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Nonequivalent Representations of Nuclear Algebras of Canonical Commutation Relations: Quantum Fields

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Non-Fock representations of the canonical commutation relations modeled over an infinite-dimensional nuclear space are constructed in an explicit form. The example of the nuclear space of smooth real functions of rapid decrease results in nonequivalent quantizations of scalar fields.

KEY WORDS: canonical commutation relations; nuclear space; Euclidean field; Wick rotation.

1. INTRODUCTION

By virtue of the well-known Stone–von Neumann uniqueness theorem, all irreducible representations of the canonical commutation relations (henceforth the CCR) for finite degrees of freedom are equivalent. On the contrary, the CCR for infinite degrees of freedom admit infinitely many nonequivalent irreducible representations (see (Florig and Summers, 2000) for a survey).

One can find the comprehensive description of representations of the CCR modeled over an infinite-dimensional nuclear space Q in Gelfand and Vilenkin (1964). These representations are associated to translationally quasi-invariant measures on the (topological) dual Q' of Q and, due to the well-known Bochner theorem, are characterized by continuous positive-definite functions on Q. In Sections 4–5 of this work, operators of these representations are written in an explicit form. For instance, the Fock representation is associated to a certain Gaussian measure on Q'. If Q is not a nuclear space, the Fock representation need not exist (see Remark 2 below).

Nuclear (non-Banach) involutive algebras are widely studied in algebraic quantum field theory since the well-known GNS construction for C^*-algebras can

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be generalized to these algebras too (Borchers, 1984; Horuzhy, 1990; Iguri and Catagnino, 1999). A Banach space is not nuclear, unless it is finite-dimensional (see Remark 1 below). A physically relevant example of an infinite-dimensional nuclear space is the space RS^4 of smooth real functions of rapid decrease on $\mathbb{R}^4$. It is the real subspace of the space $S(\mathbb{R}^4)$ of smooth complex functions of rapid decrease on $\mathbb{R}^4$ (see Remark 5 below). Its (topological) dual is the space $S'(\mathbb{R}^4)$ of tempered distributions (generalized functions) (Bogoliubov *et al.*, 1990). Of course, elements of RS^4 by no means are physical fields, but test functions. Continuous positive forms f on the tensor Borchers algebra

$$A_{RS^4} = \mathbb{R} \oplus RS^4 \oplus RS^8 \oplus \cdots \tag{1}$$

of RS^4 are expressed as

$$f(\psi_n) = \int W_n(x_1, \ldots, x_n)\psi_n(x_1, \ldots, x_n)\, d^4x_1 \cdots d^4x_n, \quad \psi_n \in RS^{4n}, \tag{2}$$

into the tempered distributions $W_n \in S'(\mathbb{R}^{4n})$ whose Fourier transform is regarded as the vacuum expectations of the plane wave operators of quantum scalar fields.

Of course, quantum fields do not constitute any CCR algebra, but there is a morphism of RS^4 to the CCR algebra over the nuclear space RS^3. It is treated as the instantaneous CCR algebra of scalar fields. Its Fock representation provides the familiar vacuum expectations of free quantum scalar fields on the Minkowski space $\mathbb{R}^4$, while the non-Fock ones lead to nonstandard quantizations of these fields (see Section 6).

In order to characterize interacting quantum fields created at some instant and annihilated at another one, one should turn to the causal forms f^c on the Borchers algebra A_{RS^4} (1). They are given by the functionals

$$f^c(\psi_n) = \int W_n^c(x_1, \ldots, x_n)\psi_n(x_1, \ldots, x_n)\, d^4x_1 \cdots d^4x_n, \quad \psi_n \in RS^{4n}, \tag{3}$$

$$W_n^c(x_1, \ldots, x_n) = \sum_{(i_1 \ldots i_n)} \theta\left(x_{i_1}^0 - x_{i_2}^0\right) \cdots \theta\left(x_{i_{n-1}}^0 - x_{i_n}^0\right) W_n(x_1, \ldots, x_n), \tag{4}$$

where $W_n \in S'(\mathbb{R}^{4n})$ are tempered distributions, θ is the step function, and the sum runs through all permutations $(i_1 \cdots i_n)$ of the tuple of numbers $1, \ldots, n$ (Bogoliubov and Shirkov, 1980). The problem is that the functionals W_n^c (4) need not be tempered distributions and, therefore, the causal forms f^c (3) are neither positive nor continuous forms on the Borchers algebra A_{RS^4}.

At the same time, the causal forms issue from the Wick rotation of Euclidean states of the Borchers algebra A_{RS^4} which describe particles in the interaction zone (see Section 7). The key point is that, since the causal forms (4) are symmetric, the Euclidean states of the Borchers algebra A_{RS^4} can be obtained as states of the corresponding commutative tensor algebra B_{RS^4} (Sardanashvily, 1991, 1994;

Sardanashvily and Zakharov, 1991). They characterize different representations of the Abelian subgroup of the CCR group modeled over the nuclear space SR^4, and are associated to different positive measures on the space of generalized functions $S'(\mathbb{R}^4)$. From the physical viewpoint, these states are Euclidean Green's functions whose Wick rotation gives complete Green's functions of interacting quantum scalar fields on the Minkowski space. Some nonperturbative phenomena, e.g., the Higgs vacuum can be studied in this manner (Sardanashvily, 1991).

For the sake of simplicity, our consideration here is restricted to scalar fields. In order to describe nonscalar fields on $\mathbb{R}^4$ with values in a vector space V, one can consider the Borchers algebra of the tensor product space $V \otimes SR^4$ (Sardanashvily, 1994). Difficulties arise if nonscalar fields are defined on a noncontractible manifold X, i.e., they are sections of a vector bundle $Y \to X$. If this is a trivial bundle, the space $Y_K(X)$ of its sections of compact support equipped with the Schwartz topology is a nuclear space. In the general case, $Y \to X$ is a Whitney summand of a trivial bundle, and the vector space $Y_K(X)$ of its sections of compact support is provided with the relative Schwartz topology, which makes it to a nuclear Schwartz manifold. However, the extension of the Bocher theorem for this nuclear manifold remains under question. Note that quantum gauge theory on compact manifolds usually deals with Sobolev spaces of gauge potentials (Kondracki and Sadowski, 1986; Mitter and Viallet, 1981). To dispose of the compactness assumption, the technique of nuclear Schwartz manifolds also has been applied to gauge theory (Abati *et al.*, 1986; Cirelli and Manià, 1985). However, it meets serious inconsistencies because of the lack of the inverse function theorem.

2. THE NUCLEAR CCR

Let us recall the notion of a nuclear space (Gelfand and Vilenkin, 1964; Pietsch, 1972). Let a complex vector space Q have a countable set of nondegenerate Hermitian forms $\langle . \mid . \rangle_k, = 1, \ldots,$ such that

$$\langle q \mid q \rangle_1 \leq \cdots \leq \langle q \mid q \rangle_k \leq \cdots$$

for all $q \in Q$. If Q is complete with respect to the (Hausdorff) topology defined by the set of norms

$$\|.\|_k = \langle . \mid . \rangle_k^{1/2}, \quad k = 1, \ldots, \tag{5}$$

it is called a countably Hilbert space. The dual Q' of Q is provided with the weak and strong topologies.

Let Q_k denote the completion of Q with respect to the norm $\|.\|_k$ (5). We have the chain of injections

$$Q_1 \supset Q_2 \supset \cdots Q_k \supset \cdots$$

together with the homeomorphism $Q = \bigcap_k Q_k$. Let T^n_m, $m \le n$, be a prolongation of the map

$$Q_n \supset Q \ni q \mapsto q \in Q \subset Q_m$$

to the continuous map of Q_n onto the dense subset of Q_m. A countably Hilbert space Q is called a nuclear space if, for any m, there exists n such that T^m_n is a nuclear map, i.e.,

$$T^n_m(q) = \sum_i \lambda_i \left\langle q \mid q^i_n \right\rangle_{Q_n} q^i_n,$$

where: (i) $\{q^i_n\}$ and $\{q^i_m\}$ are bases for the Hilbert spaces Q_n and Q_m, respectively, (ii) $\lambda_i \ge 0$, and (iii) the series $\Sigma \lambda_i$ converges.

Remark 1. A nuclear space is perfect, i.e., every bounded closed set in a nuclear space is compact. It follows that a Hilbert space is not nuclear, unless it is finite-dimensional. Furthermore, a nuclear space is separable, and the weak and strong topologies both on this space and its dual coincide.

Let a nuclear space Q be provided with still another nondegenerate Hermitian form $\langle . \mid . \rangle$ which is separately continuous. It follows that there exist numbers M and m such that

$$\langle q \mid q \rangle \le M \|q\|_m, \quad \forall q \in Q. \tag{6}$$

Let $\tilde{Q}$ denote the completion of Q with respect to this form. There are the injections

$$Q \subset \tilde{Q} \subset Q', \tag{7}$$

where Q is dense in $\tilde{Q}$, and so is $\tilde{Q}$ in Q'. The triple (7) is called the rigged Hilbert space.

Given a real nuclear space Q together with a nondegenerate separately continuous Hermitian form $\langle . \mid . \rangle$, let us consider the group $G(Q)$ of triples $g = (q_1, q_2, \lambda)$ of elements q_1, q_2 of Q and complex numbers λ of unit modulus which are subject to multiplications

$$(q_1, q_2, \lambda)(q'_1, q'_2, \lambda') = (q_1 + q'_1, q_2 + q'_2, \ \exp[i\langle q_2, q'_1\rangle]\lambda\lambda'). \tag{8}$$

It is a Lie group whose group space is a nuclear manifold modeled over $Q \oplus Q \oplus \mathbb{R}$. Let us denote

$$T(q) = (q, 0, 0), \qquad P(q) = (0, q, 0).$$

Then the multiplication law (8) takes the form

$$T(q)T(q') = T(q + q'), \qquad P(q)P(q') = P(q + q'),$$
$$P(q)T(q') = \exp[i\langle q \mid q'\rangle]T(q')P(q). \tag{9}$$

Written in this form, $G(Q)$ is called the nuclear Weyl CCR group.

The complexified Lie algebra of the nuclear Lie group $G(Q)$ is the Heisenberg CCR algebra $\mathcal{G}(Q)$. It is generated by the elements $\phi(q)$, $\pi(q)$, $q \in Q$, and I which obey the Heisenberg CCR commutation relations

$$[\phi(q), I] = [\pi(q), I] = 0, \qquad [\phi(q), \phi(q')] = [\pi(q), \pi(q')] = 0,$$
$$[\pi(q), \phi(q')] = -i\langle q \mid q'\rangle I. \tag{10}$$

There is the exponential map

$$T(q) = \exp[i\phi(q)], \qquad P(q) = \exp[i\pi(q)].$$

Because of the relation (6), the normed topology on the pre-Hilbert space Q defined by the Hermitian form $\langle . \mid . \rangle$ is coarser than the nuclear space topology. The latter is metric, separable and, consequently, second-countable. Hence, the pre-Hilbert space Q is also second-countable and, therefore, admits a countable orthonormal basis. Given such a basis $\{q_i\}$ for Q, the Heisenberg CCR (10) take the form

$$[\phi(q_j), \phi(q_k)] = [\pi(q_k), \pi(q_j)] = 0, \qquad [\pi(q_j), \phi(q_k)] = -i\delta_{jk}I.$$

3. REPRESENTATIONS OF THE NUCLEAR CCR GROUP

The CCR group $G(Q)$ contains two nuclear Abelian subgroups $T(Q)$ and $P(Q)$. Following the representation algorithm in Gelfand and Vilenkin (1964), we first construct representations of the nuclear Abelian group $T(Q)$. These representations under certain conditions can be extended to representations of the whole CCR group $G(Q)$.

One can think of the nuclear Abelian group $T(Q)$ as being the group of translations in the nuclear space Q. Its cyclic strongly continuous unitary representation π in a Hilbert space $(E, \langle . \mid . \rangle_E)$ with a (normed) cyclic vector $\theta \in E$ defines the complex function

$$Z(q) = \langle \pi(T(q))\theta | \theta\rangle_E$$

on Q. This function is proved to be continuous and positive-definite, i.e., $Z(0) = 1$ and

$$\sum_{i,j} Z(q_i - q_j)\overline{c_i}c_j \geq 0$$

for any finite set $q_1, \ldots, q_m$ of elements of Q and arbitrary complex numbers $c_1, \ldots, c_m$.

In accordance with the well-known Bochner theorem for nuclear spaces, any continuous positive-definite function $Z(q)$ on a nuclear space Q is the Fourier transform

$$Z(q) = \int \exp[i\langle q, u\rangle]\mu(u) \tag{11}$$

of a positive measure μ of total mass 1 on the dual Q' of Q (Gelfand and Vilenkin, 1964). Then the above mentioned representation π of $T(Q)$ can be given by the operators

$$T_Z(q)\rho(u) = \exp[i\langle q, u\rangle]\rho(u) \tag{12}$$

in the Hilbert space $L^2_C(Q', \mu)$ of classes of μ-equivalent square integrable complex functions $\rho(u)$ on Q'. The cyclic vector θ of this representation is the μ-equivalence class $\theta \approx_\mu 1$ of the constant function $\rho(u) = 1$. Then we have

$$Z(q) = \langle T_Z(q)\theta|\theta\rangle_\mu = \int \exp[i\langle q, u\rangle]\mu. \tag{13}$$

Conversely, every positive measure μ of total mass 1 on the dual Q' of Q defines the cyclic strongly continuous unitary representation (12) of the group $T(Q)$. By virtue of the above mentioned Bochner theorem, it follows that every continuous positive-definite function $Z(q)$ on Q characterizes a cyclic strongly continuous unitary representation (12) of the nuclear Abelian group $T(Q)$. We agree to call $Z(q)$ a generating function of this representation.

It should be emphasized that the representation (12) need not be (topologically) irreducible. For instance, let $\rho(u)$ be a function on Q' such that the set where it vanishes is not a μ-null subset of Q'. Then the closure of the set $T_Z(Q)\rho$ is a $T(Q)$-invariant closed subspace of $L^2_C(Q', \mu)$.

One can show that distinct generating functions $Z(q)$ and $Z'(q)$ determine equivalent representations T_Z and $T_{Z'}$ (12) of $T(Q)$ in the Hilbert spaces $L^2_C(Q', \mu)$ and $L^2_C(Q', \mu')$ iff they are the Fourier transform of equivalent measures on Q' (Gelfand and Vilenkin, 1964). Indeed, let

$$\mu' = s^2\mu, \tag{14}$$

where a function $s(u)$ is strictly positive almost everywhere on Q', and $\mu(s^2) = 1$. Then the map

$$L^2_C(Q', \mu') \ni \rho(u) \mapsto s(u)\rho(u) \in L^2_C(Q', \mu) \tag{15}$$

provides an isomorphism between the representations $T_{Z'}$ and T_Z.

The representation T_Z (12) of the nuclear Abelian group $T(Q)$ in the Hilbert space $L^2_C(Q', \mu)$ determined by the generating function $Z(11)$ can be extended to the CCR group $G(Q)$ if the measure μ possesses the following property.

Let u_q, $q \in Q$, be an element of Q' given by the condition

$$\langle q', u_q\rangle = \langle q' \mid q\rangle, \quad \forall q' \in Q. \tag{16}$$

These elements form the image of the monomorphism $Q \to Q'$ determined by the Hermitian form $\langle . \mid . \rangle$ on Q. Let the measure μ in (11) remains equivalent under translations

$$Q' \ni u \mapsto u + u_q \in Q', \quad \forall u_q \in Q \subset Q',$$

in Q', i.e.,

$$\mu(u + u_q) = a^2(q, u)\mu(u), \quad \forall u_q \in Q \subset Q', \tag{17}$$

where a function $a(q, u)$ is square μ-integrable and strictly positive almost everywhere on Q'. This function fulfils the relations

$$a(0, u) = 1, \qquad a(q + q', u) = a(q, u)a(q', u + u_q). \tag{18}$$

A measure on Q' obeying the condition (17) is called translationally quasi-invariant, but it does not remain equivalent under any translation in Q', unless Q is finite-dimensional.

Let a generating function Z of a cyclic strongly continuous unitary representation of the nuclear group $T(Q)$ be the Fourier transform (11) of a translationally quasi-invariant measure μ on Q'. Then one can extend the representation (12) of this group to the representation of the CCR group in the Hilbert space $L^2_C(Q', \mu)$ by operators

$$P_Z(q)\rho(u) = a(q, u)\rho(u + u_q). \tag{19}$$

Indeed, it is easily justified that the Weyl CCR (9) hold, while the equalities

$$\begin{aligned} \|\rho\|_\mu &= \int |\rho(u)|^2\mu(u) = \int |\rho(u + u_q)|^2\mu(u + u_q) \\ &= \int a^2(q, u)|\rho(u + u_q)|^2\mu(u) = \|P_Z(q)\rho\|^2_\mu, \end{aligned} \tag{20}$$

show that the operators (19) are unitary.

Let μ'(14) be a μ-equivalent positive measure of total mass 1 on Q'. The equality

$$\mu'(u + u_q) = s^{-2}(u)a^2(q, u)s^2(u + u_q)\mu'(u)$$

shows that it is also translationally quasi-invariant. Then the isomorphism (15) between representations T_Z and $T_{Z'}$ of the nuclear Abelian group $T(Q)$ is extended to the isomorphism

$$P_{Z'}(q) = s^{-1}P_Z(q)s : \rho(u) \mapsto s^{-1}(u)a(q, u)s(u + u_q)\rho(u + u_q)$$

of the corresponding representations of the CCR group $G(Q)$.

4. REPRESENTATIONS OF THE CCR ALGEBRA

Similarly to the case of a finite-dimensional Lie group, any strongly continuous unitary representation T_Z (12), P_Z (19) of the nuclear CCR group $G(Q)$

implies a representation of its Lie algebra $\mathcal{G}(Q)$ by (unbounded) operators in the same Hilbert space $L^2_C(Q', \mu)$. This representation reads

$$I = 1, \quad \phi(q)\rho(u) = \langle q, u\rangle \rho(u), \qquad \pi(q)\rho(u) = -i(\delta_q + \eta(q, u))\rho(u),$$
$$\delta_q \rho(u) = \lim_{\alpha\to 0} \alpha^{-1}[\rho(u + \alpha u_q) - \rho(u)], \quad \alpha \in \mathbb{R}, \tag{21}$$
$$\eta(q, u) = \lim_{\alpha\to 0} \alpha^{-1}[a(\alpha q, u) - 1]. \tag{22}$$

One at once derives from the relations (18) that

$$\delta_q\delta_{q'} = \delta_{q'}\delta_q, \qquad \delta_q(\eta(q', u)) = \delta_{q'}(\eta(q, u)),$$
$$\delta_q = -\delta_{-q}, \qquad \delta_q(\langle q', u\rangle) = \langle q' \mid q\rangle,$$
$$\eta(0, u) = 0, \quad \forall u \in Q', \qquad \delta_q\theta = 0, \quad \forall q \in Q.$$

With the aid of these relations, it is easily justified that the operators (21) fulfil the Heisenberg CCR (10). The unitarity condition (20) implies the conjugation rule

$$\langle q, u\rangle^* = \langle q, u\rangle, \qquad \delta_q^* = -\delta_q - 2\eta(q, u).$$

Hence, the operators (21) are Hermitian.

Let us further restrict our consideration to representations with generating functions $Z(q)$ such that

$$\mathbb{R} \ni t \to Z(tq) \tag{23}$$

is an analytic function on $\mathbb{R}$ at $t = 0$ for all $q \in Q$. Then one can show that the function $\langle q \mid u\rangle$ on Q' is square μ-integrable for all $q \in Q$ and that, consequently, the operators $\phi(q)$ (21) are bounded everywhere in the Hilbert space $L^2_C(Q', \mu)$. Moreover, the mean values of operators $\phi(q)$ can be computed by the formula

$$\langle\phi(q_1)\cdots\phi(q_n)\rangle = i^{-n}\frac{\partial}{\partial\alpha^1}\cdots\frac{\partial}{\partial\alpha^n} Z(\alpha^i q_i)|_{\alpha^i=0} = \int \langle q_1, u\rangle\cdots\langle q_n, u\rangle\mu(u). \tag{24}$$

The operators $\pi(q)$ (21) act in the subspace E_∞ of all smooth complex functions in $L^2_C(Q', \mu)$ whose derivatives of any order also belongs to $L^2_C(Q', \mu)$. However, E_∞ need not be dense in the Hilbert space $L^2_C(Q', \mu)$, unless Q is finite-dimensional. The space E_∞ is also the carrier space of a representation of the enveloping algebra $\overline{\mathcal{G}}(Q)$ of the CCR algebra $\mathcal{G}(Q)$. The representations of $\mathcal{G}(Q)$ and $\overline{\mathcal{G}}(Q)$ in E_∞ need not be irreducible. Therefore, let us consider the subspace $E_\theta = \overline{\mathcal{G}}(Q)\theta$ of E_∞, where θ is a cyclic vector for the representation of the CCR group in $L^2_C(Q', \mu)$. Obviously, the representation of the CCR algebra $\mathcal{G}(Q)$ in E_θ is (algebraically) irreducible. If θ' is another cyclic vector in $L^2_C(Q', \mu)$, the representations of $\mathcal{G}(Q)$ in E_θ and $E_{\theta'}$ are equivalent.

One also introduces creation and annihilation operators

$$a^{\pm}(q) = \frac{1}{\sqrt{2}}[\phi(q) \mp i\pi(q)] = \frac{1}{\sqrt{2}}[\mp\delta_q \mp \eta(q, u) + \langle q, u\rangle]. \tag{25}$$

They obey the conjugation rule $(a^{\pm}(q))^* = a^{\mp}(q)$ and the commutation relations

$$[a^-(q), a^+(q')] = \langle q \mid q'\rangle 1, \qquad [a^+(q), a^+(q')] = [a^-(q), a^-(q')] = 0.$$

The particle number operator N in the carrier space E_θ is defined by the conditions

$$[N, a^{\pm}(q)] = \pm a^{\pm}(q)$$

up to a summand $\lambda 1$. With respect to a countable orthonormal basis $\{q_k\}$, this operator N is given by the sum

$$N = \sum_k a^+(q_k)a^-(q_k), \tag{26}$$

but need not be defined everywhere in E_θ, unless Q is finite dimensional.

5. NON-FOCK REPRESENTATIONS OF THE NUCLEAR CCR

Gaussian measures exemplifies the physically relevant class of translationally quasi-invariant measures on the dual Q' of a nuclear space Q. The Fourier transform of a Gaussian measure reads

$$Z(q) = \exp\left[-\frac{1}{2}B(q)\right], \tag{27}$$

where $B(q)$ is a seminorm on Q' called the covariance form.

Remark 2. If Q is a Banach space provided with the norm $\|.\|$, there exists a Gaussian quasi-measure on its dual Q' with the covariance form $\|.\|$, but it is not a measure unless Q is finite-dimensional. Let T be a continuous operator in Q. The Gaussian quasi-measure on Q' with the covariance form $q \mapsto \|T_q\|$ is proved to be a measure iff T is a Hilbert–Schmidt operator. Let $Q = \mathbb{R}^n$ be a finite-dimensional vector space and B a norm on Q. Let its dual Q' be coordinated by (x_i). The Gaussian measure on Q' with the covariance form B is equivalent to the Lebesgue measure $d^n x$ on Q'. It reads

$$\mu_B = \frac{\det[B]^{1/2}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(B^{-1})^{ij}x_i x_j\right] d^n x.$$

Let μ_K denote a Gaussian measure on Q' whose Fourier transform is the generating function

$$Z_K = \exp\left[-\frac{1}{2}B_K(q)\right] \tag{28}$$

with the covariance form

$$B_K(q) = \langle K^{-1}q \mid K^{-1}q\rangle, \tag{29}$$

where K is a bounded invertible operator in the Hilbert completion $\tilde{Q}$ of Q with respect to the Hermitian form $\langle . \mid . \rangle$. The Gaussian measure μ_K is translationally quasi-invariant, i.e.,

$$\mu_K(u + u_q) = a_K^2(q, u)\mu_K(u).$$

Using the formula (24), one can show that

$$a_K(q, u) = \exp\left[-\frac{1}{4}B_K(Sq) - \frac{1}{2}\langle Sq, u\rangle\right], \tag{30}$$

where $S = KK^*$ is a bounded Hermitian operator in $\tilde{Q}$.

Let us construct the representation of the CCR algebra $\mathcal{G}(Q)$ determined by the generating function Z_K (28). Substituting the function (30) into the formula (22), we find

$$\eta(q, u) = -\frac{1}{2}\langle S_q, u\rangle.$$

Hence, the operators $\phi(q)$ and $\pi(q)$ (21) take the form

$$\phi(q) = \langle q, u\rangle, \qquad \pi(q) = -i\left(\delta_q - \frac{1}{2}\langle S_q, u\rangle\right). \tag{31}$$

Accordingly, the creation and annihilation operators (25) read

$$a^{\pm}(q) = \frac{1}{\sqrt{2}}\left[\mp\delta_q \pm \frac{1}{2}\langle Sq, u\rangle + \langle q, u\rangle\right]. \tag{32}$$

They act on the subspace E_θ, $\theta \approx_{\mu_K} 1$, of the Hilbert space $L^2_C(Q', \mu_K)$, and are Hermitian with respect to the Hermitian form $\langle . \mid . \rangle_{\mu_K}$ on $L^2_C(Q', \mu_K)$.

Remark 3. If a representation of the CCR is characterized by the Gaussian generating function (28), it is convenient for a computation to express all operator into the operators δ_q and $\phi(q)$, which obey the commutation relation

$$[\delta_q, \phi(q')] = \langle q' \mid q\rangle.$$

For instance, we have

$$\pi(q) = -i\delta_q - \frac{i}{2}\phi(Sq).$$

The mean values $\langle \phi(q_1)\cdots\phi(q_n)\delta_q\rangle$ vanishes, while the meanvalues $\langle\phi(q_1)\cdots \phi(q_n)\rangle$, defined by the formula (24), obey the Wick theorem relations

$$\langle\phi(q_1)\cdots\phi(q_n)\rangle = \sum\langle\phi(q_{i_1})\phi(q_{i_2})\rangle\cdots\langle\phi(q_{i_{n-1}})\phi(q_{i_n})\rangle, \tag{33}$$

where the sum runs through all partitions of the set $1, \dots, n$ in ordered pairs $(i_1 < i_2), \dots (i_{n-1} < i_n)$, and where

$$\langle \phi(q)\phi(q') \rangle = \langle K^{-1}q \mid K^{-1}q' \rangle.$$

In particular, put $K = \sqrt{2} \cdot 1$. Then the generating function (28) takes the form

$$Z_{\mathrm{F}}(q) = \exp\left[-\frac{1}{4}\langle q \mid q \rangle\right], \tag{34}$$

and determines the Fock representation of the CCR algebra $\mathcal{G}(Q)$. It is given by the operators

$$\phi(q) = \langle q, u \rangle, \qquad \pi(q) = -i(\delta_q - \langle q, u \rangle),$$
$$a^+(q) = \frac{1}{\sqrt{2}}[-\delta_q + 2\langle q, u \rangle], \qquad a^-(q) = \frac{1}{\sqrt{2}}\delta_q.$$

Its carrier space is the subspace $E_\theta, \theta \approx_{\mu_{\mathrm{F}}} 1$, of the Hilbert space $L^2_C(Q', \mu_{\mathrm{F}})$, where μ_{F} denotes the Gaussian measure whose Fourier transform is (34). We agree to call it the Fock measure.

The Fock representation up to an equivalence is characterized by the existence of a cyclic vector θ such that

$$a^-(q)\theta = 0, \quad \forall q \in Q. \tag{35}$$

For the representation in question, this is $\theta \approx_{\mu_{\mathrm{F}}} 1$. An equivalent condition is that the particle number operator N (26) exists and its spectrum is lower bounded. The corresponding eigenvector of N in E_θ is θ itself so that $N\theta = 0$. Therefore, one often interprets this eigenvector as a vacuum state.

A glance at the expression (32) shows that the condition (35) does not hold, unless Z_K is Z_{F} (34). For instance, the particle number operator in the representation (32) reads

$$N = \sum_j a^+(q_j)a^-(q_j) = \sum_j \Bigg[-\delta_{q_j}\delta_{q_j} + S^j_k \langle q_k, u \rangle \partial_{q_j}$$
$$+ \left(\delta_{km} - \frac{1}{4}S^j_k S^j_m\right) \langle q_k, u \rangle \langle q_m, u \rangle - \left(\delta_{jj} - \frac{1}{2}S^j_j\right)\Bigg],$$

where $\{q_k\}$ is the orthonormal basis for the pre-Hilbert space Q. One can show that this operator is defined everywhere on E_θ and is lower bounded only if the operator S is a sum of the scalar operator $2{\cdot}1$ and a nuclear operator in $\tilde{Q}$, in particular, if

$$\mathrm{Tr}\left(1 - \frac{1}{2}S\right) < \infty.$$

This condition is also sufficient for the measures μ_K and $\mu_{\rm F}$ (and, consequently, the corresponding representations) to be equivalent (Gelfand and Vilenkin, 1964). For instance, the generating function

$$Z_c(q) = \exp\left[-\frac{c^2}{2}\langle q \mid q\rangle\right], \quad c^2 \neq \frac{1}{2},$$

determines a non-Fock representation of the nuclear CCR.

Remark 4. The non-Fock representation (31) of the CCR algebra (10) in the Hilbert space $L^2_C(Q', \mu_K)$ is the Fock representation

$$\phi_K(q) = \phi(q) = \langle q, u\rangle,$$

$$\pi_K(q) = \pi(S^{-1}q) = -i\left(\delta^K_q - \frac{1}{2}\langle q, u\rangle\right), \quad \delta^K_q = \delta_{S^{-1}q},$$

of the CCR algebra $\{\phi_K(q), \pi_K(q), I\}$, where

$$[\phi_K(q), \pi_K(q)] = i\langle K^{-1}q \mid K^{-1}q'\rangle I.$$

This fact motivates somebody to regard representations of the CCR group (9) as representation of two mutually commutative Abelian groups $T(Q)$ and $(P(Q)$ up to phase multipliers (Yamashita and Ozawa, 2000).

Since the Fock measure $\mu_{\rm F}$ on Q' remains equivalent only under translations by vectors $u_q \in Q \subset Q'$, the measure

$$\mu_\sigma - \mu_{\rm F}(u - \sigma), \quad \sigma \in Q' \setminus Q,$$

on Q' determines a non-Fock representation of the nuclear CCR. Indeed, this measure is translationally quasi-invariant:

$$\mu_\sigma(u + u_q) = a^2_\sigma(q, u)\mu_\sigma(u), \qquad a_\sigma(q, u) = a_{\rm F}(q, u - \sigma),$$

and its Fourier transform

$$Z_\sigma(q) = \exp[i\langle p, \sigma\rangle]Z_{\rm F}(q)$$

is a positive-definite continuous function on Q. Then the corresponding representation of the CCR algebra is given by operators

$$a^+(q) = \frac{1}{\sqrt{2}}(-\delta_q + 2\langle q, u\rangle - \langle q, \sigma\rangle), \qquad a^-(q) = \frac{1}{\sqrt{2}}(\delta_q + \langle q, \sigma\rangle). \quad (36)$$

In comparison with all the above representations, these operators possess nonvanishing vacuum mean values

$$\langle a^\pm(q)\theta \mid \theta\rangle_{\mu_{\rm F}} = \mp\langle q, \sigma\rangle.$$

If $\sigma \in Q \subset Q'$, the representation (36) becomes equivalent to the Fock representation (32) due to the morphism

$$\rho(u) \mapsto \exp[-\langle q', u\rangle]\rho(u + u_{q'}).$$

6. FREE QUANTUM FIELDS

In this section, representations of the nuclear CCR are utilized in order to describe free quantum fields. In the framework of algebraic quantum field theory, quantum fields are characterized by a unital involutive topological algebra A and a (continuous positive) state f of A. The key point is that a quantum field algebra is never normed.

With reference to the field-particle dualism, realistic quantum field models are described by tensor algebras, as a rule. Let Q be a real (locally convex) topological vector space, endowed with an involution operation $q \mapsto q^*$, $q \in Q$. Let us consider the tensor algebra

$$A_Q = \mathbb{R} \oplus Q \oplus Q^2 \oplus \cdots, \qquad Q^n = \overset{n}{\otimes} Q, \tag{37}$$

of Q. It is a $*$-algebra with respect to the involution

$$(q^1 \cdots q^n)^* = (q^n)^* \cdots (q^1)^*.$$

The direct sum topology makes A_Q to a topological involutive algebra. A state f of this algebra is given by a tuple $\{f_n\}$ of continuous forms on the tensor products Q^n. Its value $f(q^1 \cdots q^n)$ are interpreted as the vacuum expectation of the system of fields $q^1, \dots, q^n$. Further, we choose by Q the real subspace SR^4 of the nuclear space of smooth complex functions of rapid decrease on $\mathbb{R}^4$.

Remark 5. By functions of rapid decrease on an Euclidean space $\mathbb{R}^n$ are called complex smooth functions $\psi(x)$ such that the quantities

$$\|\psi\|_{k,m} = \max_{|\alpha|\leq k} \sup_x (1 + x^2)^m |D^\alpha \psi(x)| \tag{38}$$

are finite for all $k, m \in \mathbb{N}$. Here, we follow the standard notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x^1 \ldots \partial^{\alpha_n} x^n}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

for an n-tuple of natural numbers $\alpha = (\alpha_1, \dots, \alpha_n)$. The functions of rapid decrease constitute the nuclear space $S(\mathbb{R}^n)$ with respect to the topology determined by the seminorms (38). Its dual is the space $S'(\mathbb{R}^n)$ of tempered distributions (Bogoliubov *et al.*, 1990; Gelfand and Vilenkin, 1964; Pietsch, 1972). The corresponding contraction form is written as

$$\langle \psi, h\rangle = \int \psi(x) h(x)\, d^n x, \quad \psi \in S(\mathbb{R}^n), \quad h \in S'(\mathbb{R}^n).$$

The space $S(\mathbb{R}^n)$ is provided with the nondegenerate separately continuous Hermitian form

$$\langle \psi \mid \psi' \rangle = \int \psi(x)\overline{\psi'(x)}\, d^n x.$$

The completion of $S(\mathbb{R}^n)$ with respect to this form is the space $L^2_C(\mathbb{R}^n)$ of square integrable complex functions on $\mathbb{R}^n$. We have the rigged Hilbert space

$$S(\mathbb{R}^n) \subset L^2_C(\mathbb{R}^n) \subset S'(\mathbb{R}^n).$$

Let $\mathbb{R}_n$ denote the dual of $\mathbb{R}^n$ coordinated by (p_λ). The Fourier transform

$$\psi^{\mathrm{F}}(p) = \int \psi(x)\, e^{ipx}\, d^n x, \quad px = p_\lambda x^\lambda, \tag{39}$$

$$\psi(x) = \int \psi^{\mathrm{F}}(p)\, e^{-ipx}\, d_n p, \quad d_n p = (2\pi)^{-n} d^n p, \tag{40}$$

defines an isomorphism between the spaces $S(\mathbb{R}^n)$ and $S(\mathbb{R}_n)$. The Fourier transform of tempered distributions is defined by the condition

$$\int h(x)\psi(x)\, d^n x = \int h^{\mathrm{F}}(p)\psi^{\mathrm{F}}(-p)\, d_n p,$$

and is written in the form (39)–(40). It provides an isomorphism between the spaces of tempered distributions $S'(\mathbb{R}^n)$ and $S'(\mathbb{R}_n)$.

The tensor algebra $\otimes RS^4$ of the nuclear space RS^4 is called the Borchers algebra (Borchers, 1984; Horuzhy, 1990). Since the subset $\overset{n}{\otimes} S(\mathbb{R}^k)$ is dense in $S(\mathbb{R}^{kn})$, we henceforth identify it with the algebra (1). Then any state f of this algebra is represented by a collection of tempered distributions $\{W_n \in S'(\mathbb{R}^{4n})\}$ by the formula (2). Let us focus on the states of the Borchers algebra A_{RS^4} which describe free real scalar fields of mass m.

Let us provide the nuclear space RS^4 with the positive complex bilinear form

$$\begin{aligned}(\psi \mid \psi') &= \frac{2}{i}\int \psi(x)D^-(x-y)\psi'(y)\, d^4x\, d^4y \\ &= \int \psi^{\mathrm{F}}(-\omega, -\vec{p})\psi'^{\mathrm{F}}(\omega, \vec{p})\, \frac{d_3 p}{\omega},\end{aligned}$$

$$D^-(x) = i(2\pi)^{-3}\int \exp[-ipx]\theta(p_0)\delta(p^2 - m^2)\, d^4 p, \tag{41}$$

where $D^-(x)$ is the negative frequency part of the Pauli–Jordan function, p^2 is the Minkowski square, and

$$\omega = (\vec{p}^2 + m^2)^{1/2}.$$

Since the function $\psi(x)$ is real, its Fourier transform satisfies the equality $\psi^{\rm F}(p) = \overline{\psi}^{\rm F}(-p)$.

The bilinear form (41) is degenerate because the Pauli–Jordan function $D^-(x)$ obeys the mass shell equation

$$(\Box + m^2)D^-(x) = 0.$$

It takes nonzero values only at elements $\psi^{\rm F} \in RS_4$ which are not zero on the mass shell $p^2 = m^2$. Therefore, let us consider the quotient space $\gamma : RS^4 \to RS^4/J$, where $J = \{\psi \in RS^4 : (\psi \mid \psi) = 0\}$ is the kernel of the square form (41). The map γ assigns to each element $\psi \in RS^4$ with the Fourier transform $\psi^{\rm F}(p_0, \vec{p}) \in RS_4$ the couple of functions $(\psi^{\rm F}(\omega, \vec{p}), \psi^{\rm F}(-\omega, \vec{p}))$. Let us equip the factor space RS^4/J with the real bilinear form

$$(\gamma\psi \mid \gamma\psi')_L = \mathrm{Re}(\psi \mid \psi') = \frac{1}{2}\int [\psi^{\rm F}(-\omega, -\vec{p})\psi'^{\rm F}(\omega, \vec{p}) + \psi^{\rm F}(\omega, -\vec{p})\psi'^{\rm F}(-\omega, \vec{p})]\, \frac{d_3\vec{p}}{\omega}. \tag{42}$$

Then it is decomposed into the direct sum $RS^4/\mathcal{J} = L^+ \oplus L^-$ of the subspaces

$$L^{\pm} = \left\{\psi^{\rm F}_{\pm}(\omega, \vec{p}) = \frac{1}{2}(\psi^{\rm F}(\omega, \vec{p}) \pm \psi^{\rm F}(-\omega, \vec{p}))\right\},$$

which are mutually orthogonal with respect to the bilinear form (42).

There exist continuous isometric morphisms

$$\gamma_+ : \psi^{\rm F}_+(\omega, \vec{p}) \mapsto q^{\rm F}(\vec{p}) = \omega^{-1/2}\psi^{\rm F}_+(\omega, \vec{p}),$$
$$\gamma_- : \psi^{\rm F}_-(\omega, \vec{p}) \mapsto q^{\rm F}(\vec{p}) = -i\omega^{-1/2}\psi^{\rm F}_-(\omega, \vec{p})$$

of spaces L^+ and L^- to the nuclear space RS^3 endowed with the nondegenerate separately continuous Hermitian form

$$\langle q \mid q'\rangle = \int q^{\rm F}(-\vec{p})q'^{\rm F}(\vec{p})\, d_3p. \tag{43}$$

It should be emphasized that the images $\gamma_+(L^+)$ and $\gamma_-(L^-)$ in RS^3 are not orthogonal with respect to the scalar form (43). Combining γ and $\gamma_\pm$, we obtain the continuous morphisms $\tau_\pm RS^4 \to RS^3$ given by the expressions

$$\tau_+(\psi) = \gamma_+(\gamma\psi)_+ = \frac{1}{2\omega^{1/2}}\int [\psi^{\rm F}(\omega, \vec{p}) + \psi^{\rm F}(-\omega, \vec{p})]\, \exp[-i\vec{p}\vec{x}]\, d_3p,$$
$$\tau_-(\psi) = \gamma_-(\gamma\psi)_- = \frac{1}{2i\omega^{1/2}}\int [\psi^{\rm F}(\omega, \vec{p}) - \psi^{\rm F}(-\omega, \vec{p})]\, \exp[-i\vec{p}\vec{x}]\, d_3p.$$

Now let us consider the CCR algebra

$$\mathcal{G}(RS^3) = \{(\phi(q), \pi(q), I), q \in RS^3\} \tag{44}$$

modeled over the nuclear space RS^3, which is equipped with the Hermitian form (43). Using the morphisms $\tau_\pm$, let us define the map

$$RS^4 \ni \psi \mapsto \phi(\tau_+(\psi)) - \pi(\tau_-(\psi)) \in \mathcal{G}(RS^3). \tag{45}$$

With this map, one can think of (44) as being the algebra of the instantaneous CCR of scalar fields on the Minkowski space $\mathbb{R}^4$. Owing to the map (45), any representation of the nuclear CCR algebra $\mathcal{G}(RS^3)$ determined by a translationally quasi-invariant measure μ on $S'(\mathbb{R}^n)$ induces a state

$$f(\psi^1 \cdots \psi^n) = \langle \phi(\tau_+(\psi^1)) + \pi(\tau_-(\psi^1))] \cdots [\phi(\tau_+(\psi^n)) + \pi(\tau_-(\psi^n))] \rangle \tag{46}$$

on the Borchers algebra A_{RS^4} of scalar fields. Furthermore, one can justify that the corresponding distributions W_n fulfil the mass shell equation and that the following commutation relation holds:

$$W_2(x, y) - W_2(y, x) = -iD(x - y),$$

where

$$D(x) = i(2\pi)^{-3} \int \exp[-ipx](\theta(p_0) - \theta(-p_0))\delta(p^2 - m^2)\, d^4 p,$$

is the Pauli–Jordan commutation function. Thus, the states (46) describe real scalar fields of mass m.

For instance, let us take the Fock representation (31) of the CCR algebra $\mathcal{G}(RS^3)$. Using the formulae in Remark 3 where the form $\langle q \mid q' \rangle$ is given by the expression (43), one observes that the states f_{F} (46) satisfy the Wick theorem relations

$$f_{\mathrm{F}}(\psi^1 \cdots \psi^n) = \sum_{(i_1 \ldots i_n)} f_2(\psi^{i_1}\psi^{i_2}) \cdots f_2(\psi^{i_{n-1}}\psi^{i_n}), \tag{47}$$

while the state f_2 is given by the Wightman function

$$W_2(x, y) = \frac{1}{i} D^-(x - y). \tag{48}$$

Thus, the state f_{F} describes standard quantum free scalar fields of mass m.

Similarly, one can obtain states of the Borchers algebra A_{RS^4} generated by non-Fock representations of the instantaneous CCR algebra $\mathcal{G}(RS^3)$, e.g., if $K^{-1} = c\mathbf{1} \neq 2^{-1/2}\mathbf{1}$. These states fail to be given by Wightman functions. In particular, they are not covariant under time translations.

7. EUCLIDEAN SCALAR FIELDS

As was mentioned above, the causal forms (3) on the Borchers algebra A_{RS^4} are neither positive nor continuous. At the same time, they issue from the Wick rotation of Euclidean states of the commutative tensor algebra B_{RS^4}. These states play the role of Green's functions in Euclidean quantum field theory. It should be emphasized that they do not coincide with the Schwinger functions in axiomatic quantum field theory whose Minkowski partners are the Wightman functions, but not causal forms.

Let Q be a real nuclear space as above and A_Q its tensor algebra (37). We abbreviate with B_Q the complexified quotient of A_Q with respect to the ideal generated by the elements $q \otimes q' - q' \otimes q$ for all $q, q' \in Q$. It is the commutative tensor algebra of Q. Provided with the direct sum topology, B_Q becomes a topological involutive algebra. It coincides with the enveloping algebra of the Lie algebra of the additive Lie group $T(Q)$ of translations in Q. Therefore, we can obtain the states of the algebra B_Q by constructing cyclic strongly continuous unitary representations of the nuclear Abelian group $T(Q)$. As was stated in Section 3, such a representation is characterized by a positive-definite continuous generating function Z on Q which is the Fourier transform (11) of a bounded positive measure μ of total mass 1 on the (topological) dual Q' of Q. The corresponding cyclic strongly continuous unitary representation of the nuclear Abelian group $T(Q)$ is given by the operators (12) in the Hilbert space $L^2_C(Q', \mu)$ of square μ-integrable complex functions $\rho(u)$ on Q. If the function (23) is analytic at $t = 0$ for all $\phi \in \Phi$, a state $F(q_1 \cdots q_n)$ of B_Q is given by the expression (24).

A glance at this expression shows that, in applications to quantum field theory where $Q = RS^4$, the generating function Z plays the role of a generating functional represented by the functional integral (11), while the values (24) of the state F are vacuum expectations of Euclidean fields.

For instance, let μ be a Gaussian measure on Q' whose Fourier transform reads

$$Z(\varphi) = \exp\left[-\frac{1}{2}M(\varphi)\right],$$

where the covariance form $M(\phi_1, \phi_2)$ is a nondegenerate separately continuous Hermitian form on RS^4. This generating function defines a Gaussian state F of the algebra B_{RS^4} such that

$$F_1(\phi) = 0, \qquad F_2(\phi_1\phi_2) = M(\phi_1, \phi_2),$$

while $F_{n>2}$ obey the Wick relations (33). Furthermore, a covariance form M on RS^4 is uniquely determined as

$$M(\phi_1, \phi_2) = \int W_2(x_1, x_2)\phi_1(x_1)\phi_2(x_2)\, d^n x_1\, d^n x_2. \tag{49}$$

by a tempered distribution $W_2 \in S'(\mathbb{R}^8)$.

In particular, let a tempered distribution $M(\phi, \phi')$ in the expression (49) be Green's function of some positive elliptic differential operator $\mathcal{E}$, i.e.,

$$\mathcal{E}_{y_1} W_2(y_1, y_2) = \delta(y_1 - y_2),$$

where δ is Dirac's δ-function. Then the distribution W_2 reads

$$W_2(y_1, y_2) = w(y_1 - y_2), \tag{50}$$

and we obtain the form

$$\begin{aligned} F_2(\phi_1\phi_2) = M(\phi_1, \phi_2) &= \int w(y_1 - y_2)\phi_1(y_1)\phi_2(y_2)\, d^4y_1\, d^4y_2 \\ &= \int w(y)\phi_1(y_1)\phi_2(y_1 - y)\, d^4y\, d^4y_1 = \int w(y)\varphi(y)\, d^4y \\ &= \int w^{\mathrm{F}}(q)\varphi^{\mathrm{F}}(-q)\, d_4q, \quad y = y_1 - y_2, \\ \varphi(y) &= \int \phi_1(y_1)\phi_2(y_1 - y)\, d^4y_1. \end{aligned}$$

For instance, if

$$\mathcal{E}_{y_1} = -\Delta_{y_1} + m^2,$$

where Δ is the Laplacian, then

$$w(y_1 - y_2) = \int \frac{\exp(-iq(y_1 - y_2))}{q^2 + m^2}, \tag{51}$$

where q^2 is the Euclidean square, is the propagator of a massive Euclidean scalar field. Note that, restricted to the domain $(y_1^0 - y_2^0) < 0$, it coincides with the Schwinger function $s_2(y_1 - y_2)$.

Let the Fourier transform w^{F} of the distribution w (50) satisfy the condition (57) below. Then its Wick rotation (61) is the functional

$$\hat{w}(x) = \theta(x) \int_{\bar{Q}_+} w^{\mathrm{F}}(q)\ \exp(-qx)\, dq + \theta(-x) \int_{\bar{Q}_-} w^{\mathrm{F}}(q)\ \exp(-qx)\, dq$$

on scalar fields on the Minkowski space. For instance, let $w(y)$ be the Euclidean propagator (51) of a massive scalar field. Then due to the analyticity of

$$w^{\mathrm{F}}(q) = (q^2 + m^2)^{-1}$$

on the domain $\mathrm{Im}\, q \cdot \mathrm{Re}\, q > 0$, one can show that $\hat{w}(x) = -iD^c(x)$ where $D^c(x)$ is familiar causal Green's function.

8. THE WICK ROTATION

Let us describe the above mentioned Wick rotation of Euclidean states in the previous section. We start from the basic formulae of the Fourier–Laplace transform (Bogoliubov *et al.*, 1990). It is defined on Schwartz distributions, but we focus on the tempered ones.

Throughout, $\mathbb{R}^n_+$ and $\overline{\mathbb{R}}^n_+$ denote the subset of points of $\mathbb{R}^n$ with strictly positive Cartesian coordinates and its closure, respectively. Let $f \in S'(\mathbb{R}^n)$ be a tempered distribution and $\Gamma(f)$ the convex subset of points $q \in \mathbb{R}_n$ such that

$$e^{-qx} f(x) \in S'(\mathbb{R}^n). \tag{52}$$

In particular, $0 \in \Gamma(f)$. Let Int $\Gamma(f)$ and $\partial\Gamma(f)$ denote the interior and the boundary of $\Gamma(f)$, respectively.

The Fourier–Laplace (henceforth FL) transform of a tempered distribution $f \in S'(\mathbb{R}^n)$ is said to be the tempered distribution

$$f^{\mathrm{FL}}(p + iq) = (e^{-qx} f(x))^{\mathrm{F}}(p) = \int f(x)\, e^{i(p+iq)x}\, d^n x \in S'(\mathbb{R}_n), \tag{53}$$

which is the Fourier transform of the distribution (52) depending on q as parameters.

If Int $\Gamma(f) \neq \emptyset$, the FL transform $f^{\mathrm{FL}}(k)$ is a holomorphic function of complex arguments $k = p + iq$ on the open tube $\mathbb{R}_n + i\mathrm{Int}\Gamma(f) \subset \mathbb{C}_n$ over Int $\Gamma(f)$. Moreover, for any compact subset $Q \subset$ Int $\Gamma(f)$, there exist strictly positive numbers A and m, depending of Q and f, such that

$$|f^{\mathrm{FL}}(p + iq)| \leq A(1 + |p|)^m, \quad p \in \mathbb{R}_n, \quad q \in Q. \tag{54}$$

The evaluation (54) is equivalent to the fact that the function $h(p + iq)$ defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}_n)$ of the variables p depending continuously on parameters $q \in S$.

Let us notice that, if $0 \in$ Int $\Gamma(f)$, then

$$f^{\mathrm{FL}}(p + i0) = \lim_{q \to 0} f^{\mathrm{FL}}(p + iq)$$

coincides with the Fourier transform $f^{\mathrm{F}}(p)$ of f. The case of $0 \notin$ Int $\Gamma(f)$ is more intricate. Let S be a convex domain in $\mathbb{R}^n$ such that $0 \in \partial S$, and let $h(p + iq)$ be a holomorphic function on the tube T^S which defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}_n)$, depending on parameters q. One says that $h(p + iq)$ has a generalized boundary value $h_0(p) \in S'(\mathbb{R}_n)$ if, for any frustum $K^r \subset S \cup \{0\}$ of the cone $K \subset \mathbb{R}_n$ (i.e., $K^r = \{q \in K : |q| \leq r\}$), one has

$$h_0(\psi(p)) = \lim_{|q| \to 0, q \in K^r \setminus \{0\}} h_q(\psi(p))$$

for all functions $\psi \in S(\mathbb{R}_n)$ of rapid decrease. Then the following assertion holds.

Proposition 1. *Let* $f \in S'(\mathbb{R}^n)$, Int $\Gamma(f) \neq \emptyset$ *and* $0 \notin$ Int $\Gamma(f)$. *Then a generalized boundary value of the* FL *transform* $f^{\mathrm{FL}}(k)$ *in* $S'(\mathbb{R}_n)$ *exists and coincides with the Fourier transform* $f^{\mathrm{F}}(p)$ *of the distribution* f.

Let us apply this result to the following important case. The support of a tempered distribution f is defined as the complement of the maximal open subset U where f vanishes, i.e., $f(\psi) = 0$ for all $\psi \in S(\mathbb{R}_n)$ of support in U. Let $f \in S'(\mathbb{R}^n)$ be of support in $\overline{\mathbb{R}}^n_+$. Then $\mathbb{R}_{n+} \subset \Gamma(f)$, and the FL transform f^{FL} is a holomorphic function on the tube over $\mathbb{R}_{n+}$, while its generalized boundary value in $S'(\mathbb{R}_n)$ is given by the equality

$$h_0(\psi(p)) = \lim_{|q|\to 0, q\in\mathbb{R}_{n+}} f_q^{\mathrm{FL}}(\psi(p)) = f^{\mathrm{F}}(\psi(p)), \quad \forall\psi \in S(\mathbb{R}_n).$$

Conversely, one can restore a tempered distribution f of support in $\overline{\mathbb{R}}^n_+$ from its FL transform $h(k) = f^{\mathrm{FL}}(k)$ even if this function is known only on $i\mathbb{R}_{n+}$. Indeed, the formulae

$$\begin{aligned}\tilde{h} &= \int_{\mathbb{R}_{n+}} h(iq)\phi(q)\, d_n q = \int_{\mathbb{R}_{n+}} d_n q \int_{\overline{\mathbb{R}}^n_+} e^{-qx} f(x)\phi(q)\, d^n x \\ &= \int_{\overline{\mathbb{R}}^n_+} f(x)\hat{\phi}(x)\, d^n x, \quad \phi \in S(\mathbb{R}_{n+}),\end{aligned} \tag{55}$$

$$\hat{\phi}(x) = \int_{\mathbb{R}_{n+}} e^{-qx}\phi(q)\, d_n q, \quad x \in \overline{\mathbb{R}}^n_+, \quad \hat{\phi} \in S(\overline{\mathbb{R}}^n_+), \tag{56}$$

define a linear continuous functional $\tilde{h}(q) = h(iq)$ on the space $S(\mathbb{R}_{n+})$. It is called the Laplace transform $f^{\mathrm{L}}(q) = f^{\mathrm{FL}}(iq)$ of a tempered distribution f.

The image of the space $S(\mathbb{R}_{n+})$ with respect to the mapping $\phi(q) \mapsto \hat{\phi}(x)$ (56) is dense in $S(\overline{\mathbb{R}}^n_+)$. Then the family of seminorms $\|\phi\|'_{k,m} = \|\hat{\phi}\|_{k,m}$, where $\|.\|_{k,m}$ are seminorms (38) on $S(\overline{\mathbb{R}}^n)$, determines new coarser topology on $S(\mathbb{R}_{n+})$ such that the functional (55) remains continuous with respect to this topology. Then the following is proved (Bogoliubov *et al.*, 1990).

Theorem 1. *The mappings* (55) *and* (56) *provide one-to-one correspondence between the Laplace transforms* $f^{\mathrm{L}}(q) = f^{\mathrm{FL}}(iq)$ *of tempered distributions* $f \in S'(\overline{\mathbb{R}}^n_+)$ *and the elements of* $S'(\mathbb{R}_{n+})$ *which are continuous with respect to the coarser topology on* $S(\mathbb{R}_{n+})$.

This Theorem enables one to define the above mentioned Wick rotation of Euclidean states to causal forms on the Minkowski space.

Since the Minkowski space X and its Euclidean partner Y in $\mathbb{C}^4$ have the same spatial subspace, we further omit the dependence on spatial coordinates. Therefore, let us consider the complex plane $\mathbb{C}^1 = X \oplus iZ$ of the time x and the Euclidean time z and the complex plane $\mathbb{C}_1 = P \oplus iQ$ of the associated momentum coordinates p and q.

Let $W(q) \in S'(Q)$ be a tempered distribution such that

$$W = W_+ + W_-, \qquad W_+ \in S'(\overline{Q}_+), \qquad W_- \in S'(\overline{Q}_-). \tag{57}$$

For instance, $W(q)$ is an ordinary function at 0. For every test function $\psi_+ \in S'(X_+)$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\overline{Q}_+} W(q)\hat{\psi}_+(q)\,dq &= \frac{1}{2\pi}\int_{\overline{Q}_+} dq \int_{X_+} dx[W(q)\,\exp(-qx)\psi_+(x)]\\ &= \frac{1}{(2\pi)^2}\int_{\overline{Q}_+} dq\int_P dp\int_{X_+} dx[W(q)\psi_+^{\mathrm{F}}(p)\\ &\quad\times\exp(-ipx-qx)] = \frac{-i}{(2\pi)^2}\int_{\overline{Q}_+} dq\\ &\quad\times\int_P dp\left[W(q)\frac{\psi_+^{\mathrm{F}}(p)}{p-iq}\right]\\ &= \frac{1}{2\pi}\int_{\overline{Q}_+} W(q)\psi_+^{\mathrm{L}}(iq)\,dq,\end{aligned} \tag{58}$$

due to the fact that the FL transform $\psi_+^{\mathrm{FL}}(p+iq)$ of the function $\psi_+ \in S(X_+) \subset S'(X_+)$ exists and that it is holomorphic on the tube $P + iQ_+$, $Q_+ \subset Q_{\phi+}$. Moreover, $\psi_+^{\mathrm{FL}}(p+i0) = \phi_+^{\mathrm{F}}(k)$, and the function $\hat{\psi}_+(q) = \psi_+^{\mathrm{FL}}(-q)$ can be regarded as the Wick rotation of the test function $\psi_+(x)$. The equality (58) can be brought into the form

$$\begin{aligned}\frac{1}{2\pi}\int_{\overline{Q}_+} W(q)\hat{\psi}_+(q)\,dq &= \int_{X_+}\widehat{W}_+(x)\psi_+(x)\,dx,\\ \widehat{W}_+(x) &= \frac{1}{2\pi}\int_{\overline{Q}_+}\exp(-qx)W(q)\,dq, \quad x\in X_+.\end{aligned} \tag{59}$$

It associates to a distribution $W(q) \in S'(Q)$ the distribution $\widehat{W}_+(x) \in S'(X_+)$, continuous with respect to the coarser topology on $S(X_+)$.

For every test function $\psi_- \in S(X_-)$, the similar relations

$$\begin{aligned}\frac{1}{2\pi}\int_{\overline{Q}_-} W(q)\hat{\phi}_-(q)\,dq &= \int_{X_-}\widehat{W}_-(x)\phi_-(x)\,dx,\\ \widehat{W}_-(x) &= \frac{1}{2\pi}\int_{\overline{Q}_-}\exp(-qx)W(q)\,dq, \quad x\in X_-,\end{aligned} \tag{60}$$

hold. Combining (59) and (60), we obtain

$$\frac{1}{2\pi}\int_Q W(q)\hat{\psi}(q)\,dq = \int_X \widehat{W}(x)\psi(x)\,dx, \tag{61}$$

$$\hat{\psi} = \hat{\psi}_+ + \hat{\psi}_-, \qquad \psi = \psi_+ + \psi_-,$$

where $\widehat{W}(x)$ is a linear functional on functions $\psi \in S(X)$, which together with all derivatives vanish at $x = 0$. One can think of $\widehat{W}(x)$ as being the Wick rotation of the distribution (57). One should additionally define $\widehat{W}$ at the point $x = 0$ in order to make it to a functional on the whole space $S(X)$. This is the well-known ambiguity of chronological forms in quantum field theory.

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