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Equivalence of Two Definitions of a Generalized L_p Solution to the Initial–Boundary Value Problem for the Wave Equation

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Abstract—In our previous papers, we introduced the notion of a generalized solution to the initial–boundary value problem for the wave equation with a boundary function $\mu(t)$ such that the integral $\int_0^T (T-t)|\mu(t)|^p\,dt$ exists. Here we prove that this solution is a unique solution to the problem in L_p that satisfies the corresponding integral identity.

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Just as in [1–7], we will consider an initial–boundary value problem for the wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \tag{1}$$

in the rectangle $\overline{Q}_T = [0 \le x \le l] \times [0 \le t \le T]$ with the zero initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0 \tag{2}$$

and with the boundary conditions of the first kind

$$u(0,t) = \mu(t), \qquad u(l,t) = 0; \tag{3}$$

without loss of generality, we assume that the second condition is homogeneous.[1]

In [1], we used the following definition of a solution to the initial–boundary value problem (1)–(3).

Definition 1. A function $u(x,t)$ is called a *generalized solution in $L_p(Q_T)$ to the initial–boundary value problem* (1)–(3) if it belongs to the class $L_p(Q_T)$ and satisfies the integral identity

$$\int_0^l \int_0^T u(x,t)\bigl[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)\bigr]\,dx\,dt = \int_0^T \mu(t)\Phi_x(0,t)\,dt \tag{4}$$

for any test function $\Phi(x,t)$ subject to the conditions

$$\begin{gathered} \Phi(x,t) \in C^2(\overline{Q}_T), \qquad \Phi(x,T) \equiv 0 \quad \text{and} \quad \Phi_t(x,T) \equiv 0 \quad \text{for all } \ 0 \le x \le l, \\ \Phi(0,t) \equiv 0 \quad \text{and} \quad \Phi(l,t) \equiv 0 \quad \text{for all } \ 0 \le t \le T. \end{gathered} \tag{5}$$

Let us represent an arbitrary $T > 0$ as $T = 2ln - \Delta$ with a positive integer n and $0 \le \Delta < 2l$. In [5], we proved that if the boundary function $\mu(t)$ belongs to the class L_p on the whole interval $[0,T]$, then the solution $u(x,t)$ introduced in the above definition exists and is defined by the

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[1]The case of two inhomogeneous boundary conditions can be reduced to the composition of two problems of the form under consideration.

equality

$$u(x,t) = \sum_{k=0}^{n-1} \underline{\mu}(t - x - 2lk) - \sum_{k=1}^{n} \underline{\mu}(t + x - 2lk) \tag{6}$$

in which $\underline{\mu}(\tau)$ coincides with $\mu(\tau)$ for $\tau \geq 0$ and vanishes for $\tau < 0$.

However, the condition that $\mu(t)$ belongs to L_p on the whole interval $[0,T]$ is only sufficient but not necessary for the existence of a solution in $L_p(Q_T)$ in the sense of the above definition. In [2], under the condition that $\mu(t)$ belongs to the class L_1 on the interval $[0, T-\varepsilon]$ for all ε such that $0 < \varepsilon < T$, we introduced the following notion of a generalized solution to the initial–boundary value problem (1)–(3) in the rectangle $Q_T = [0 \leq x \leq l] \times [0 \leq t < T)$, which is open on one side: A solution is a function $u(x,t)$ that coincides, for any ε such that $0 < \varepsilon < T$, with the certainly existing solution in the class $L_1(Q_{T-\varepsilon})$ to the initial–boundary value problem (1)–(3) in the closed rectangle $\overline{Q}_{T-\varepsilon} = [0 \leq x \leq l] \times [0 \leq t \leq T-\varepsilon]$.

In [6] we proved that the solution $u(x,t)$ thus defined is specified in Q_T by formula (6), and a necessary and sufficient condition for it to belong to the class $L_p(Q_T)$ is the existence of the integral

$$\int_0^T (T-t)|\mu(t)|^p \, dt. \tag{7}$$

Here the following question remained open: Under the condition that the integral (7) exists, is the solution $u(x,t)$ introduced in [2] a generalized solution in $L_p(Q_T)$ satisfying the conditions of the above definition, i.e., satisfying the integral identity (4) for any test function $\Phi(x,t)$ subject to conditions (5).

In the present paper we answer this question affirmatively. The results of the paper were announced in [4]. Here we give an expanded and technically improved account of these results.

Theorem. *For any $T > 0$ and $p \geq 1$, provided that the integral* (7) *exists, the initial–boundary value problem* (1)–(3) *has a unique generalized solution $u(x,t)$ in $L_p(Q_T)$ that satisfies the integral identity* (4) *for any test function $\Phi(x,t)$ subject to conditions* (5). *This solution is defined by equality* (6).

Proof. To prove the theorem, we first notice that, as established in [6], if (and only if) the integral (7) exists, the function $u(x,t)$ defined by equality (6) for $T = 2ln - \Delta$, $0 \leq \Delta < 2l$, belongs to the class $L_p(Q_T)$. Therefore, it suffices to prove that the function $u(x,t)$ defined by the indicated equality satisfies the integral identity (4) for any test function $\Phi(x,t)$ subject to conditions (5). This follows from the fact that for any $p \geq 1$ there may exist only one function $u(x,t) \in L_p(Q_T)$ satisfying identity (4) for any test function $\Phi(x,t)$ subject to conditions (5).[2]

To prove the validity of identity (4), we employ here a result from [2] stating that the boundary function $\mu(t)$ can be decomposed into a sum of two terms, $\mu_1(t) + \mu_2(t)$, such that the solution $u_1(x,t)$ corresponding to the first term $\mu_1(t)$ consists of a single term $\mu_1(t-x)$ and is different from zero only for the values of t in an interval adjoining the boundary point T, while the solution $u_2(x,t)$ corresponding to the second term $\mu_2(t)$ vanishes for the values of t in this interval adjoining the point T.

Just as in [6], we consider two cases:

(a) $T > l$ and

(b) $0 < T \leq l$.

These cases are examined in similar ways, and we restrict the analysis to case (a).

[2]The uniqueness of such a function for any $p \geq 1$ follows either from a considerably simplified scheme of arguments used in [8, Ch. 2, § 9] or from the result established in [9].

In this case, the interval $[0,T)$ is decomposed into the union of the interval $[0,T-l]$ and the interval $(T-l,T)$, and we represent the boundary function $\mu(t)$ as a sum $\mu(t)=\mu_1(t)+\mu_2(t)$ with the following terms:

$$\mu_1(t)=\begin{cases}0 & \text{if } t\le T-l,\\ \mu(t) & \text{if } T-l<t<T,\end{cases}\qquad \mu_2(t)=\begin{cases}\mu(t) & \text{if } 0\le t\le T-l,\\ 0 & \text{if } T-l<t<T.\end{cases}$$

As established in [6], the solutions $u_1(x,t)$ and $u_2(x,t)$ corresponding to these terms under such a decomposition are defined as

$$u_1(x,t)=\begin{cases}0 & \text{if } t-x\le T-l,\\ \mu_1(t-x)=\mu(t-x) & \text{if } T-l<t-x<T,\end{cases}\tag{8}$$

$$u_2(x,t)=\sum_{k=0}^{n-1}\underline{\mu}_2(t-x-2lk)-\sum_{k=1}^{n}\underline{\mu}_2(t+x-2lk);\tag{9}$$

here $\underline{\mu}_2(\tau)$ stands for the function that coincides with $\mu_2(\tau)$ for $0\le\tau<T$ and vanishes for $\tau<0$.

Since $\mu_2(t)$ belongs to the class L_p on the whole interval $[0,T]$, it follows from [5, Theorem 1] that the function $u_2(x,t)$ is a generalized solution in $L_p(Q_T)$ to the initial–boundary value problem (1)–(3) and the integral identity

$$\int_0^l\int_0^T u_2(x,t)\big[\Phi_{tt}(x,t)-\Phi_{xx}(x,t)\big]\,dx\,dt=\int_0^T\mu_2(t)\Phi_x(0,t)\,dt$$

holds for any test function $\Phi(x,t)$ subject to conditions (5).

Therefore, it suffices to prove that for any test function $\Phi(x,t)$ subject to conditions (5), provided that the integral (7) exists, the function $u_1(x,t)$ defined by equality (8) satisfies the integral identity

$$\int_0^l\int_0^T u_1(x,t)\big[\Phi_{tt}(x,t)-\Phi_{xx}(x,t)\big]\,dx\,dt=\int_0^T\mu_1(t)\Phi_x(0,t)\,dt,$$

which for the function (8) takes the form

$$\int_0^l\int_{T-l}^T \mu_1(t-x)\big[\Phi_{tt}(x,t)-\Phi_{xx}(x,t)\big]\,dx\,dt=\int_{T-l}^T\mu_1(t)\Phi_x(0,t)\,dt.\tag{10}$$

To prove the validity of identity (10), we denote by $\{\varepsilon_m\}$ a sequence of numbers lying in the interval $(0,l)$ and converging to zero. Using the number sequence $\{\varepsilon_m\}$, we introduce a sequence of functions $\{\widehat{\mu}_m(t)\}$ by defining it as

$$\widehat{\mu}_m(t)=\begin{cases}\mu_1(t) & \text{if } 0\le t\le T-\varepsilon_m,\\ 0 & \text{if } T-\varepsilon_m<t<T\end{cases}$$

and noticing that every function $\widehat{\mu}_m(t)$, just as $\mu_1(t)$, vanishes for $0\le t\le T-l$. Since each function $\widehat{\mu}_m(t)$ belongs to the class L_p on the interval $[0,T]$, it follows from [5, Theorem 1] that the corresponding solution $\widehat{u}_m(x,t)$ in $L_p(Q_T)$ to the initial–boundary value problem (1)–(3) exists and satisfies the integral identity

$$\int_0^l\int_0^T \widehat{u}_m(x,t)\big[\Phi_{tt}(x,t)-\Phi_{xx}(x,t)\big]\,dx\,dt=\int_0^T\widehat{\mu}_m(t)\Phi_x(0,t)\,dt\tag{11}$$

for any test function $\Phi(x,t)$ subject to conditions (5). Since $\widehat{\mu}_m(t)$ vanishes for $0 \le t \le T-l$, the function $\widehat{u}_m(x,t)$ also vanishes for these values of t and the equality $\widehat{u}_m(x,t) = \widehat{\mu}_m(t-x)$ holds. Therefore, identity (11), which is valid for any test function $\Phi(x,t)$ subject to conditions (5), can be rewritten as

$$\int_0^l \int_{T-l}^T \widehat{\mu}_m(t-x)\big[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)\big]\,dx\,dt = \int_{T-l}^T \widehat{\mu}_m(t)\Phi_x(0,t)\,dt. \tag{12}$$

From (12) we conclude that to prove the validity of identity (10), it suffices to establish the following two equalities (i.e., the existence and vanishing of the limits):

$$\lim_{m\to\infty} \int_0^l \int_{T-l}^T \big|\mu_1(t-x) - \widehat{\mu}_m(t-x)\big| \cdot \big|\Phi_{tt}(x,t) - \Phi_{xx}(x,t)\big|\,dx\,dt = 0, \tag{13}$$

$$\lim_{m\to\infty} \int_{T-l}^T \big[\mu_1(t) - \widehat{\mu}_m(t)\big]\Phi_x(0,t)\,dt = 0. \tag{14}$$

Since $\Phi(x,t) \in C^2(\overline{Q}_T)$, to prove equality (13), it suffices to apply the Hölder inequality and to derive the equality

$$\lim_{m\to\infty} \int_0^l \int_{T-l}^T \big|\mu_1(t-x) - \widehat{\mu}_m(t-x)\big|^p\,dx\,dt = 0. \tag{15}$$

To prove equalities (14) and (15), we use the fact that the definition of the functions $\widehat{\mu}_m(t)$ and the existence of the integral (7) imply the equality

$$\lim_{m\to\infty} \int_{T-l}^T (T-t)|\mu_1(t) - \widehat{\mu}_m(t)|^p\,dt = 0 \tag{16}$$

and, in particular, the equality

$$\lim_{m\to\infty} \int_{T-l}^T (T-t)|\mu_1(t) - \widehat{\mu}_m(t)|\,dt = 0. \tag{17}$$

In view of (16), to prove equality (15), it suffices to verify that for any number m the double integral on the left-hand side of (15) exists and is bounded from above by the integral on the left-hand side of (16). To this end, using the property of complete additivity of the Lebesgue integral of a nonnegative function, it suffices to prove that the following inequality holds on the set of all sufficiently small $\varepsilon > 0$:

$$\int_0^l \left[\int_{T-l}^{T-\varepsilon} \big|\mu_1(t-x) - \widehat{\mu}_m(t-x)\big|^p\,dt\right] dx \le \int_{T-l}^{T-\varepsilon} (T-t)|\mu_1(t) - \widehat{\mu}_m(t)|^p\,dt. \tag{18}$$

Denoting by $I(\varepsilon)$ the integral on the left-hand side of (18), making the change $\tau = t - x$ of the variable in it, and taking into account that $\mu_1(t)$ and $\widehat{\mu}_m(t)$ vanish for $t \le T - l$ and that

$T-\varepsilon-x \geq T-l$ only for $x \leq l-\varepsilon$, we obtain the equality

$$I(\varepsilon)=\int\limits_0^l\left[\int\limits_{T-l}^{T-\varepsilon-x}|\mu_1(\tau)-\widehat{\mu}_m(\tau)|^p\,d\tau\right]dx=\int\limits_0^{l-\varepsilon}\left[\int\limits_{T-l}^{T-\varepsilon-x}|\mu_1(\tau)-\widehat{\mu}_m(\tau)|^p\,d\tau\right]dx.$$

Calculating the last integral by parts, we find

$$I(\varepsilon)=\left[x\int\limits_{T-l}^{T-\varepsilon-x}|\mu_1(\tau)-\widehat{\mu}_m(\tau)|^p\,d\tau\right]_{x=0}^{x=l-\varepsilon}+\int\limits_0^{l-\varepsilon}x\big|\mu_1(T-\varepsilon-x)-\widehat{\mu}_m(T-\varepsilon-x)\big|^p\,dx. \qquad (19)$$

For any $\varepsilon>0$, both substitutions in (19) vanish; so, making the change $y=x+\varepsilon$ in the last integral in (19), we obtain

$$I(\varepsilon)=\int\limits_\varepsilon^l(y-\varepsilon)\big|\mu_1(T-y)-\widehat{\mu}_m(T-y)\big|^p\,dy\leq\int\limits_\varepsilon^l y\big|\mu_1(T-y)-\widehat{\mu}_m(T-y)\big|^p\,dy. \qquad (20)$$

Making the change $t=T-y$ in the integral on the right-hand side of (20), we get the inequality

$$I(\varepsilon)\leq\int\limits_{T-l}^{T-\varepsilon}(T-t)|\mu_1(t)-\widehat{\mu}_m(t)|^p\,dt,$$

which coincides with (18) and thus finishes the proof of (15).

To complete the analysis of case (a), we should prove equality (14).

Fix an arbitrary test function $\Phi(x,t)$ subject to all conditions (5). Then, first, there exists a constant M such that

$$|\Phi_x(0,t)|\leq M \qquad (21)$$

for all t in the interval $[0,T]$. Second, in view of the existence of the zero derivative $\Phi_{xt}(0,T)$ and the equality $\Phi_x(0,T)=0$, there exists a $\delta>0$ such that

$$|\Phi_x(0,t)|\leq|T-t| \qquad (22)$$

for all t in the interval $T-\delta<t<T$. From (21) and (22) we conclude that

$$\begin{aligned}\left|\int\limits_{T-l}^{T}\big[\mu_1(t)-\widehat{\mu}_m(t)\big]\Phi_x(0,t)\,dt\right|&\leq\int\limits_{T-l}^{T-\delta}|\mu_1(t)-\widehat{\mu}_m(t)|\cdot|\Phi_x(0,t)|\,dt+\int\limits_{T-\delta}^{T}|\mu_1(t)-\widehat{\mu}_m(t)|\cdot|\Phi_x(0,t)|\,dt\\&\leq M\int\limits_{T-l}^{T-\delta}|\mu_1(t)-\widehat{\mu}_m(t)|\,dt+\int\limits_{T-\delta}^{T}(T-t)|\mu_1(t)-\widehat{\mu}_m(t)|\,dt\\&\leq\left(\frac{M}{\delta}+1\right)\int\limits_{T-l}^{T}(T-t)|\mu_1(t)-\widehat{\mu}_m(t)|\,dt;\end{aligned}$$

hence, the existence and vanishing of the limit in (14) follows from the existence and vanishing of the limit in (17). This completes the proof of the theorem in case (a).

In case (b), when $0 < T \leq l$, the solution to the initial–boundary value problem (1)–(3) is defined by the formula

$$u(x,t) = \underline{\mu}(t - x).$$

Then, denoting by $\{\varepsilon_m\}$ a number sequence in the interval $(0,T)$ that converges to zero, we carry out the proof according to the above-described scheme.

Remark 1. The results of [6] imply the converse statement: If the function $\mu(t)$ belongs to the class $L_1[0, T - \varepsilon]$ for any ε in the interval $0 < \varepsilon < T$ and if the conditions of Definition 1 hold for the function $u(x,t)$, then the integral (7) exists and $u(x,t)$ is defined by equality (6) for $T = 2ln - \Delta$, $0 \leq \Delta < 2l$. This also implies the uniqueness of the solution $u(x,t)$ introduced by Definition 1.

Remark 2. The solution $u(x,t)$ considered in [3, 7], which belongs to the class $W_p^1(Q_T)$ provided that the integral $\int_0^T (T-t)|\mu'(t)|^p\,dt$ exists, is also a generalized solution in $L_p(Q_T)$ to the initial–boundary value problem (1)–(3), i.e., a solution satisfying identity (4) for any test function $\Phi(x,t)$ subject to conditions (5). This follows from the analytic form of the solution and the results of [5] and [7, Lemma].

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