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Proceeding Paper

On Single Server Queues with Batch Arrivals †

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Abstract: We consider a queuing system $GI^{\nu}|M|1|\infty$ with arrival of customers in batches, general renewal arrivals, exponential service times, single service channels and an infinite number of waiting positions, where customers are serviced in the order of their arrival. In the stationary case, new forms of the probability generating functions of the number of clients in the system are derived. These new forms are written in terms of the p.g.f. of the tail distribution function of the number of customers per group and of the p.g.f. of an embedded discrete time homogeneous Markov chain. In a queuing system with a batch Poisson arrival flow $M^{\nu}_{\lambda}|M_{\mu}|1|\infty$, the number of customers in the system can be obtained from the normalized tail distribution.

Keywords: queueing system; infinite capacity; server; batch arrivals; renewal process; probability generating functions; embedded Markov chain; distribution of the number of customers

1. Introduction

Many practical applications in communication systems, production systems, transportation and stocking systems, information processing systems, etc., can be modeled as a queueing system. Therefore, queuing theory is very useful for solving this problem. One of the most important types of queueing systems are bulk queuing systems [1]. Batch queues are a class of queues in which arrival or service (or both) are in bulk. Many scientific publications are devoted to this type of queuing system [2,3].

In this manuscript, we consider a batch queueing system $GI^{\nu}|M|1|\infty$. It was considered in the works [4,5]. A brief description of this system is as follows. Customer arrival moments $0 < t_1 < t_2 < ... < t_n < ...$ constitute a renewal process [6] with the probability generating function $P\{t_n - t_{n-1} < t\} = F(t)$. Customers arrive in batches at a single server queue. At every moment t_n, a group of ν_n customers arrives. The collection of these random variables ν_n is independent and identically distributed. Additionally, suppose that ν_n is bounded and

$$\alpha(z) = Mz^{\nu_n} = \alpha_1 z + \alpha_2 z^2 + ... + \alpha_m z^m \,, \; \alpha_m \neq 0$$

is its generating function. The system has a single service channel and the service time is exponentially distributed with parameter μ. The queue has infinite capacity and customers are serviced in the order of their arrival.

Let a stochastic process $\xi(t)$ denote the number of customers in the queueing system at time t. The stationary distribution of this process can be described using the probability generating function

$$P(z) = \lim_{t\to\infty} Mz^{\xi(t)} = \sum_{n=0}^{\infty} p_n z^n \,. \tag{1}$$

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The probability p_n can be interpreted as the fraction of time that n customers are in the system. Consider process $\xi(t)$ at the arrival moments of batches of customers and denote

$$\xi_n = \xi(t_n - 0),\ n = 1, 2, \dots,\ \xi_1 = 0\,.$$

Then, ξ_n describes the number of customers in the system at the arrival moment of batches of customers t_n. It is obvious [4] that the sequence of ξ_n constitutes a homogenous Markov chain.

The stationary distribution of the chain ξ_n can be described using the probability generating function

$$\pi(z) = \lim_{n\to\infty} Mz^{\xi_n} = \sum_{k=0}^{\infty} \pi_k z^k\,. \tag{2}$$

We will calculate the stationary distribution of the process $\xi(t)$ by calculating the corresponding distribution in the embedded Markov chain ξ_n.

It is known [4,5] that Markov chain ξ_n has a stationary distribution if and only if

$$\nu = \sum_{k=0}^{m} k\alpha_k < \mu T\,, \tag{3}$$

where $T = \int_0^\infty t dF(t)$ is the average inter-arrival time and

$$\nu = \mathsf{M}\nu_n = \alpha'(z)\big|_{z=1}$$

is the average number of customers in an arriving batch. The steady state condition (3) of the queue can be written in the form of the traffic rate

$$\rho = \frac{\nu}{\mu T} < 1 \tag{4}$$

where ρ is the traffic rate, generalized here for batch systems. Next, we suppose that inequality (4) holds.

2. Results

As we will see below, some normalized tail probabilities [7] only connect the stationary distribution of the stochastic process $\xi(t)$ with the stationary distribution of the chain ξ_n. Therefore, it will be convenient to use a notation for the distribution tails of ν_n. Thus, we shall write

$$A_k = \mathsf{P}\{\nu_n \geqslant k\} = \sum_{l=k}^{m} \alpha_l\,,\ k = 1, \dots, m$$

and

$$A(z) = \frac{1}{\nu} \sum_{k=1}^{m} A_k z^k\,. \tag{5}$$

In this case, it is easy to see that $A(z)$ is the probability generating function for some discrete random variable ζ with the probability mass function

$$q_k = \mathsf{P}\{\zeta = k\} = A_k/\nu\,,\ k = 1, \dots, m\,.$$

Let us call the probability distributions given by $\{q_k\}$ the normalized distribution tails and $A(z)$ the probability generating function of the normalized distribution tails.

For $A(z)$, it can be shown that

$$A(z) = \frac{z}{\nu} \frac{1 - \alpha(z)}{1 - z}\,.$$

Thus, we have the following chain of equalities:

$$\frac{\nu}{z}A(z) = \sum_{l=k}^{m} A_k z^{k-1} =$$

$$= \alpha_1 + \alpha_2(1+z) + \cdots + \alpha_m\left(1 + z + \cdots + z^{m-1}\right) =$$

$$= \frac{\alpha_1(1-z) + \alpha_2\left(1-z^2\right) + \cdots + \alpha_m(1-z^m)}{1-z} =$$

$$= \frac{1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_m z^m}{1-z} = \frac{1-\alpha(z)}{1-z}.$$

This approach allows us to formulate two theorems.

Theorem 1. *The condition stationary distribution of process $\xi(t)$ exists and it may be defined by the generating function*

$$P(z) = \rho\pi(z)A(z) + 1 - \rho\,, \tag{6}$$

where $A(z)$ is the probability generating function (5), ρ is the traffic rate (4) and $\pi(z)$ is defined by (2).

Remark 1. *Formula (6) shows that the distribution of the probability p_n is a mixture of the degenerate distributions. The distribution can be represented as a convolution of two distributions, one of which is the distribution of the nested Markov chain and the other is the normalized tail distribution of the arriving batch sizes.*

Now, let us consider a queuing system $M_\lambda^\nu|M_\mu|1|\infty$ with batch Poisson arrival flow, which contains the arrival rate constant λ, i.e., the batch of customers arrives with exponential inter-arrival times with mean $T = \frac{1}{\lambda}$. In this case, the following theorem holds.

Theorem 2. *For the system $M_\lambda^\nu|M_\mu|1|\infty$, under (4), the condition stationary distribution of process $\xi(t)$ exists and it may be defined by the generating function*

$$P(z) = \pi(z) = \frac{1-\rho}{1-\rho A(z)} \tag{7}$$

where $A(z)$ is from (5), ρ is the traffic rate (4) and $\pi(z)$ is defined by (2).

3. Conclusions

In this article, we studied relationships between probability distributions in a single server batch queueing model $GI_\lambda^\nu|M_\mu|1|\infty$. Future research may be devoted to the search for similar probabilistic relationships between the target sequence of probabilities and the corresponding nested Markov chains in other queuing systems.

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Abbreviations

The following abbreviations are used in this manuscript:

p.g.f.	probability generating functions

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