabolic systems, and symmetric hyperbolic–elliptic systems.

For example, consider the following symmetric hyperbolic system:

$$A^0u_t+\sum_{j=1}^{N}A^ju_{xj}+Lu=0, \tag{103}$$

where $u=u(x,t)$ is an m-vector function of $x=(x_1,...,x_N)\in\mathbb{R}^N$, $t\ge 0$, and the coefficient matrices satisfy the following conditions:

(a) A^0 is symmetric and positive definite;
(b) A^j is symmetric for any j;
(c) L is symmetric and non-negative definite.

The Fourier transform of (103) leads to the equation

$$A^0\hat{u}_t+i\xi A(\omega)\hat{u}+L\hat{u}=0 \tag{104}$$

where

$$A(\omega)=\sum_{j=1}^{N}A^j\omega_j,\quad \omega_j=\xi/|\xi|\in S^{N-1}.$$

Assuming that (SK) condition (see [43]) is satisfied for system (104), that is,

If $\mu\in\mathbb{R}$, $\varphi\in\mathbb{R}^m$, $\omega\in S^{N-1}$, then

$$\mu A^0\varphi+A(\omega)\varphi=0 \text{ and } L\varphi=0$$

implies $\varphi=0$,

then the pointwise estimates for system (104) is given by (see [43, Theorem 2])

$$\left|\hat{u}(\xi,t)\right|\le Ce^{-\rho_1(\xi)t}\left|\hat{u}_0(\xi)\right|.$$

An analysis of the behavior of the Fourier image $\hat{u}(\xi,t)$ of the solution in the low-frequency region ($|\xi|\le 1$) leads, following the proof of Theorem 3.1, to the following faster decay estimates:

$$\left\|\partial_x^k u(t)\right\|_2\le Ce^{-ct}\left\|\partial_x^k u_0\right\|_2+C(1+t)^{-N/4-(k+\gamma)/2}\|u_0\|_{L^{1,\gamma}(\mathbb{R}^N)},$$

where $\gamma\in[0,1]$, provided that $\int_{\mathbb{R}^N}u_0(x)\,dx=0$ and $k\ge 0$. Clearly, this is an improvement of the estimate (8) given in [43].

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