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Imaging of Layered Media in Inverse Scattering Problems for an Acoustic Wave Equation

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Abstract—Two-dimensional (2D) inverse scattering problems for the acoustic wave equation consisting of obtaining the density and acoustic impedance of the medium are considered. A necessary and sufficient condition for the unique solvability of these problems in the form of the law of energy conservation has been established. It is proved that this condition is that for each pulse oscillation source located on the boundary of a half-plane, the energy flow of the scattered waves is less than the energy flux of waves propagating from the boundary of this half-plane. This shows that for inverse dynamic scattering problems in acoustics and geophysics when the law of energy conservation holds it is possible to determine the elastic density parameters of the medium. The obtained results significantly increase the class of mathematical models currently used in solving multidimensional inverse scattering problems. Some specific aspects of interpreting inverse problems solutions are considered.

Keywords: acoustic equations, Klein–Gordon equations, Gelfand–Levitan equations, Galerkin method, eikonal, density, acoustic impedance

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INTRODUCTION

We consider two-dimensional (2D) inverse dynamic scattering problems for a wave equation describing the propagation of bulk waves in an inhomogeneous medium. There is extensive scientific literature on inverse dynamic coefficient problems for hyperbolic equations [4–23] starting from [1–3]. To date, the development of methods for the solution of one-dimensional inverse problems have been practically accomplished. The most effective among those are Volterra's method for integral and functional equations [7, 8], the method of the Gelfand–Levitan–Kreyn integral equations [9–12], the method of inverting a difference scheme [13, 14], variational methods [15, 16], and the boundary control (*BC*) method [17, 18].

The transition to the solution of multidimensional inverse dynamic problems of scattering in one form or another was due to the reduction of multidimensional inverse problems to one-dimensional ones based on the mapping approach [19–21]. First of all, it was successfully done by reconstructing the coefficients in the lower part of the equations [19]. The further sophistication of the model was based on coordinate systems connected with the notions of wave fronts and rays. This significantly narrowed the range of the considered equations. The best studied is the acoustic equation where the inverse problem is reduced to the reconstruction of the medium's density [5].

This work is an attempt to extend the range of mathematical models, for which the statement of inverse problems seems practically feasible. We considered a 2D medium characterized by two parameters, including density and the elastic parameter, i.e., the volume compressibility coefficient. Other pairs of functions can also be considered, such as the density and the propagation velocity of the compressional waves or density and acoustic impedance. The paper shows that the exact scattering data make it possible to reconstruct alongside the density also the two remaining parameters in the observer's frame of reference. From the practical viewpoint, a medium can be interpreted as a substance. This result can prove useful, in particular, in interpreting the data of a geophysical survey.

One of the main tasks of a geophysical survey is subsurface imaging. This is especially important for deep-seated deposits of hydrocarbons. The main component of comprehensive geophysical prospecting is a seismic survey, which supplies up to 90% of the useful information about the structure of the layers of the earth's strata. At the same time, the method of seismic reflection (MSR) and the refracted waves method (RWM) are based on the solution of the inverse dynamic problem of scattering, when the geological structure of the medium is to be reconstructed using the field of elastic waves scattered by the near-surface layers of earth formations (up to 4–5 km).

The classical scheme of observations in exploration geophysics is two-dimensional land seismic profiling, in which a local source of seismic waves (explosive, acoustic, vibrational, etc.) is located on the surface, and the registration of the reflected or refracted waves is linear or areal. In the first case, the task is two-dimensional seismic imaging. There, the source is located at M different points and, therefore, MN seismograms are recorded where N is the number of seismic receivers in the geophone streamer.

For areal observations, the number of seismograms is about M^2N, and the task is to construct a $3D$ model of the geological environment. It should be noted that for various reasons this research is confined to constructing a series of independent longitudinal sections followed by the reconstruction of a $3D$ model based on interpolation methods. Moreover, 2D imaging can be implemented in field conditions on medium-powered workstations.

Thus, the considered statements of direct and inverse scattering problems are the basis for the relevant mathematical models of modern seismic exploration.

1. BODY WAVES IN LAYERED MEDIA

The propagation of seismic waves in the geological environment is extremely complex and, therefore, mathematical models describing this process mostly offer a limited scope of application. However, long years of geophysical research have firmly established the understanding of the earth's crust as a layered structure where the wave propagation is described by a system of partial differential equations.

Even in the simplest case of an isotropic medium, the equations of elasticity theory determine two different velocities of the propagating elastic (compressional and shear) waves, which results in the converted waves. Despite this, many inverse problems of seismic explorations are solved using acoustic approximation, which only involves the consideration of compressional or P-waves. This approach is justified by the fact that most seismic sources directly generate only P-waves, and since under the waves' incidence on the layer boundaries close to normal, the mode conversion can be neglected, the elastic problem leads to the corresponding acoustic statement.

Thus, the considered problem of 2-D imaging within the acoustic approximation is meaningful and informative. Note that the mathematical formulation of the problem and methods for its study are extended to the multidimensional case.

We consider a system of equations describing changes in the pressure fluctuations $p(\mathbf{r},t)$ and in the displacement velocity $\mathbf{v}(\mathbf{r},t)$ in a two-dimensional acoustic medium

$$\rho(\mathbf{r})\mathbf{v}_t = -\nabla p, \quad p_t = -\rho(\mathbf{r})a^2(\mathbf{r})\operatorname{div}\mathbf{v}, \quad \mathbf{r} \in \mathbb{R}^2,$$

where ρ is the density and a is the kinematic (sound) velocity. The pressure in this case is governed by the acoustic wave equation, i.e.,

$$p_{tt} = \rho a^2 \operatorname{div}(1/\rho \operatorname{grad} p).$$

As is known [5, 22], the equation in some (semigeodesic) coordinate system is reduced to a special form, for which the description of the wave propagation is one-dimensional. In the two-dimensional case, this coordinate system can be constructed explicitly and has a transparent physical meaning. This naturally yields a system of coordinate curves, i.e., wave fronts.

Further consideration will be within the frame of an layered $2D$ acoustic medium, which fills a half-space $\xi \geq 0$, where ξ, η is the observation system of the Cartesian coordinate system. According to the geophysical tradition, ξ here acts as the depth, $\xi = 0$ is the daylight surface, and η denotes a lateral variable along the direct recording of observations on the daylight surface. It is assumed that $a(0,\eta) = a_0(\eta)$ and $\rho(0,\eta) = \rho_0(\eta)$ are known. In our consideration, we also use functions $\sigma(\xi,\eta) = a(\xi,\eta)\rho(\xi,\eta)$, i.e., the hardness (acoustic impedance) of the medium, and, accordingly, $\sigma_0(\eta) = \sigma(0,\eta)$.

Let there exist a function $x(\xi,\eta)$, which is the solution of the following initial eikonal problem:

$$a^2(\xi,\eta)|\nabla x|^2 = 1, \quad \xi > 0, \quad x(0,\eta) = 0, \quad -\infty < \eta < \infty.$$

The physical meaning of the eikonal $x(\xi,\eta)$ is the time of signal propagation from the point (ξ,η) to the surface $\xi = 0$.

We consider a set of curves $y(\xi,\eta) = \text{const}$, which are orthogonal to the curves $x(\xi,\eta) = \text{const}$. From the orthogonality condition we have

$$y_\xi = -Gx_\eta, \quad y_\eta = Gx_\xi, \tag{1}$$

Hence, for $G(\xi,\eta)$ we obtain the Cauchy problem for the transport equation

$$\nabla G \nabla x + G\Delta x = 0, \quad G(0,\eta) = 1/\rho_0(\eta),$$

where $\nabla = \{\partial_\xi, \partial_\eta\}$, $\Delta = \partial_\xi^2 + \partial_\eta^2$, and the initial condition is selected for the convenience of further consideration.

Thus, a semigeodesic orthogonal coordinate system has been constructed. Since the differential operators in this coordinate system take the form

$$\nabla = \left\{\frac{1}{a.}\frac{\partial}{\partial x}, \frac{G.}{a.}\frac{\partial}{\partial y}\right\}, \quad \text{div} = \left\{\frac{G.}{a.^2}\frac{\partial}{\partial x}\frac{a.}{G.}, \frac{G.}{a.^2}\frac{\partial}{\partial y}a.\right\},$$

the initial acoustic wave equation in the new coordinates is written as follows:

$$\frac{\partial^2 v}{\partial t^2} = \rho.G.\frac{\partial}{\partial x}\left(\frac{1}{\rho.G.}\frac{\partial v}{\partial x}\right) + \rho.G.\frac{\partial}{\partial y}\left(\frac{G.\partial v}{\rho.\ \partial y}\right),$$

where $v(x,y,t) = p(\xi(x,y),\eta(x,y),t)$, and when the coordinates are changed the coefficients are marked by a dot subscript.

We assume $1/\sqrt{\rho.G.} = z(x,y)$, then $z.(\xi,\eta) = 1/\sqrt{\rho G}$ and $z.(0,\eta) = z(0,y) = 1$. This last equation can be represented in the form

$$v_{tt} = v_{xx} + 2/z\,(z_x v_x + G.^2 z_y v_y) + (G.^2 v_y)_y. \tag{2}$$

Coefficients ρ, G, z have a definite physical meaning and are interconnected. Indeed, by definition $\rho G z.^2 = 1$, and the properties of the transport equation solution imply the equality $Gs/a = \text{const}$, which holds true along the ray tube where s is the magnitude of its cross section depending on the space dimensionality (for a 2D medium it is the linear dimension). Under a slight change in the section along the ray, the following approximated equalities hold true:

$$z. \approx \sqrt{\sigma_0/\sigma}, \quad G \approx a/\sigma_0.$$

Summing up, the proposed mathematical model takes into account the following main properties of the studied object:

—the inhomogeneous medium is characterized by two functions, i.e., density and compressional wave velocity, as well as the acoustic impedance depending on it, which are known on the daylight surface;

—the medium is layered; i.e., the changes in the parameters along the daylight surface are small in comparison to their depth changes;

—the ray approximation there holds; within it, only effects along the ray's tube are significant, i.e., scattering and geometrical divergence;

—the ray's propagation is close to vertical due to the method of the excitation of vibrations and the layered medium;

—the law of the energy conservation holds and consists in the energy flow of the waves scattered by the medium being less than the energy flow of the waves leaving the daylight surface.

It should be noted that this physical and, accordingly, mathematical model is common in geophysics. At the same time, within this model, we have obtained new results, mostly on the solvability of the inverse dynamic seismic problem based on energy considerations and the possibility of constructing geological cross sections for various physical parameters (density, velocity, impedance) in the observation coordinate system (the depth profiles). The latter circumstance is crucial for enhancing the resolution of the existing methods for processing and interpreting seismic data.

2. SCATTERING PROBLEMS IN GALERKIN'S APPROXIMATION

We write (2) as

$$zv_{tt} = zv_{xx} + 2(z_x v_x + cz_y v_y) + z(cv_y)_y, \tag{3}$$

where $c(x,y) = G.^2(x,y)$, $z(x,y) > 0$. The simplifying replacement $u = zv$ in (3) yields the Klein–Gordon equation describing the propagation of bulk waves of pressure fluctuations in the layered seismic acoustic medium

$$u_{tt} = u_{xx} + (cu_y)_y - qu \tag{4}$$

where $q = q(x,y)$ is determined by the equality

$$q = (z_{xx} + (cz_y)_y)/z. \tag{5}$$

Evidently, Eq. (4) can also be considered in the case of an arbitrary continuous potential $q(x,y)$, i.e., unrepresentable in the above form, just as Eq. (3), in which coefficient $z(x,y)$ is not bound by equality (5).

In the domain $D = \{x, y| \, x > 0, -\infty < y < \infty\}$ in Eq. (4) let $c \in C^{0,1}(D)$ and $q \in C(D)$. We impose the following initial and boundary conditions for (4):

$$u(x,y,0) = u_t(x,y,0) = 0, \quad u_x(0,y,t) - h(y)u(0,y,t) = -\varphi(y,t), \tag{6}$$

where $h \in C(\partial D)$ and $\varphi \in C^{2,1}(\partial D \times [0,\infty))$. Further, we assume that functions c, q, h, and φ are $2L$-periodic in y. Let them be representable in the interval $[-L, L]$ by a finite Fourier series by the complex system $\psi_n(y) = \exp\{i\mu_n y\}$, $\mu_n = \pi n/L$, i.e.,

$$\begin{gathered} c(x,y) = \sum_{|n| \le N} c_n(x)\psi_n(y), \quad q(x,y) = \sum_{|n| \le N} q_n(x)\psi_n(y), \\ h(y) = \sum_{|n| \le N} h_n \psi_n(y), \quad \varphi(y,t) = \sum_{|n| \le N} \varphi_n(t)\psi_n(y). \end{gathered} \tag{7}$$

Further, both the direct and inverse problems for (4) are considered within the projected approach illustrated by the Galerkin method as an example. It will be shown that in this case problem (4) and (6) has a unique solution also representable as a finite series similar to (7), i.e.,

$$u(x,y,t) = \sum_{|n| \le N} u_n(x,t)\psi_n(y). \tag{8}$$

From (4), (6), and (7), we find that $u_n(x,t)$, $n \in \mathbf{N}$, where, at $x, t > 0$, $\mathbf{N} = \{0, \pm 1, , \ldots, \pm N\}$ are the complex solution of the following initial boundary-value problem for a second-order hyperbolic system of α equations ($\alpha = (2N+1)$):

$$\begin{gathered} \partial_t^2 u_n = \partial_x^2 u_n - \sum_{|j| \le N} (\mu_j \mu_n c_{n-j} + q_{n-j}) u_j, \\ u_n(x,0) = \partial_t u_n(x,0) = 0, \quad \partial_x u_n(0,t) - \sum_{|j| \le N} h_j u_{n-j}(0,t) = -\varphi_n(t). \end{gathered} \tag{9}$$

It is readily shown that the scattering problem (9) has a unique classical solution when the matching conditions $\varphi_n(0) = \varphi_n'(0) = 0$ are fulfilled. For this purpose, we introduce complex vector functions $u(x,t) = \{u_n(x,t)\}$, $\mathbf{\Phi}(t) = \{\varphi_n(t)\}$ and $\alpha \times \alpha$ Hermitian matrices $C(x) = \|c_{nj}(x)\|$, $Q(x) = \|q_{nj}(x)\|$, $H = \|h_{nj}\|$, where $c_{nj} = c_{n-j}$, $q_{nj} = q_{n-j}$, and $h_{nj} = h_{n-j}$ at $|n - j| \le N$; and $c_{nj} = q_{nj} = h_{nj} = 0$ at $|n - j| > N$; and the diagonal matrix $M = \mathrm{diag}\{\mu_n\}$, here $n, j \in \mathbf{N}$ everywhere.

Definition 1.1. *Matrices of the type $C, H, Q, \ldots$ form a linear space $\mathcal{L}^\alpha$.*

We write (9) in vector form (further, vectors are understood as column-vectors)

$$\begin{gathered} \mathbf{u}_{tt} = \mathbf{u}_{xx} - P(x)\mathbf{u}, \quad x, t > 0, \quad \mathbf{u}(x,0) = \mathbf{u}_t(x,0) = 0, \quad x \ge 0, \\ \mathbf{u}_x(0,t) - H\mathbf{u}(0,t) = -\mathbf{\Phi}(t), \quad t \ge 0, \end{gathered} \tag{10}$$

where $P(x) = Q(x) + MC(x)M$. Then the existence and uniqueness of the classical solution of problem (10) is proved similarly to the scalar case [23].

Of much greater interest are the direct and inverse scattering problems for the case where generalized solutions are considered. We introduce vector function $\mathbf{\Theta}_j(t) = \{\delta_{nj}\theta(t)\}$, $n, j \in \mathbf{N}$, where θ is the Heaviside step function, $\theta(+0) = \theta(0) = 1$, and δ_{nj} is the Kronecker symbol. Obviously, by definition, $\theta'(\cdot) = \delta(\cdot)$ is the Dirac delta function.

From the columns of the initial conditions $\mathbf{\Theta}'_j(t)$, $j \in \mathbf{N}$, we form a singular matrix $\Theta'(t) = \theta'(t)E$ and consider the following initial-boundary-value problem with respect to the matrix $U(x,t)$ of the fundamental solutions of the boundary-value problem for equations of the Klein-Gordon type, i.e.,

$$\begin{gathered} U_{tt} = U_{xx} - P(x)U, \quad x,t > 0, \quad U(x,0) = U_t(x,0) = 0, \quad x \geq 0, \\ U_x(0,t) - HU(0,t) = -\Theta'(t), \quad t \geq 0. \end{gathered} \tag{11}$$

The statement and solution of this problem are understood in accordance with [15].

The following statements whose proofs are similar to the considerations from [23] hold true.

1. Let $P \in C[0,T]$. The solution of problem (11) exists, is unique, and can be represented as

$$U(x,t) = \Theta(t-x) + \hat{U}(x,t), \quad 0 \leq x + t \leq 2T,$$

under any $T > 0$ where $\hat{U}$ is a continuous function.

2. We introduce domains $D_0(T) = \{x,t | x + t \leq 2T, x > t \geq 0\}$ and $D_1(T) = \{x,t | x + t \leq 2T, t \geq x \geq 0\}$. Then $U = 0$ in $D_0(T)$, $\hat{U} \in C^2(D_1(T))$ for any $T > 0$.

3. There the following equalities hold true:

$$\hat{U}_x(x,x) = -\hat{U}_t(x,x) = H + \frac{1}{2}\int_0^x P(s)\mathrm{d}s, \quad 0 < x \leq T. \tag{12}$$

4. Let $F(t) = U(0,t)$. Then $F \in C^2[0,2T] \cap C(0,2T]$, $F(+0) = \Theta(0)$, $F'(+0) = -H$, and matrix F has a central symmetry property (I-property): $F = \overline{IF}$, where I is the inversion about the center of the matrices and the upper bar denotes the complex conjugation.

We formulate the inverse scattering problem in terms of generalized solutions:

by tracing $U(0,t) = F(t)$, $t \in [0,2T]$, *the solution of problem (11) is determined by the continuous matrices* $C(x)$ *and* $Q(x)$, $x \in [0,T]$.

Theorem 2.1. *The solution of the inverse scattering problem for the Klein–Gordon equation (11) on* $[0,T]$ *is unique for any* $T > 0$.

Proof. It follows from [23] that continuous matrix $P(x)$, $x \in [0,T]$ is reconstructed uniquely from tracing solution $F(t)$, $t \in [0,2T]$. It will be shown that from $P(x) = Q(x) + MC(x)M$, the Toeplitz matrices $C(x)$ and $Q(x)$ are reconstructed separately on $[0,T]$. Indeed, for the elements of these matrices, we have

$$p_{nj} = q_{n-j} + \mu_j\mu_n c_{n-j},$$

from where, assuming $n - j = k$, first we find

$$c_k = (p_{n+1\,j+1} - p_{nj}) / (\mu_{n+1}\mu_{j+1} - \mu_n\mu_j),$$

and then also q_k. Hence, it follows that in the inverse problem functions $c(x,y)$ and $q(x,y)$ are uniquely reconstructed.

Now we consider the statement of direct and inverse scattering problems for (3) within the Galerkin method. It should be noted that system (4) and (5), taking into account the equality $u = zv$, is equivalent to (3). We introduce the Hermitian matrix $Z(x) = \|z_{nj}(x)\| > 0$ such that $z_{nj} = z_{n-j}$ at $|n-j| \leq N$, $z_{n-j} = 0$ at $|n-j| > N$, $n, j \in \mathbf{N}$, where $z_n(x)$ are the Fourier coefficients of the function $z(x,y) = \sum_{|n| \leq N} z_n(x)\psi_n(y)$, and we assume that $Z(0) = E$ and $Z'(0) = H$.

Let the coefficient $z(x,y)$ be related to $q(x,y)$ and $c(x,y)$ by Eq. (5). In this case we have

$$Z''Z^{-1} = Q(x) + MC(x)M = P(x), \quad x > 0, \quad Z(0) = E, \quad Z'(0) = H. \tag{13}$$

It should be noted that problem (13) is equivalent to the following Cauchy problem for the Riccati equation in the matrix $Y(x) = Z'Z^{-1}$:

$$Y' + Y^2 = P(x), \quad x > 0, \quad Y(0) = H. \tag{14}$$

We associate equality $u = zv$ with the equality $U = ZV$, where $V = V(x,t)$ is a matrix corresponding to $v(x, y, t)$. By substituting $U = ZV$ in (11), we arrive at the following initial-boundary-value problem in $V(x,t)$:

$$\begin{gathered} ZV_{tt} = ZV_{xx} + 2Z'V_x + [V, MC(x)MZ], \quad x,t > 0, \\ V(x,0) = V_t(x,0) = 0, \quad x \geq 0, \quad V_x(0,t) = -\Theta'(t), \quad t \geq 0, \end{gathered} \tag{15}$$

where $[K,J] = KJ - JK$. It is easy to see that (15) corresponds to the initial equation (3).

Since the solutions of problems (11) and (15) are related by the equality $U = ZV$, the properties of $V(x,t)$ are similar to the properties of $U(x,t)$. In particular, if $C(x)$ and $Z(x)$ satisfy Eq. (13) at $x \in [0,T]$, then, with allowance for the equality $Z(0) = E$, we have

$$V(0,t) = U(0,t) = F(t), \quad 0 < t \leq 2T. \tag{16}$$

The inverse scattering problem for (15) is given below:

from the trace $V(0,t) = F(t)$, $t \in [0,2T]$, *find the matrix-valued functions* $C(x)$ *and* $Z(x)$, $x \in [0,T]$.

Theorem 2.2. *The inverse scattering problem for the acoustic wave equation* (15) *on* $[0,T]$ *has no more than one continuous solution for any* $T > 0$.

Proof. Since the traces of problems (11) and (15) coincide, by theorem 2.1, the matrices $C(x)$ and $Q(x)$ are determined uniquely, and from the condition $F'(+0) = -H$, we find matrix H. Thus, matrix $Z(x)$ is uniquely determined as the solution of the Cauchy problem (13).

3. SOLVABILITY OF INVERSE SCATTERING PROBLEMS

The existence of solution is one of the key issues in studying inverse scattering problems. It is established in [12, 23] that the solvability of inverse problems is directly connected with the fulfillment of certain energy relations.

First we present the relevant result on a qualitative level. Suppose that the source in the boundary condition is expanded in spatial harmonics. For each of these harmonics any excitation is possible in the considered time interval. The totality of such space-time sources generates a set of solutions for scattering problems. At the same time, every solution corresponds to an energy flux at the boundary over the time of consideration.

We assume by definition that the flux is positive if the scattered field energy is lower than the energy of the field emerging from the border to the medium. The solvability criterion for any inverse scattering problem is the positive value of the flow for any spatial harmonic and any time representation of the source.

Now the necessary and sufficient condition for the solvability of the inverse scattering problem for the Klein-Gordon equation is obtained. We consider for (11) an auxiliary Cauchy problem with a timelike variable $x \geq 0$ with respect to matrix $W(x,t)$

$$W(0,t) = \Theta'(t), \quad W_x(0,t) = H\Theta'(t), \quad -\infty < t < \infty. \tag{17}$$

The following properties of the function $W(x,t)$ hold true:

1. Let $P \in C[0,\infty]$. Then the generalized solution of problem (11) and (17) can be represented in the form

$$W(x,t) = \frac{1}{2}(\Theta'(t-x) + \Theta'(t+x)) + \widehat{W}(x,t),$$

where $\widehat{W}$ is its regular part, which is even with respect to t.

2. $\widehat{W} \in C^1(\overline{K}_0)$, where $\overline{K}_0$ is the closure of the cone $K_0 = \{x,t \mid 0 < |t| < x\}$ and $\widehat{W} = 0$ outside $\overline{K}_0$.

3. There holds the formula

$$\widehat{W}(x,x) = \frac{1}{2}H + \frac{1}{4}\int_0^x P(s)ds, \quad x \geq 0.$$

We extend function $U(x,t)$, the solution of problem (11), in an odd way with respect to t to the entire half-plane $x \geq 0$ while retaining the designation. Moreover, the trace of the solution $F(t) = U(0,t)$ is also extended in an odd way. We express $U(x,t)$ in terms of $F(t)$ by means of the matrix of fundamental solutions $W(x,t)$. Thanks to the principle of superposition for $(x,t) \in \overline{K}_0$, we have

$$U(x,t) = \frac{1}{2}(F(t-x) + F(t+x)) + \int_{-x}^{x} \widehat{W}(x,\tau)F(t-\tau)d\tau = 0. \tag{18}$$

Matrix $F(t)$ in the neighborhood of zero can be represented as $F(t) = \Theta(t) - \Theta(-t) + \text{smooth}$ part. Taking this into account, by differentiating (18) with respect to t, at $|t| \leq x$ we arrive at a matrix integral equation of the Gelfand-Levitan type in $\widehat{W}(x,t)$, i.e.,

$$\widehat{W}(x,t) + \frac{1}{2}\int_{-x}^{x} \widehat{W}(x,\tau)F'(t-\tau)d\tau + \frac{1}{4}(F'(t-x) + F'(t+x)) = 0. \tag{19}$$

The kernel of this equation is continuous and symmetric due to the evenness of $F'(t)$. Moreover, function $F'(t)$ is smooth when $t \neq 0$, which implies that the solution of Eq. (19) is even with respect to t and continuously differentiable in K_0 (and also after the closure in $\overline{K}_0$). Hence, it follows, in particular, that $\widehat{W}_t(x,0) = 0$.

According to the properties of the solution of (11), the conditions presented below are necessary for the solvability of the inverse scattering problem for the Klein-Gordon equation on the interval $[0,T]$:

$$F = \overline{IF}, \quad F \in C^2[0,2T] \cap C(0,2T], \quad F(+0) = \Theta(0), \quad F'(+0) = H. \tag{20}$$

We also assume that function $F(t)$ in Eq. (19) is extended in an odd way retaining the designation. In this case, we have the following necessary and sufficient condition for the solvability of the inverse scattering problem whose proof can be found in [23].

Theorem 3.1. *Let the conditions of* (20) *be fulfilled. In this case, the inverse scattering problem for* (11) *is uniquely solvable in the class of the matrices* $P(x) = \overline{IP(x)}$ *continuous in the interval* $[0,T]$ *when and only when* Eq. (19) *is uniquely solvable for any* $x \in [0,T]$.

We consider the homogeneous parametric equation (19) and write it in operator form

$$A_x\Psi = -\Psi, \quad \Psi \in L_2^{\alpha}[-x,x], \quad 0 < x \leq T,$$

where A_x is a completely continuous operator acting in the space $L_2^{\alpha}[-x,x]$ of the functional matrices from $\mathcal{L}^{\alpha}$ with elements from $L_2[-x,x]$.

Definition 3.1. *The following scalar products are introduced in spaces* $\mathcal{L}^{\alpha}$ and $L_2^{\alpha}(\Omega)$. *For matrices* $K(\omega)$, $J(\omega) \in \mathcal{L}^{\alpha}$, *and* $\omega \in \Omega$, *with elements from* $L_2(\Omega)$, *it is assumed, accordingly, that*

$$\langle K,J\rangle = \mathrm{S}p(K^*J), \quad \langle K,J\rangle_{L_2^{\alpha}(\Omega)} = \int_{\Omega}\langle K(\omega),J(\omega)\rangle d\omega.$$

It is readily verified that the Hermitian form $\langle\Psi,A_x\Psi\rangle_{L_2^{\alpha}[-x,x]}$ holds true. In addition, operator A_x is continuously dependent on parameter x so that $\|A_x\| \to 0$ when $x \to 0$. Hence and from theorem 3.1 it follows that the inverse scattering problem in the interval $[0,T]$ is solvable when and only when for any $x \in [0,T]$ and $\Psi \neq 0$ from $L_2^{\alpha}[-x,x]$ the following inequality is fulfilled:

$$\langle\Psi,\Psi\rangle_{L_2^{\alpha}[-x,x]} + \langle\Psi,A_x\Psi\rangle_{L_2^{\alpha}[-x,x]} > 0. \tag{21}$$

The obtained solvability condition imposes certain restrictions on the class of the matrices $P(x)$ for which the solution of the inverse scattering problem exists. We show that this singles out of all admissible $P(x)$ those for which the law of energy conservation is observed.

Consider the classical statement of the mixed problem for (11)

$$U(x,0) = U_t(x,0) = 0, \quad U_x(0,t) - HU(0,t) = -\Phi(t).$$

We further multiply Eq. (11) on the left by U_t^*, we multiply the one conjugated to it on the right by U_t, and add the resulting equalities. This yields the next conservation law at an arbitrary point $(x,t) \in D_1$, i.e.,

$$\partial_t(\langle U_t, U_t\rangle + \langle U_x, U_x\rangle + \langle MU, C(x)MU\rangle + \langle U, Q(x)U\rangle) = 2\,\mathrm{Re}\,\partial_x\langle U_t, U_x\rangle.$$

We integrate the obtained equality on the plane x,t over the characteristic triangle $\Delta_{0,T}$ with vertices at points $(0,0)$, (T,T), and $(0,2T)$ and apply Green's formula in order to reduce the double integral to the contour one. This yields the conservation law for Hermitian forms, which can naturally be called the law of energy conservation,

$$\begin{gathered}0.5(\langle U_t - U_x, U_t - U_x\rangle_{L_2^\alpha(\Gamma)} + \langle MU, C(x)MU\rangle_{L_2^\alpha(\Gamma)} + \langle U, Q(x)U\rangle_{L_2^\alpha(\Gamma)} \\ + \langle U(0,2T), HU(0,2T)\rangle) = -\langle U_t(0,t), U_x(0,t) - HU(0,t)\rangle_{L_2^\alpha[0,2T]} \equiv \Pi,\end{gathered} \tag{22}$$

where Γ is the interval (segment) $\{x,t \mid x+t = 2T, 0 \le x \le T\}$, which can be passed in the direction of the increasing x. It can easily be shown that thanks to the *I-property,* the scalar product in the right-hand side of the last equality holds true.

The quantity $\Pi = \Pi(\Phi)$ is by definition an energy flux of waves at the boundary $x = 0$, which are excited by the source $\Phi(t)$, over time $2T$. For the thus introduced functional $\Pi(\Phi)$, it is assumed that

$$\Pi_T = \inf_{\Phi}\{\Pi(\Phi) \mid \langle\Phi,\Phi\rangle_{L_2^\alpha[0,2T]} = 1\}. \tag{23}$$

Intuitively, it is clear that for the physically realizable model of the medium the law of energy conservation must be observed, which is expressed by the inequality $\Pi_T > 0$. It appears that this inequality at the same time determines the sufficient solvability condition of the inverse problem.

Theorem 3.2. *The inverse scattering problem for Eq.* (11) *is solvable on* $[0,T]$ *when and only when*, $\Pi_T > 0$.

Proof. In (21), we make a substitution $\Psi(t) = \Phi(t-T)$, $t \in [0,2T]$ and write this inequality as follows:

$$\mathrm{Sp}\int_0^{2T}\Phi(t)\Phi^*(t)dt + \frac{1}{2}\mathrm{Sp}\int_0^{2T}\Phi(t)\int_0^{2T}\Phi^*(\tau)F'(t-\tau)d\tau dt > 0.$$

Hence, taking into account the evenness of $F'(t)$ and the possibility for a change in the integration order, we have (the sign $*$ denotes convolution)

$$\langle\Phi,\Phi\rangle_{L_2^\alpha[0,2T]} + \langle\Phi, F' * \Phi\rangle_{L_2^\alpha[0,2T]} > 0.$$

Now we turn to the energy conservation law (22) and substitute the following boundary equalities into it:

$$U_x(0,t) - HU(0,t) = -\Phi(t),\quad U_t(0,t) = \Phi(t) + (F' * \Phi)(t),$$

where the latter equality is fulfilled for the solution of the boundary-value problem thanks to the superposition principle. As a result, we have for any matrix-valued function $\Phi \in L_2^\alpha[0,2T]$

$$\Pi(\Phi) = \langle\Phi,\Phi\rangle_{L_2^\alpha[0,2T]} + \langle\Phi, F' * \Phi\rangle_{L_2^\alpha[0,2T]} > 0.$$

Since the integral operator A_T is completely continuous, we achieve $\inf_\Phi\{\Pi(\Phi)\}$ in (23) and, thereby, $\Pi_T > 0$.

The statement analogous to theorem 3.2 also holds for the case of the acoustic wave equation. It can easily be shown that for (15) the energy conservation law is also observed. Indeed, at an arbitrary point $(x,t) \in D_1$ the conservation law holds in the form

$$\partial_t(\langle ZV_t, ZV_t\rangle + \langle ZV_x, ZV_x\rangle + \langle ZMV, C(x)ZMV\rangle) = 2\,\mathrm{Re}\,\partial_x\langle ZV_t, ZV_x\rangle.$$

By integrating this equality over the characteristic triangle $\Delta_{0,T}$ and applying Green's formula, we obtain the energy conservation law for (15):

$$\begin{gathered}0.5(\langle Z(V_t - V_x), Z(V_t - V_x)\rangle_{L_2^\alpha(\Gamma)} + \langle ZMV, C(x)ZMV\rangle_{L_2^\alpha(\Gamma)}) \\ = -\langle V_t(0,t), V_x(0,t)\rangle_{L_2^\alpha[0,2T]} \equiv \Pi(\Phi).\end{gathered}$$

Using considerations similar to those presented above for the inverse scattering problem for (11), we prove that the necessary and sufficient condition of the solvability of the inverse scattering problem for (15) is expressed by the same criterion as for (11), that is $\Pi_T > 0$.

Theorem 3.3. *The inverse scattering problem for the Klein–Gordon equation* (11) *is solvable in the class of such matrices* $P(x)$ *that on* $[0,T]$ *problem* (14) *is solvable. The equality* $\Pi_T = 0$ *holds true when and only when* $\det Z(T) = 0$.

Proof. Let the inverse problem for (11) be solvable. Then, since $Z(0) = E$, the Cauchy problem (13) uniquely defines the matrix $Z(x) > 0$, $x \in [0,T]$. In contrast, if on $[0,T]$ the inverse problem is solvable for (15), Eq. (13) defines the matrix $P(x)$.

The second part of the statement is a direct corollary of the solvability criterion of the inverse scattering problem for (15) and of the continuity of functions Π_T and $Z(T)$ with respect to variable T.

We present the class of the matrices $C(x)$ and $Q(x)$, for which the inverse scattering problem is solvable for any N. The validity of the following statements immediately follows from theorem 3.2.

Corollary 3.1. *The inverse scattering problem for Klein–Gordon equation* (11) *for any* $N \in \mathbb{N}$ *is solvable in the class of continuous functional matrices* $C(x), Q(x) \in L_2^\alpha[0,T]$ *and* $H \in \mathcal{L}^\alpha$ *such that for any matrix* $\Psi \in L_2^\alpha[0,T]$, $\Psi \neq 0$, *there holds the inequality*

$$\langle \Psi, (Q(x) + MC(x)M)\Psi \rangle_{L_2^\alpha[0,T]} + \langle \Psi(0), H\Psi(0) \rangle > 0. \tag{24}$$

Corollary 3.2. *The inverse scattering problem for the acoustic wave equation* (15) *for any* $N \in \mathbb{N}$ *is solvable in the class of continuous functional matrices* $C(x), Z(x) \in L_2^\alpha[0,T]$ *such that for any matrix* $\Psi \in L_2^\alpha[0,T]$, $\Psi \neq 0$, *there holds the inequality*

$$\langle Z(x)\Psi, C(x)Z(x)\Psi \rangle_{L_2^\alpha[0,T]} > 0. \tag{25}$$

4. INTERPRETATION OF SOLUTIONS AND IMAGING OF GEOLOGICAL ENVIRONMENT

Since the purpose of the interpretation is to obtain the material parameters of the geomedium and its imaging, it is natural to impose the following conditions on the solutions of the above-mentioned inverse problems: the validity of inequalitues $c(x,y) > 0$ for (4) and $c(x,y)$, $z(x,y) > 0$ for (3) in $D_T = [0,T]\times[-L,L]$. It will be shown that for sufficiently large N it is attainable practically everywhere in D_T.

Theorem 4.1. *Let* $C(x)$ *and* $Q(x)$ *be the solution of the inverse problem for* (11) *and let the inequality* (24) *be fulfilled. Then we can find sufficiently large* N *such that* $c(x,y) \geq 0$ *almost everywhere in* D_T. *Similarly, let* $C(x)$, $Z(x)$ *be the solution of the inverse problem for* (15) *and let the inequality* (25) *be fulfilled. Then at a sufficiently large* N *inequalities* $c(x,y), z(x,y) \geq 0$ *are fulfilled almost everywhere in* D_T.

Proof. By virtue of Parseval equality, inequality (24) is equivalent to inequality

$$\iint_{D_T} (c(x,y)\psi_y^2(x,y) + q(x,y)\psi^2(x,y))dxdy + \int_{-L}^{L} h(y)\psi^2(0,y)dy > 0, \tag{26}$$

which holds true for any real continuous function $\psi(x,y) \neq 0$ $2L$-periodic in y and such that $\psi_y \in L_2(D_T)$.

Let us assume that there is a point $(x_0,y_0) \in D_T$ such that $c(x_0,y_0) < 0$. Then for any $\varepsilon > 0$ there is a sufficiently large $N(\varepsilon)$ such that as $\psi(x,y)$ a cup function can be selected, which oscillates with respect to y and can be represented as a finite Fourier series such that

$$\iint_{D_T} \psi^2(x,y)dxdy < \varepsilon, \quad \iint_{\mathbb{O}(x_0,y_0)} \psi_y^2(x,y)dxdy = 1, \quad \iint_{D_T\setminus\mathbb{O}(x_0,y_0)} \psi_y^2(x,y)dxdy < \varepsilon,$$

where $\mathbb{O}(x_0,y_0)$ is the vicinity of the point (x_0,y_0) of the nonzero Lebesgue measure, in which $c(x,y) < 0$. The contradiction in (26) proves that $c(x,y) \geq 0$ in D_T almost everywhere.

Since the inverse problem (15) is solvable, then except (25), for any matrix $\widehat{\Psi} \in L_2^\alpha[0,T]$, $\widehat{\Psi} \neq 0$, the following inequality is also fulfilled:

$$\langle \widehat{\Psi}, Z(x)\widehat{\Psi} \rangle_{L_2^\alpha[0,T]} > 0.$$

At the same time, thanks to the Parseval equality, for any real continuous functions $\psi(x,y), \widehat{\psi}(x,y) \neq 0$ from $L_2(D_T)$, which are $2L$-periodic in y, the following inequalities are fulfilled:

$$\iint\limits_{D_T} c(x,y) z^2(x,y) \psi^2(x,y) dx dy > 0, \quad \iint\limits_{D_T} z(x,y) \widehat{\psi}^2(x,y)) dx dy > 0.$$

Further we repeat the above-presented considerations selecting cup functions as $\Psi, \widehat{\Psi}$.

Now we study which physical parameters can be obtained from the scattering data. It should be recalled that in the inverse scattering problem for (15) density $\rho.(x,y)$ is uniquely defined. However, this function is not defined in the observation coordinates ξ, η but in the semigeodesic frame of reference x, y.

It will be shown that when $G.(x,y)$ is known, we can find the relationship between the initial Cartesian coordinates ξ, η and coordinates x, y, and also reconstruct the density $\rho(\xi,\eta)$ and kinematic velocity $a(\xi,\eta)$ in the medium.

Theorem 4.2. *Let $G.(x,y) > 0$ be the solution of inverse scattering problems for (3) and (4) in D_T, and let functions $a_0(\eta)$, $\rho_0(\eta) > 0$ be specified. Then we have defined the one-to-one mapping $\mathbf{r}(x,y) = \{\xi(x,y), \eta(x,y)\}$ determined by system (1). In addition, uniquely defined on $\mathbf{r}(D_T)$ are the density $\rho(\xi,\eta)$ and velocity $a(\xi,\eta)$ such that $\rho(0,\eta) = \rho_0(\eta)$ and $a(0,\eta) = a_0(\eta)$*

Proof. Since curvilinear coordinates x, y are orthogonal, the tangent vectors $\{\xi_x, \eta_x\}$ and $\{\xi_y, \eta_y\}$ to the coordinate curves are orthogonal, i.e.,

$$\xi_y = -A\eta_x, \quad \eta_y = A\xi_x, \tag{27}$$

hence for the Jacobian $J = D(\xi,\eta)/D(x,y)$, we have

$$\det J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = A(\xi_x^2 + \eta_x^2) = Aa^2.$$

On the other hand, since by construction

$$y_\xi = -Gx_\eta, \quad y_\eta = Gx_\xi,$$

for Jacobian $J^{-1} = D(x,y)/D(\xi,\eta)$ we have

$$\det J^{-1} = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = G(x_\xi^2 + x_\eta^2) = G/a^2.$$

This implies that $A = 1/G$. Recalling that $a_0(\eta)\rho_0(\eta) = \sigma_0(\eta)$, from (27), we obtain a system of equations in $\xi(x,y), \eta(x,y)$, i.e.,

$$\begin{aligned} &(\xi_x/G.)_x + (G.\xi_y)_y = 0, \quad \xi(0,y) = 0, \quad \xi_x(0,y) = a_0(\eta(y)), \\ &(\eta_x/G.)_x + (G.\eta_y)_y = 0, \quad \eta_x(0,y) = 0, \quad \eta_y(0,y) = \sigma_0(\eta(y)), \end{aligned} \tag{28}$$

where the function $\eta(y)$ is still unknown.

In order to define $\eta(y)$, we use function $\sigma_0(\eta)$ which is known when $\xi = x = 0$. Thus, we have

$$\left.\frac{dy}{d\eta}\right|_{x=0} = \frac{1}{\sigma_0(\eta)}, \quad y(\eta)|_{x=0} = \int_0^\eta \frac{d\nu}{\sigma_0(\nu)} = g(\eta),$$

hence $\eta|_{x=0} = g^{-1}(y)$, where g^{-1} is a function inverse to g. Thereby, in system (28) we define the functions $a_0(\eta(y)) = a_0(g^{-1}(y))$ and $\rho_0(\eta(y)) = \rho_0(g^{-1}(y))$.

Thanks to the initial assumption about the possibility of constructing semigeodesic coordinates, the solution of (28) exists, and by the Cauchy-Kovalevskaya theorem, the solution of the initial problem for

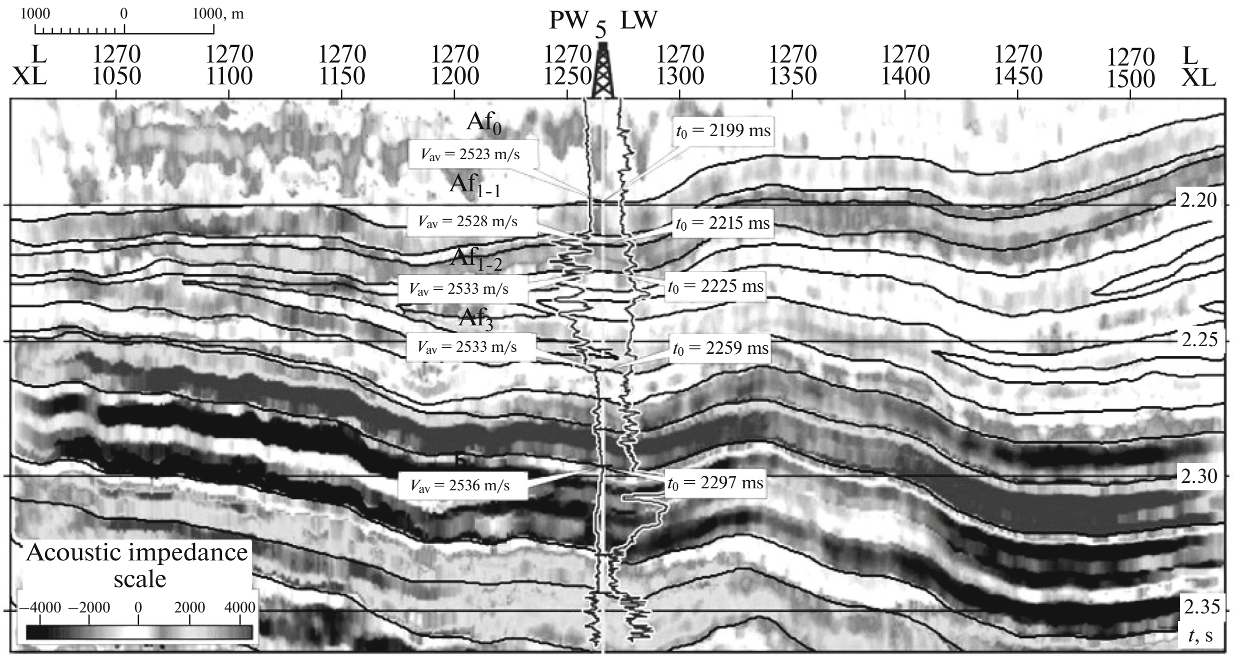

Fig. 1. The acoustic impedance section in the *inline* 1270 direction.

this system is unique. Knowing the transformation $\mathbf{r}(x,y) = \{\xi(x,y), \eta(x,y)\}$ and, consequently, the inverse to it $\{x(\xi,\eta), y(\xi,\eta)\}$, we finally get $\rho(\xi,\eta)$ and $a(\xi,\eta) = 1/|\nabla x(\xi,\eta)|$.

The following note appears useful for the practical solution of the problem.

Remark 1. The solution of inverse scattering problems is invariant with respect to the choice of the initial condition $G(0,\eta)$ in the Cauchy problem for the transport equation. In particular, it can be assumed that $G(0,\eta) = 1$.

This remark under certain conditions makes it possible to solve the problems of reconstructing the function $z(x,y)$ using the obtained $G.(x,y)$ and $q(x,y)$. As is known, the initial-boundary-value problem $2L$-periodic in y

$$z_{xx} + (G.^2 z_y)_y = qz, \quad x > 0, \quad z(0,y) = z_0(y), \quad z_x(0,y) = z_1(y),$$

where $z_0(y) = 1/\sqrt{\rho_0(\eta(y))}$ and $z_1(y) = h(y)z_0(y)$, is ill posed both because the solution is unbounded and due to its unstable initial data. It is especially important because the function $h(y)$ is unknown a priori and obtaining it from the equality $u_t(0,y,+0) = -h(y)$ by differentiating the trace of solution $u(0,y,t)$ is also unstable. It will be shown that within the natural assumptions, function $z(x,y)$ can be found even when $h(y)$ is unknown.

We assume that the sought function $z(x,y)$ is bounded under each $x > 0$. Then it will be sought as a solution of the boundary-value problem for Eq. (5) with the boundedness condition for $|z(x,y)|$ at $x \to +\infty$.

We also assume that the solution $u(x,y,t)$, corresponding to coefficients $c(x,y)$ and $q(x,y)$ is such that at $\operatorname{Re} s \geq 0$ its Laplace transform $\tilde{u}(x,y,s) \risingdotseq u(x,y,t)$ is determined. Then we obtain from (4) at $s = 0$ that

$$\tilde{u}_{xx} + (c\tilde{u}_y)_y - q\tilde{u} = 0,$$

which coincides with Eq. (5). Therefore, the bounded solution of the boundary-value problem for (5) can be constructed by solving the problem for the Klein-Gordon equation (4) with the conditions

$$u(x,y,t) = 0, \quad t < 0, \quad u(0,y,t) = f(y,t), \quad \int_0^\infty |u(+\infty,y,t)|\,dt < \infty, \tag{29}$$

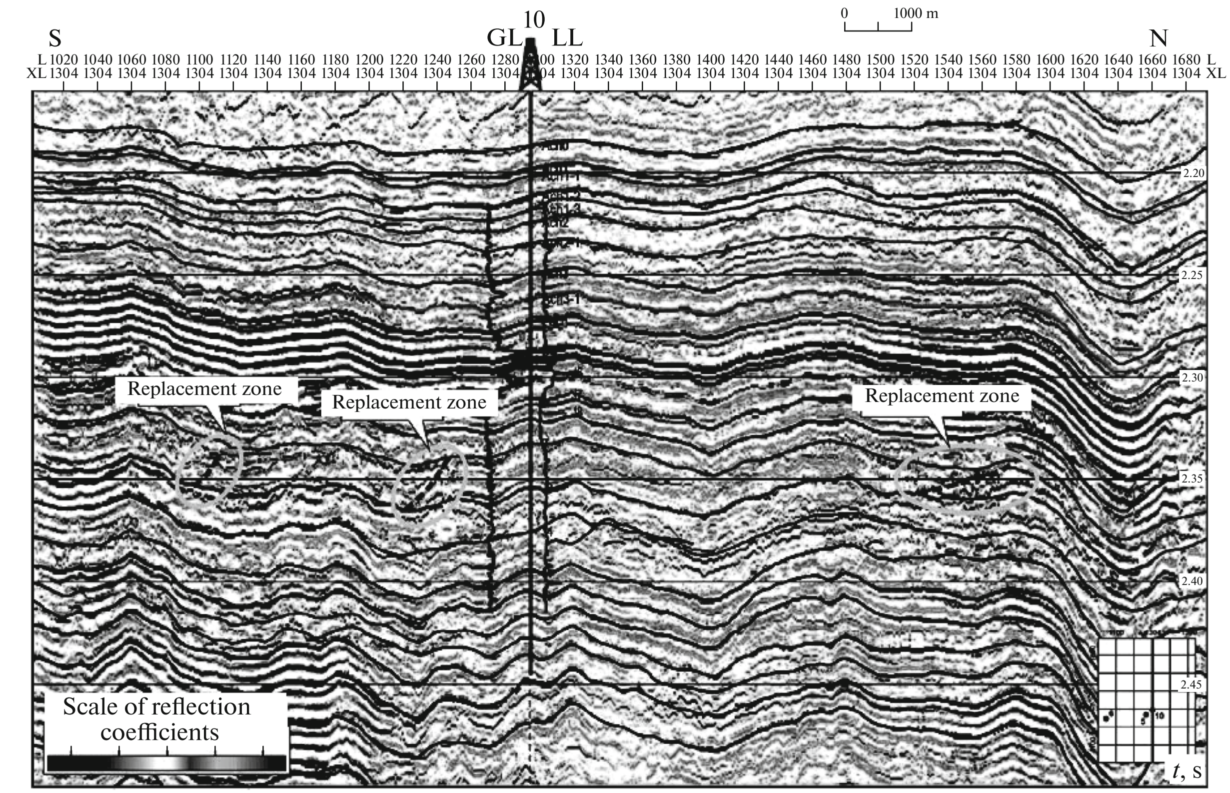

Fig. 2. Section of reflection coefficients in the *crossline* 1304 direction.

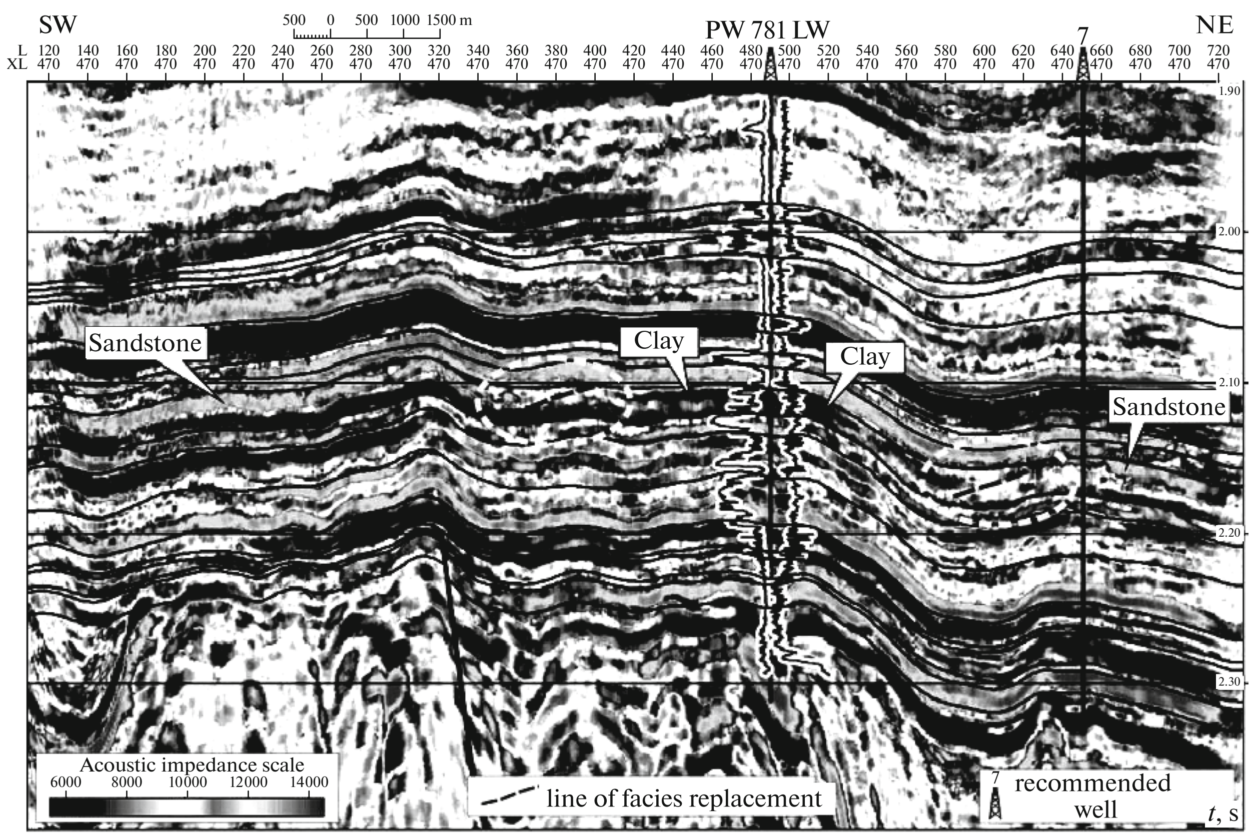

Fig. 3. Fragment of the *crossline* 470 acoustic impedance section.

where function $f(y,t)$ satisfies the equality $\widetilde{f}(y,0) = z_0(y)$.

It is necessary that the numerical solution of problem (5) and (29) could be algorithmically organized parallel to solving problem (11).

Example 1. Let $G.(x,y) = 1$, $q(x,y) = 0$, $z_0(y) = 1 + \cos(my)/m$, $m \in \mathbb{N}$, $m > 1$, and $y \in [-\pi,\pi]$. We assume

$$f(y,t) = \delta(t) + \mathrm{J}_0(mt)\cos(my),$$

where J_0 is the Bessel function of the 0th order. In this case, as the solution of the initial problem for (4) with the boundary conditions (29), we obtain a generalized function

$$u(x,y,t) = \delta(t-x) + \mathrm{J}_0(m\sqrt{t^2 - x^2})\cos(my),$$

Hence, we find

$$z(x,y) = 1 + \exp\{-mx\}\cos(my)/m$$

and, accordingly, when it is necessary,

$$h(y) = -m\cos(my)/(m + \cos(my)).$$

Remark 2. Clearly, the particular choice of $f(y,t)$ does not affect the ultimate result when the equality $\widetilde{f}(y,0) = z_0(y)$ is satisfied and is only determined by considerations of convenience.

CONCLUSIONS

In conclusion, we consider some of the results of mathematical modeling in the studied class of environments. As an example of real geological structures and the results of their imaging based on processing the geophysical survey data, we present the geological sections of uncovered wells in one of the areas of Western Siberia. The seismic data processing results were taken from [24–26].

In our short explanations to the presented images of the environment, we retain the conventional terms of geophysical exploration. Figure 1 shows an acoustic impedance section, i.e., functions $\sigma.(x,y)$, along the *inline* 1270 layout of the seismic receivers. The vertical line x represents the time (eikonal) in *ms*, and the horizontal line y shows the distance in *m*.

Figure 2 presents the stacked seismic data (time section) for the coefficients of the reflection, i.e., functions $0.5|\nabla \ln z(x,y)|$, in the direction transverse to the *inline* 1270 profile along the *crossline* 1304. These sections give an idea of the structure of the boundaries of the geological layers. They are used as a basis for the construction of the acoustic impedance sections.

Finally, Fig. 3 presents a detailed fragment of the *crossline* 470 impedance section. This section demonstrates the results of the geological interpretation aimed at the identification of the promising layers of the hydrocarbon content. In addition to the logging well LW and the production well PW, well number 7 in Fig. 3 designates a well recommended for drilling.

All the presented sections are related to the wells, thus making it possible to coordinate the results obtained by the solution of inverse problems, with the measurement data based on techniques of production well logging (PWL) and acoustic logging (AL). The presented illustrations show the relevance of the model of the layered geological environment adopted in this paper with the properties weakly changing horizontally, and demonstrate the possibility of constructing an image of such a medium by mathematical modeling.

It should be noted that the sections in Figs. 1–3 are constructed in the semigeodesic coordinates x, y. In order to represent these sections in real depths, the quantity inverse to the velocity is integrated over the depth at constant y, thereby obtaining the respective value of the eikonal. In this case, the velocity itself is taken from the data obtained by PWL and AL or from the a priori velocity model. Typically, these models are very approximate, since they either only correspond to the borehole environment or are described by parameters of the thick-layer medium model. Nevertheless, this method for the construction of the depth profiles is common for the existing software packages for the processing of seismic data. Obviously, the resulting depth profiles inevitably contain artefacts.

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