In Memoriam
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[1]

DERIVED SERIES OF THE MULTIDIMENSIONAL JENNINGS GROUP

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Abstract. The derived series of the d-dimensional analogue $\mathcal{J}^{(d)}(\mathbf{k})$ of the Jennings group ($d = 1$) of substitutions of mappings of formal power series with coefficients in an arbitrary field $\mathbf{k}$ is calculated. More precisely, we find the commutators of some subgroups in the group $\mathcal{J}^{(d)}(\mathbf{k})$ for $d \geq 2$.

1. Introduction

In 1954, Jennings introduced [1] the group $\mathcal{J}(\mathbf{k})$ of formal power series in one variable

$$f(x) = x + \alpha_2 x^2 + \ldots + \alpha_n z^n + \ldots \ , \quad \alpha_n \in \mathbf{k}, \tag{1}$$

with coefficients in an abelian ring with identity $\mathbf{k}$. This set becomes a group if we use substitution as a group operation: $f \circ g(x) = f\big(g(x)\big)$. The group $\mathcal{J}(\mathbf{k})$ is non commutative. But Jennings group has some eliments of commutativity and compactness – she is amenable [2]. Commutators of subgroups of Jennings group has a different character depending on the characteristics p of the coefficient field $\mathbf{k}$. A description of the commutators subgroups of the Jennings group in different cases ($p = 0$, p is odd, and p is even) is contained in the papers [1, Theorem 2.1.3], [3, Lemma 1.2.9], [4, Theorem 2], [2, Theorem 1.1, Propositions 2.1 and 2.9], and [5]. A brief overview of these results is

[1] Mitrofan and I are both from Moldova, but we first met in Moscow, when Mitrofan was a graduate student, and myself an undergraduate, at the Department of Higher Geometry and Topology, Faculty of Mechanics and Mathematics, Moscow State University. Mitrofan always had a strong personality, and next to him it was difficult to maintain independence rather than becoming subsumed into it. Therefore, our substantive communication really began in adulthood. Both in Moscow and in Chisinau, we primarily discussed issues of general human interest, rather than mathematics. Mitrofan was fond of collecting minerals and it was a pleasure for me to add to his collection, bringing specimens from my travels on the Kola Peninsula, in the Urals and the Pamirs. At what was to be our last meeting, I had the pleasure to gift Mitrofan the album of the minerals. I want to reiterate that Mitrofan's forceful personality demanded commensurate concentration whenever we communicated, but was always stimulating.

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presented in [6]. The article [6] considers the commutators of some special subgroups of the multidimensional Jennings group. Below we use the symbolism of article [6] and for the convenience of the reader we repeat word for word the notation from it. In addition, we considered it possible to repeat the formulation of the computational Proposition 2.3.

For a commutative ring $\mathbf{k}$ with unit, the d-dimensional Jenning's group $\mathcal{J}^{(d)}(\mathbf{k})$ means the group of formal power self mappings of $\mathbf{k}^d$ with fixed point zero and the linear part equal to identity. That is

$$\mathcal{J}^{(d)}(\mathbf{k}) = \{F = (f_1, ..., f_d)\colon f_r(x_1, ..., x_d) = x_r + \sum_{i_1+...+i_d \geq 2} a^{(r)}_{i_1,...,i_d} x_1^{i_1} \cdot ... x_d^{i_d},$$

$$a^{(r)}_{i_1,...,i_d} \in \mathbf{k},\ r = 1, ..., d\ \}.$$

For a fixed, clearly indicated value of d, we will simply write $\mathcal{J}(\mathbf{k})$. If it is necessary to take two elements $F, G \in \mathcal{J}(\mathbf{k})$, the coefficients of the second element are denoted by letters of the Greek alphabet. The coefficients of the commutator $[F, G] = FGF^{-1}G^{-1}$ are denoted with capital Latin letters. In further calculations, we use the equality $[F, G](GF) = FG$, which avoids the calculation of inverse elements. For two subgroups H_1, H_2, the reciprocal commutator $[H_1, H_2]$ is a subgroup generated by commutators of elements from H_1 and H_2, respectively. Since $(ab)^{-1} = b^{-1}a^{-1}$, $[a, b]^{-1} = [b, a]$ and the reciprocal commutator $[H_1, H_2]$ consists of finite products of commutators of elements from H_1 and H_2, respectively, but taken in an arbitrary order.

Introduce the subgroup

$$J_m = \big\{F \in \mathcal{J}(\mathbf{k})\colon a^{(r)}_{i_1,...,i_d} = 0 \text{ for } \ 2 \leq i_1 + i_2 + ... + i_d \leq m, \quad r = 1, ..., d\big\}.$$

In the group $\mathcal{J}^{(2)}(\mathbf{k})$ for even m the following subgroups are important for us

$$J^{=}_m = \big\{F \in J_m\colon a_{2r+1,m-2r} = b_{2r,m+1-2r} \text{ for every } r = 0, 1, \ldots, \frac{m}{2}\big\},$$

$$J^{=,0}_m = \big\{F \in J^{=}_m\colon a_{2r,m+1-2r} = b_{m+1-2r,2r} = 0 \text{ for every } r = 0, 1, \ldots, \frac{m}{2}\big\}.$$

A natural ultrametric is defined in the group $\mathcal{J}(\mathbf{k})$. For the elements $F \neq G$, take the maximum m such that

$$a^{(r)}_{i_1,...,i_d} = \alpha^{(r)}_{i_1,...,i_d} \text{ for } i_1 + \ldots + i_d \leq m, 1 \leq r \leq d.$$

The number m is determined by the condition $FG^{-1} \in J_m \setminus J_{m+1}$. The function $\varrho(F, G) = 2^{-m}$ sets an ultra metric on Jenning's group and J_m is a normal closed subgroup for any m. In the multidimensional case, the description of commutators is simpler. The results [6, Theorem 2.11, Theorems 2.7, 2.8, and 2.13] can be combined into the following compact formula.

Theorem 1.1. *If $d \geq 2$ and $\mathbf{k}$ is a field of characteristic p then for any natural numbers m and n the following equality holds*

$$\overline{[J_m, J_n]} = \begin{cases} J^{=}_{m+n} & \textit{if } d = 2,\ p = 2, \quad \textit{and} \quad m \equiv n \equiv 0 \mod 2; \\ J_{m+n} & \textit{otherwise.} \end{cases}$$

To every topological group G there are associated two series of commutators subgroups – the lower central series and derived series. The lower central series of a topological group G is defined as follows:

$$G_1 = G, \quad G_{n+1} = \overline{[G_n, G]}.$$

The derived series of a topological group G is defined as follows:

$$G^{(0)} = G, \quad G^{(n)} = \overline{[G^{(n-1)}, G^{(n-1)}]}.$$

Theorem 1.1 allows us to describe the lower central series of the multidimensional Jennings group and the derived series is described always except for the case $d = 2$, $p = 2$. Theorems 1.1, 3.4, and 3.3 allow to describe the derived series in the remaining case $d = 2$, $p = 2$.

2. Calculations

Everywhere below, it is assumed that $d = 2$ and the coefficient field has characteristic $p = 2$. It is also assumed that $F = (f_1, f_2) \in J_m$ and $G = (g_1, g_2) \in J_n$. We need to calculate the composition $F \circ G$. Since $J_1 = \mathcal{J}(\mathbf{k})$, the resulting formulas can be applied in the most general case. Let

$$f_1 = x + \tilde{f}_1 = x + \sum_{i_1+i_2 \geq m+1} a_{i_1,i_2} x^{i_1} y^{i_2},$$

$$f_2 = y + \tilde{f}_2 = y + \sum_{i_1+i_2 \geq m+1} b_{i_1,i_2} x^{i_1} y^{i_2} \ ;$$

$$g_1 = x + \tilde{g}_1 = x + \sum_{j_1+j_2 \geq n+1} \alpha_{j_1,j_2} x^{j_1} y^{j_2},$$

$$g_2 = y + \tilde{g}_2 = y + \sum_{j_1+j_2 \geq n+1} \beta_{j_1,j_2} x^{j_1} y^{j_2} \ .$$

Let $F \circ G = \{q_1, q_2\}$, then

$$q_1 = x + \tilde{g}_1 + \sum_{i_1+i_2 \geq m+1} a_{i_1,i_2} (x + \tilde{g}_1)^{i_1} (y + \tilde{g}_2)^{i_2} = x + \tilde{g}_1 +$$

$$\sum_{i_1+i_2 \geq m+1} a_{i_1,i_2} \left(x^{i_1} + \sum_{k=1}^{i_1} \binom{i_1}{k} x^{i_1-k} \tilde{g}_1^k \right) \left(y^{i_2} + \sum_{l=1}^{i_2} \binom{i_2}{l} y^{i_2-l} \tilde{g}_2^l \right) = x + \tilde{g}_1 + \tilde{f}_1 +$$

$$\sum_{i_1+i_2 \geq m+1} a_{i_1,i_2} \left(x^{i_1} \sum_{l=1}^{i_2} \binom{i_2}{l} y^{i_2-l} \tilde{g}_2^l + y^{i_2} \sum_{k=1}^{i_1} \binom{i_1}{k} x^{i_1-k} \tilde{g}_1^k + \sum_{k=1}^{i_1} \sum_{l=1}^{i_2} \binom{i_1}{k} \binom{i_2}{l} x^{i_1-k} y^{i_2-l} \tilde{g}_1^k \tilde{g}_2^l \right).$$

Here $\binom{i}{k}$ is the binomial coefficient.

The formula for q_2 is obtained by replacing a_{i_1,i_2} with b_{i_1,i_2}. Accordingly, the coefficients of the composition $G \circ F$ are obtained by replacing Latin letters with Greek ones, the numbers $m \leftrightarrow n$, and the summation indices $i_1 \leftrightarrow j_1$ and $i_2 \leftrightarrow j_2$. However, later for the summation indices, you can take other designations.

From the formulas written out we obtain

Proposition 2.1. *(A partial case for $d = 2$ in [6, Proposition 2.1]) If $2 \le k_1+k_2 \le m+n$, then*

$$a_{k_1,k_2}(F \circ G) = a_{k_1,k_2}(F) + \alpha_{k_1,k_2}(G).$$

Corollary 2.2. *([6, Proposition 2.2]) For any commutative ring $\mathbf{k}$ the following inclusion is true*

$$\overline{[J_m, J_n]} \subset J_{m+n}.$$

The smaller terms of the composition can be written out more specifically. The smallest possible "non-additive" degree of the composition $F \circ G$ is $m+n+1$.

Proposition 2.3. *([6, Proposition 2.4]) If $k_1 + k_2 = m + n + 1$, then*

$$a_{k_1,k_2}(F \circ G) = a_{k_1,k_2} + \alpha_{k_1,k_2} + \sum_{\substack{i_1+i_2=m+1,\\ j_1+j_2=n+1,\\ i_1+j_1=k_1+1,\\ i_2+j_2=k_2}} i_1 a_{i_1,i_2}\alpha_{j_1,j_2} + \sum_{\substack{i_1+i_2=m+1,\\ j_1+j_2=n+1,\\ i_1+j_1=k_1,\\ i_2+j_2=k_2+1}} i_2 a_{i_1,i_2}\beta_{j_1,j_2}$$

$$= a_{k_1,k_2} + \alpha_{k_1,k_2} + \sum_{i=\max\{0,k_1-n\}}^{\min\{k_1+1,m+1\}} i a_{i,m+1-i}\alpha_{k_1+1-i,n-k_1+i}$$

$$+ \sum_{i=\max\{0,k_1-n-1\}}^{\min\{k_1,m+1\}} (m+1-i)a_{i,m+1-i}\beta_{k_1-i,n+1-k_1+i} \ .$$

Proposition 2.4. *If $p = 2$ and $k_1 + k_2 = m + n + 1$, then*

$$a_{k_1,k_2}(F \circ G) = a_{k_1,k_2} + \alpha_{k_1,k_2} + \sum_{l=\max\left\{0,\left[\frac{k_1-n}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\left[\frac{m}{2}\right]\right\}} a_{2l+1,m-2l}\alpha_{k_1-2l,n+1-k_1+2l}$$

$$+ \sum_{l=\max\left\{0,\left[\frac{m-k_1+1}{2}\right]\right\}}^{\min\left\{\left[\frac{m+n+1-k_1}{2}\right],\left[\frac{m}{2}\right]\right\}} a_{m-2t,2t+1}\beta_{k_1-m+2t,m+n+1-k_1-2t} \ . \qquad (a_{k_1,k_2}(F \circ G))$$

Proof. In the Proposition 2.3 only the terms with odd indices i_1 and i_2 remain. □

The formula for $b_{k_1,k_2}(F \circ G)$ is obtained from $(a_{k_1,k_2}(F \circ G))$ by replacing a_{i_1,i_2} with b_{i_1,i_2}. In this case, the summation indices can be denoted for reasons of convenience.

Proposition 2.5. *If $p = 2$, $m \equiv 0 \mod 2$ and $k_1 + k_2 = m + n + 1$, then*

$$a_{k_1,k_2}(F \circ G) = a_{k_1,k_2} + \alpha_{k_1,k_2} + \sum_{l=\max\left\{0,\left[\frac{k_1-n}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{m}{2}\right\}} a_{2l+1,m-2l}\alpha_{k_1-2l,n+1-k_1+2l}$$

$$+ \sum_{l=\max\left\{0,\left[\frac{k_1-n}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{m}{2}\right\}} a_{2l,m+1-2l}\beta_{k_1-2l,n+1-k_1+2l} \ .$$

Proof. In the second sum of the Proposition 2.4 we denote $m - 2t$ by $2l$. □

Proposition 2.6. *If $p = 2$, $m \equiv n \equiv 0 \mod 2$ and $k_1 + k_2 = m + n + 1$, then* $A_{k_1,k_2}([F,G]) =$

$$\sum_{l=\max\left\{0,\left[\frac{k_1-n}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{m}{2}\right\}} a_{2l+1,m-2l}\alpha_{k_1-2l,n+1-k_1+2l} + \sum_{l=\max\left\{0,\left[\frac{k_1-n}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{m}{2}\right\}} a_{2l,m+1-2l}\beta_{k_1-2l,m+1-k_1+2l}$$
$$+ \sum_{l=\max\left\{0,\left[\frac{k_1-m}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{n}{2}\right\}} \alpha_{2l+1,n-2l}a_{k_1-2l,n+1-k_1+2l} + \sum_{l=\max\left\{0,\left[\frac{k_1-m}{2}\right]\right\}}^{\min\left\{\left[\frac{k_1}{2}\right],\frac{n}{2}\right\}} \alpha_{2l,n+1-2l}b_{k_1-2l,m+1-k_1+2l}.$$

Proof. According to the Corollary 2.2 $[F,G] \in J_{m+n}$. Since $[F,G]\circ(G\circ F) = (F\circ G)$, then according to Proposition 2.1 $A_{k_1,k_2}([F,G]) = a_{k_1,k_2}(F\circ G) + a_{k_1,k_2}(G\circ F)$. □

Corollary 2.7. *If $p = 2$, $m \equiv n \equiv 0 \mod 2$, then*

$$A_{2r+1,m+n-2r}([F,G]) = B_{2r,m+n+1-2r}([F,G]) =$$
$$\sum_{l=\max\left\{0,r-\frac{n}{2}\right\}}^{\min\left\{r,\frac{m}{2}\right\}} a_{2l,m+1-2l}\beta_{2r+1-2l,n-2r+2l} + \sum_{l=\max\left\{0,r-\frac{m}{2}\right\}}^{\min\left\{r,\frac{n}{2}\right\}} \alpha_{2l,n+1-2l}b_{2r+1-2l,m-2r+2l}$$

for every $r = 0,\dots,\frac{m+n}{2}$.

Proof. Change $r-l$ on the t in the third sum in the Proposition 2.6. It turns out that the first sum is the same as the third, so they are reduced. Similarly, two sums are reduced in the representation $B_{2r,m+n+1-2r}([F,G])$ and the same expressions remain. □

Corollary 2.8. *If $d = 2$, $p = 2$, and $m \equiv n \equiv 0 \mod 2$, then the following inclusion holds*

$$\overline{[J_m, J_n]} \subseteq J^{=}_{m+n}.$$

This corollary is contained in Theorem 1.1 (and is in [6]), but we present it as an important illustration of the previous formulas.

Corollary 2.9. *If $p = 2$, $m \equiv n \equiv 0 \mod 2$, then* $A_{2r,m+n+1-2r}([F,G]) =$

$$\sum_{l=\max\left\{0,r-\frac{n}{2}\right\}}^{\min\left\{r,\frac{m}{2}\right\}} a_{2l+1,m-2l}\alpha_{2r-2l,n+1-2r+2l} + \sum_{l=\max\left\{0,r-\frac{n}{2}\right\}}^{\min\left\{r,\frac{m}{2}\right\}} a_{2l,m+1-2l}\beta_{2r-2l,n+1-2r+2l}$$
$$+ \sum_{l=\max\left\{0,r-\frac{n}{2}\right\}}^{\min\left\{r,\frac{m}{2}\right\}} a_{2l,m+1-2l}\alpha_{2r-2l+1,n-2r+2l} + \sum_{l=\max\left\{0,r-\frac{n}{2}\right\}}^{\min\left\{r,\frac{m}{2}\right\}} b_{2l,m+1-2l}\alpha_{2r-2l,n+1-2r+2l}.$$

Proof. Change $r-l$ on the t in the third and fourth sums in the Proposition 2.6. Next, we rename t to l. □

Now let's create all the "bricks", the product of which approximates any element of the subgroup J_{m+n} as accurately as we want.

Proposition 2.10. *Let F and G have an elementary form:*

$$f_1 = x + ax^i y^{m+1-i}, \quad f_2 = y + bx^v y^{m+1-v}; \qquad g_1 = x + \alpha x^j y^{n+1-j}, \quad g_2 = y + \beta x^w y^{n+1-w}.$$

Then their commutator $H = [F, G]$ has the form

$$h_1 = x + (i-j)a\alpha x^{i+j-1} y^{m+n+2-i-j}$$

$$+(m+1-i)a\beta x^{i+w} y^{m+n+1-i-w} + (n+1-j)b\alpha x^{v+j} y^{m+n+2-v-j} + \dots\ ,$$

$$h_2 = y + (m+n+v+w)b\beta x^{v+w} y^{m+n+1-v-w}$$

$$+vb\alpha x^{v+j-1} y^{m+n+2-v-j} + wa\beta x^{i+w-1} y^{m+n+2-i-w} + \dots\ .$$

Proof. The first non-additive coefficients can be calculated using Proposition 2.3, but in this simple case it is easy to repeat the calculations. So,

$$q_1 = x + \alpha x^j y^{n+1-j} + a(x + \alpha x^j y^{n+1-j})^i (y + \beta x^w y^{n+1-w})^{m+1-i}.$$

Then for the smallest nonlinear degree the following equality holds

$$\hat{q}_1^{(m+n+1)}(F \circ G) = ia\alpha x^{i+j-1} y^{m+n+2-i-j} + (m+1-i)a\beta x^{i+w} y^{m+n+1-i-w}.$$

For another order we get

$$\hat{q}_1^{(m+n+1)}(G \circ F) = j\alpha a x^{j+i-1} y^{m+n+2-i-j} + (n+1-j)\alpha b x^{j+v} y^{m+n+1-j-v}.$$

From the equality for the commutator $h_1^{(n+m+1)} + q_1^{(n+m+1)}(G \circ F) = q_1^{(n+m+1)}(F \circ G)$ (Proposition 2.1) we obtain the equality

$$h_1^{(n+m+1)} + j\alpha a x^{j+i-1} y^{m+n+2-i-j} + (n+1-j)\alpha b x^{j+v} y^{m+n+1-j-v}$$

$$= ia\alpha x^{i+j-1} y^{m+n+2-i-j} + (m+1-i)a\beta x^{i+w} y^{m+n+1-i-w}.$$

Thence $h_1^{(n+m+1)} =$

$$(i-j)a\alpha x^{i+j-1} y^{m+n+2-i-j} + (m+1-i)a\beta x^{i+w} y^{m+n+1-i-w} - (n+1-j)\alpha b x^{j+v} y^{m+n+1-j-v}.$$

For $i = j = 0$ one should assume that the corresponding monomial is absent. □

Corollary 2.11. *Let F and G are as in Propisition 2.10, $m \equiv 0 \mod 2$, $i = 2r$, $w = 2l+1$, and $b = \alpha = 0$. Then their commutator $H = [F, G]$ has the form*

$$h_1 = x + (m+1-i)a\beta x^{i+w} y^{m+n+1-i-w} + \dots\ , \quad h_2 = y + wa\beta x^{i+w-1} y^{m+n+2-i-w} + \dots\ .$$

Corollary 2.12. *Let F and G are as in Propisition 2.10, $m \equiv 0 \mod 2$, $i = 2r+1$, $v = 2r$, and $a = b$. Then their commutator $H = [F, G]$ has the form*

$$h_1 = x + na\alpha x^{2r+j} y^{m+n+1-2r-j} + \dots\ , \quad h_2 = y + na\beta x^{2r+w} y^{m+n+1-2r-w} + \dots\ .$$

3. Main results

Theorem 3.1. *If $d = 2$, $p = 2$, $m \equiv 0 \mod 2$, and $n \equiv 1 \mod 2$, then the following equality holds*

$$\overline{[J_m^{=,0}, J_n]} = J_{m+n}.$$

Proof. Inclusion $\overline{[J_m^{=,0}, J_n]} \subseteq \overline{[J_m, J_n]} \subseteq J_{m+n}$ follows from the Corollary 2.2.

Since $J_m^{=,0} \supseteq J_{m+1}$, then $\overline{[J_m^{=,0}, J_n]} \supseteq \overline{[J_{m+1}, J_n]}$. According to Theorem 1.1 the last set is equal to J_{m+n+1}.

It remains to realize in the form of a commutator the beginning of a common element from $J_{m+n} \setminus J_{m+n+1}$. According to the Proposition 2.1 it is necessary (enough) to implement as a commutator an element with a unique monomial of degree $m + n + 1$, that is, single coefficient $a_{i,m+n+1-i}$.

It is easy to see that the conditions from Corollary 2.12 are satisfied for any element $F \in J_m^{=,0}$. For $\beta = 0$ and odd n the commutator has the form $h_1 = x + a\alpha x^{2r+j} y^{m+n+1-2r-j} + \dots$, $h_2 = y + \dots$. Combinations $2r = 0, \dots, m$ и $j = 0, 1, \dots, n+1$ implement all "bricks" $a_{k_1,m+n+1-k_1}$. $\square$

Corollary 3.2. *If $d = 2$, $p = 2$, $m \equiv 0 \mod 2$, and $n \equiv 1 \mod 2$, then the following equality holds*

$$\overline{[J_m^{=}, J_n]} = J_{m+n}.$$

Theorem 3.3. *If $d = 2$, $p = 2$, and $m \equiv n \equiv 0 \mod 2$, then the following equality holds*

$$\overline{[J_m^{=,0}, J_n^{=,0}]} = J_{m+n+1}.$$

Proof. We substitute in the formula of Corollary 2.7 the conditions $a_{2l,m+1-2l} = b_{m+1-2l,2l} = 0$ and $\alpha_{2l,n+1-2l} = \beta_{n+1-2l,2l} = 0$: $A_{2r+1,m+n-2r}([F,G]) = B_{2r,m+n+1-2r}([F,G]) =$

$$\sum_{l=\max\left\{0, r-\frac{n}{2}\right\}}^{\min\left\{r, \frac{m}{2}\right\}} a_{2l,m+1-2l}\beta_{2r+1-2l,n-2r+2l} + \sum_{l=\max\left\{0, r-\frac{m}{2}\right\}}^{\min\left\{r, \frac{n}{2}\right\}} b_{2r+1-2l,m-2r+2l}\alpha_{2l,n+1-2l} = 0.$$

Inclusion $\overline{[J_m^{=,0}, J_n^{=,0}]} \supseteq \overline{[J_m^{=,0}, J_{n+1}]} = J_{m+n+1}$ follows from the Theorem 3.1. $\square$

Theorem 3.4. *If $d = 2$, $p = 2$, and $m \equiv n \equiv 0 \mod 2$, then the following equality holds*

$$\overline{[J_m^{=}, J_n^{=}]} = J_{m+n}^{=,0}.$$

Proof. Inclusion $\overline{[J_m^{=}, J_n^{=}]} \subseteq \overline{[J_m, J_n]} \subseteq J_{m+n}^{=}$ retrieved from Corollary 2.8.

Substitute in the formula of Corollary 2.9 the conditions $b_{2l,m+1-2l} = a_{2l+1,m-2l}$ and $\beta_{2l,n+1-2l} = a_{2l+1,n-2l}$: $A_{2r,m+n+1-2r}([F,G]) =$

$$\sum_{l=\max\left\{0, r-\frac{n}{2}\right\}}^{\min\left\{r, \frac{m}{2}\right\}} a_{2l+1,m-2l}\alpha_{2r-2l,n+1-2r+2l} + \sum_{l=\max\left\{0, r-\frac{n}{2}\right\}}^{\min\left\{r, \frac{m}{2}\right\}} a_{2l,m+1-2l}\alpha_{2r-2l+1,n-2r+2l}$$

$$+ \sum_{l=\max\left\{0, r-\frac{n}{2}\right\}}^{\min\left\{r, \frac{m}{2}\right\}} a_{2l,m+1-2l}\alpha_{2r-2l+1,n-2r+2l} + \sum_{l=\max\left\{0, r-\frac{n}{2}\right\}}^{\min\left\{r, \frac{m}{2}\right\}} a_{2l+1,m-2l}\alpha_{2r-2l,n+1-2r+2l} = 0.$$

Similarly, it is proved that $B_{2r+1,m+n-2r}([F,G])=0$.

Inclusion $\overline{[J_m^{=},J_n^{=}]} \supseteq \overline{[J_m^{=},J_{n+1}]} = J_{m+n+1}$ follows from the Corollary 3.2.

It remains to realize in the form of a commutator the beginning of a common element from $J_{m+n}^{=,0} \setminus J_{m+n+1}$. According to the Proposition 2.1 it is necessary (enough) to implement as a commutator an element with a pair of equal coefficients $a_{2r+1,m+n-2r} = b_{2r,m+n+1-2r}$.

It is easy to see that the elements F and G from Corollary 2.11 for $b=\alpha=0$, $i=2l$, and $w=2t+1$ lie in the subgroup $J_m^{=}$. In this case, the commutator has the form $h_1 = x + a\beta x^{2r+2t+1}y^{m+n-2r-2t} + \dots$, $h_2 = y + a\beta x^{2r+2t}y^{m+n+1-2r-2t} + \dots$. Combinations $2l = 0,\dots,m$ and $2t+1 = 1,3,\dots,n+1$ implement all "bricks" $a_{2r+1,m+n-2r} = b_{2r,m+n+1-2r}$, $2r+1 = 1,3,\dots,m+n+1$. $\square$

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