Ergod. Th. & Dynam. Sys. (2009), **0**, 1–21
doi:10.1017/S0143385709000431

Symbolic images and invariant measures of dynamical systems

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(*Received* 8 *October* 2008 *and accepted in revised form* 9 *March* 2009)

Abstract. Let f be a homeomorphism of a compact manifold M. The Krylov–Bogoloubov theorem guarantees the existence of a measure that is invariant with respect to f. The set of all invariant measures $\mathcal{M}(f)$ is convex and compact in the weak topology. The goal of this paper is to construct the set $\mathcal{M}(f)$. To obtain an approximation of $\mathcal{M}(f)$, we use the symbolic image with respect to a partition $C = \{M(1), M(2), \ldots, M(n)\}$ of M. A symbolic image G is a directed graph such that a vertex i corresponds to the cell $M(i)$ and an edge $i \to j$ exists if and only if $f(M(i)) \cap M(j) \neq \emptyset$. This approach lets us apply the coding of orbits and symbolic dynamics to arbitrary dynamical systems. A flow on the symbolic image is a probability distribution on the edges which satisfies Kirchhoff's law at each vertex, i.e. the incoming flow equals the outgoing one. Such a distribution is an approximation to some invariant measure. The set of flows on the symbolic image G forms a convex polyhedron $\mathcal{M}(G)$ which is an approximation to the set of invariant measures $\mathcal{M}(f)$. By considering a sequence of subdivisions of the partitions, one gets sequence of symbolic images G_k and corresponding approximations $\mathcal{M}(G_k)$ which tend to $\mathcal{M}(f)$ as the diameter of the cells goes to zero. If the flows m^k on each G_k are chosen in a special manner, then the sequence $\{m^k\}$ converges to some invariant measure. Every invariant measure can be obtained by this method. Applications and numerical examples are given.

1. *Introduction*

Let $f : M \to M$ be a homeomorphism of a compact manifold $M \subset \mathbb{R}^d$ that generates the discrete dynamical system $\{f^k : k \in \mathbb{Z}\}$. Let $C = \{M(1), \ldots, M(n)\}$ be a finite covering of M. The set $M(i)$ is called the *cell* (or box) of index i. Let G be a directed graph with vertices $\{i\}$ corresponding to the cells $\{M(i)\}$. Two vertices i and j of G are connected by the directed edge $i \to j$ if and only if $f(M(i)) \cap M(j) \neq \emptyset$. The graph G is called the *symbolic image* of f with respect to the covering C. **Q1**

The notion of symbolic image of a dynamical system was introduced in [**12**]. It is a powerful tool for investigating global dynamics and the structure of orbits. It enables us to apply the coding of orbits and symbolic dynamics to arbitrary dynamical systems. Symbolic image methods and their applications are discussed in detail in [**14**].

Here we consider the construction of invariant measures based on the concept of symbolic image. The Krylov–Bogolubov theorem [**10**] guarantees the existence of a probability measure μ that is invariant with respect to f. The collection of all f-invariant measures $\mathcal{M}(f)$ forms a convex compact set in the weak topology [**9**]. Ulam [**17**] proposed a method for constructing a sequence of measures by approximation of the Frobenius–Perron operator; such a sequence converges to an invariant measure. In particular, Sinai–Bowen–Ruelle (SBR) measures were constructed via the Ulam method in a variety of settings [**4–6**]. The paper [**3**] describes numerical methods and results relating to the approximation of SBR measures. However, despite the fact that any map f has a set of invariant measures $\mathcal{M}(f)$, it is not necessarily the case that f will have SBR measure. Extreme points of the convex set $\mathcal{M}(f)$ are ergodic measures.

Our aim in this paper is to construct the set $\mathcal{M}(f)$ for any f. To obtain an approximation of $\mathcal{M}(f)$, we use the notion of symbolic image with respect to a partition of M. In the general case, there are no restrictions on the properties of the cells. However, we will focus on coverings C with connected Lebesgue-measurable cells. In numerical experiments, these cells are parallelepipeds that intersect in the boundary discs. When the covering C is a partition, the cells are semi-open parallelepipeds and the boundary discs belong to one of the cells.

To understand the proposed construction, suppose that the transformation f has an invariant measure μ; then each edge $i \to j$ of the symbolic image G gets the measure

$$m_{ij} = \mu(M(i) \cap f^{-1}(M(j))) = \mu(f(M(i)) \cap M(j)), \tag{1}$$

Q2 where the second equality comes from the invariance of μ. In addition, we have

$$\sum_k m_{ki} = \sum_k \mu(f(M(k)) \cap M(i)) = \mu(M(i)) = \sum_j \mu(M(i) \cap f^{-1}(M(j))) = \sum_j m_{ij}.$$

The sum $\sum_k m_{ki}$ is called the incoming flow at vertex i, and the sum $\sum_j m_{ij}$ is the outgoing flow from i. The equality

$$\sum_k m_{ki} = \sum_j m_{ij} \tag{2}$$

can be treated as a Kirchhoff-type law. In addition, we have the equality

$$\sum_{ij} m_{ij} = \mu(M) = 1, \tag{3}$$

which means that the distribution m_{ij} is normalized. So an invariant measure μ generates on the symbolic image a distribution m_{ij} which satisfies the conditions (2) and (3). This observation leads us to the following definition.

Definition 1. Let G be a directed graph. A distribution $\{m_{ij}\}$ on the edges $\{i \to j\}$ is called a *flow* on G if:

- $m_{ij} \geq 0$;
- $\sum_{ij} m_{ij} = 1$;
- $\sum_k m_{ki} = \sum_j m_{ij}$ for each vertex $i \in G$.

The last property may be thought of as invariance of the flow. The second property, i.e. normalization, can be rewritten in the form $m(G) = 1$, where the measure of G is the sum of the measures of its edges. In graph theory, such a distribution is called a closed or invariant flow. For the flow $\{m_{ij}\}$ on G, we define the measure of the vertex i to be

$$m_i = \sum_k m_{ki} = \sum_j m_{ij}.$$

In this case, $\sum_i m_i = m(G) = 1$. Thus, each invariant measure generates a flow on the symbolic image. Now consider the converse construction. Let $m = \{m_{ij}\}$ be a flow on a symbolic image G. Then a measure μ^* can be defined by the formula

$$\mu^*(A) = \sum_i m_i v(A \cap M(i))/v(M(i)), \tag{4}$$

where v is a normalized Lebesgue measure. It is assumed that $v(M(i)) \neq 0$ for each cell. By the above definition, the measure of $M(i)$ coincides with the measure of the vertex i:

$$\mu^*(M(i)) = m_i.$$

The inequalities

$$\sum_{k:M(k)\subset A} m_k \leq \mu^*(A) \leq \sum_{i:M(i)\cap A\neq\emptyset} m_i$$

follow from (4). They may be treated as lower and upper estimates for the invariant measure constructed through the distribution (flow) m. In general, the constructed measure μ^* is not invariant with respect to f. However, as will be shown later, this measure is an approximation of an invariant measure.

The set of flows $\{m = (m_{ij})\}$ on the symbolic image G forms a convex polyhedron $\mathcal{M}(G)$ which is an approximation of the set of invariant measures $\mathcal{M}(f)$. By considering a sequence C_k of subdivisions of the partitions, one gets a sequence of symbolic images G_k and corresponding approximations $\mathcal{M}(G_k)$ which tend to $\mathcal{M}(f)$ as the diameters of the cells go to zero. This technique allows us to get an individual measure. If the flows m^k on each G_k are chosen in a special manner, then the sequence $\{m^k\}$ converges to some invariant measure μ. Moreover, every invariant measure can be obtained by this method.

2. *Flows on graphs*

Let G be a directed graph with n vertices. Consider the space $\mathcal{M}(G) = \{m\}$ of all flows on G. Let $m^1 = \{m^1_{ij}\}$ and $m^2 = \{m^2_{ij}\}$ be two flows, and let $\alpha, \beta \geq 0$ be such that $\alpha + \beta = 1$. Then it is easy to check that the distribution

$$m = \alpha m^1 + \beta m^2 = \{\alpha m^1_{ij} + \beta m^2_{ij}\}$$

is a flow as well. In this case we say that the flow m is the sum of the flows m^1 and m^2 with weights α and β. Thus, the space of all flows $\mathcal{M}(G)$ is a convex set.

Let us investigate the structure of $\mathcal{M}(G)$. Suppose that $\omega = (i_1 \to i_2 \to \cdots \to i_k \to i_1)$ is a simple periodic path (cycle); this means that all vertices $\{i_t : t = 1, 2, \ldots, k\}$ are different. To construct a simple flow $m(\omega)$ located on the cycle, we put $m_{ij} = 1/k$ for all edges from ω and $m_{ij} = 0$ for all other edges. It is evident that $m(\omega)$ is invariant, unique, located on the cycle ω and not decomposable into the sum of other flows. In other words, $m(\omega)$ is an extreme point of the set $\mathcal{M}(G)$. Since the number of vertices is finite, the number of simple flows (cycles) is finite as well.

PROPOSITION 1. *Any flow $m \in \mathcal{M}(G)$ can be decomposed into the sum of simple flows.*

Proof. Let $m = \{m_{ij}\}$ be a flow on G. Consider a set of the edges $D = \{i \to j\}$ such that $m_{ij} > 0$. Kirchhoff's law (2) holds on this set. It follows that there is a path of infinite length, $\omega^* \in D$, that goes through each edge from D. In fact, if there is an edge $k \to i$ with $m_{ki} > 0$, then by (2) there must be an edge $i \to j$ with $m_{ij} > 0$; hence we can continue the path in D. Since G has a finite number of edges, ω^* contains a periodic path ω which can be considered as a simple one. Let p be the minimal period of ω and define

$$m_{\min} = \min\{m_{ij} \mid i \to j \in \omega\} > 0$$

to be the minimal measure of edges from ω. Let $\alpha > 0$ be a number such that

$$\alpha = p m_{\min} \quad \text{or} \quad \alpha/p = m_{\min}.$$

We construct a new distribution m^* on the edges of G. For each edge $i \to j$ in ω we define a new measure $m^*_{ij} = m_{ij} - \alpha/p \geq 0$. If an edge $i \to j$ is not included in ω, then $m^*_{ij} = m_{ij}$. It is clear that the sum of measures of all edges is $\sum_{ij} m^*_{ij} = 1 - \alpha$.

Let us show that Kirchhoff's law (2) holds for the distribution m^*. If a vertex i is not in ω, then $m^*_{ij} = m_{ij}$, $m^*_{ki} = m_{ki}$ and the equality (2) holds. Let i lie in the simple cycle ω. Then, in ω, there exists an edge $k^* \to i$ coming in at i and an edge $i \to j^*$ going out from i; all other edges from ω are free of connection with the vertex i, since ω is a simple cycle. In this case, the left- and right-hand sides of the equality

$$\sum_k m_{ki} = \sum_j m_{ij} \tag{5}$$

contain, respectively, the terms m_{k^*i} and m_{ij^*} generated by the edges $k^* \to i$ and $i \to j^*$; all the other terms are free of connection with the cycle ω. Thus we have

$$m_{k^*i} + \sum_{k \neq k^*} m_{ki} = m_{ij^*} + \sum_{j \neq j^*} m_{ij}. \tag{6}$$

Subtracting $\alpha/p = m_{\min}$ from both sides of (6), we get the equality

$$m_{k^*i} - \alpha/p + \sum_{k \neq k^*} m_{ki} = m_{ij^*} - \alpha/p + \sum_{j \neq j^*} m_{ij}$$

or

$$\sum_k m^*_{ki} = \sum_j m^*_{ij}.$$

It follows that the new distribution m^* satisfies Kirchhoff's law but that the set of edges $D^* = \{i \to j\}$ with $m^*_{ij} > 0$ does not contain some edges from the cycle ω, since

$$m^*_{\min} = m_{\min} - \alpha/p = 0$$

on ω.

By repeating this process of eliminating simple cycles from D, we obtain, in a finite number of steps, the zero distribution. In this case, the initial flow can be represented in the form

$$m = \sum_{\omega} \alpha_\omega m_\omega,$$

where $\alpha_\omega \geq 0$, m_ω is a simple flow, and the sum is taken over all simple cycles. It follows from the equality $m(G) = m_\omega(G) = 1$ that $\sum_\omega \alpha_\omega = 1$. □

Remark. It follows from the proof that the measure m_{ij} may be positive on an edge $i \to j$ when a periodic path passes through it.

A vertex i is called recurrent if a periodic path passes through it. Two recurrent vertices i and j are equivalent if there is a periodic path that contains both i and j. The set of recurrent vertices is decomposed into classes of equivalent recurrent vertices, called strongly connected components in graph theory. The strong components of a symbolic image generate an isolating neighborhood of the chain-recurrent set of a dynamical system [**14**]. It is known [**9**] that an invariant measure equals zero outside the chain-recurrent set. Hence, to construct invariant measures, it is enough to study an isolated component of the chain-recurrent set. Because of this, without loss of generality we can suppose that the graph G consists of a single strong component.

It follows from Proposition 1 that the family of flows $\mathcal{M}(G)$ is a convex polyhedron which is the hull of the simple flows. This means that any flow can be constructed by the following method. Let $P = \{\omega_z\}$ be the set of all simple cycles and $\{m^z_{ij}\}$ the set of simple flows. The set of simple flows is finite and each simple flow is uniquely defined. By Proposition 1, any flow $m = \{m_{ij}\}$ is determined by a collection of values $a_z \geq 0$ such that $\sum_z a_z = 1$, in which case $m_{ij} = \sum_z a_z m^z_{ij}$. The coefficients $\{a_z\}$ are called the weights of $\{\omega_z\}$. Consequently, the flow m can be represented by a point on the standard simplex

$$\Delta = \left\{ a \in \mathbb{R}^N \;\middle|\; a_z \geq 0, \sum_z a_z = 1 \right\},$$

where N is the number of simple cycles on G. The method of construction using all simple cycles can require a lot of computation time, since, as a rule, the number of all simple cycles is huge. For example, the full graph (in which each vertex is connected to every one) with n vertices has n^2 edges and $N = 2^n - 1$ simple cycles. Of course, we can use a partial collection of the cycles by setting the weights of the untapped cycles to zero.

Definition 2. Let Q and G be directed graphs; then $s : Q \to G$ is said to be a *mapping of the graphs* (or *graph mapping*) if it transforms the vertices and edges of Q into the vertices and edges of G in a consistent way. In other words, if k and l are vertices of Q between which there is an edge $k \to l$, and if $s(k) = i$ and $s(l) = j$, then the edge $i \to j$ exists in G and $s(k \to l) = i \to j$. The converse must hold as well: if $s(k \to l) = i \to j$, then $s(k) = i$ and $s(l) = j$.

A graph mapping generates a mapping of (admissible) paths, and a periodic path is transformed into a periodic path. It should be noted that under such a transformation the period may decrease. Recurrent vertices are transformed into recurrent ones and equivalent recurrent vertices are transformed into equivalent recurrent vertices [**14**]. Hence, strong components are transformed into strong components.

PROPOSITION 2. *Let Q and G be directed graphs, let $s : Q \to G$ be a mapping of the graphs and suppose that there exists a flow m on Q. Then, on G, the flow $m^* = s^*(m)$ is generated such that the measure of edge $i \to j \in G$ is*

$$m^*_{ij} = \sum_{s(p \to q) = i \to j} m_{pq},$$

where the sum is taken over all edges $p \to q$ which are transformed into $i \to j$. If an edge $i \to j$ does not have a preimage, then $m^*_{ij} = 0$.

Proof. It is enough to check two of the properties of flow for m^*. Since summation over all vertices of G coincides with summation over all vertices of Q, the normalization property holds:

$$\sum_{ij} m^*_{ij} = \sum_{pq} m_{pq} = 1.$$

To check the invariance property, we fix a vertex $i \in G$. If i does not have a preimage, i.e. if $s^{-1}(i) = \emptyset$, then both incoming edges and outgoing ones do not have preimages either. Hence, the measure equals zero and the invariance condition holds in i. If $s^{-1}(i) \neq \emptyset$, we consider all vertices p in the preimage $s^{-1}(i)$. The invariance condition is valid for each vertex p, so that

$$\sum_r m_{rp} = \sum_t m_{pt}.$$

By summing over $p \in s^{-1}(i)$ and taking into account the fact that $m^*_{rl} = 0$ for edges $r \to l$ with $s^{-1}(r \to l) = \emptyset$, we get the desired equality

$$\sum_k m^*_{ki} = \sum_j m^*_{ij}$$

for $i \in G$. □

PROPOSITION 3. *Suppose that an N-periodic path ω exists on the graph G. Then, on G, there is a flow m^* such that $m^*_{ij} = k_{ij}/N$, where k_{ij} is the number of passages of ω through the edge $i \to j$.*

Proof. Let $\omega = \{i_1 \to i_2 \to \cdots \to i_N \to i_1\}$. Construct the graph Q consisting of one simple cycle of period N, i.e. let $Q = \{1 \to 2 \to \cdots \to N \to 1\}$. On Q there exists a unique flow m such that $m_{pq} = 1/N$. Let the mapping $s : Q \to G$ stack the cycle Q on the periodic path ω, that is, $s(k) = i_k$. According to Proposition 2, a flow is generated on G

such that the measure of edge $i \to j$ is the sum of the measures of the edges in $s^{-1}(i \to j)$; in other words, $m^*_{ij} = k_{ij}/N$, where k_{ij} is the number of passages of ω through the edge $i \to j$. □

Let G consist of one strong component; then any two vertices are connected by an admissible path. Hence there exists a periodic path Ω passing through all vertices. The path Ω can be called 'dense on vertices'. According to Proposition 3, there is a flow $m = \{m_{ij}\}$ on G with positive measure $m_i = \sum_j m_{ij} > 0$ for each vertex. Similarly, there exists a periodic path Ω^* passing through each edge; this path Ω^* can be called 'dense on edges'. Evidently, a path that is dense on edges is dense on vertices. According to Proposition 3, Ω^* generates a flow with $m_{ij} > 0$ on each edge $i \to j$. Thus, on any graph there exists a flow which is positive on each recurrent vertex or edge.

PROPOSITION 4. *Suppose that on G there exists a family of periodic paths $\omega_1, \ldots, \omega_r$ with periods $p_1, \ldots, p_r$. Set $N = p_1 + \cdots + p_r$. Then there exists a flow m on G such that $m_{ij} = k_{ij}/N$, where k_{ij} is the number of passages of the paths $\omega_1, \ldots \omega_r$ through the edge $i \to j$.*

Proof. The proof of this proposition essentially repeats the proof of Proposition 3. Let us assume the hypotheses of the proposition. Note that Proposition 2 does not require that the graphs G and Q be connected. Construct the graph Q consisting of the disconnected union of r simple cycles $\Omega_1, \ldots, \Omega_r$ with periods $p_1, \ldots, p_r$; then Q has N vertices and N edges, where $N = p_1 + \cdots + p_r$. It is easy to check that on Q there is a flow m^* with the measure of edges given by $m^*_{pq} = 1/N$. The mapping $s : Q \to G$ stacks the cycles $\Omega_1, \ldots \Omega_r$ on the periodic paths $\omega_1, \ldots, \omega_r$, respectively. According to Proposition 2, the measure of edge $i \to j$ is the sum of the measures of its preimages, i.e. $m_{ij} = k_{ij}/N$ where k_{ij} is the number of passages of the paths $\omega_1, \ldots \omega_r$ through the edge $i \to j$. □

Proposition 4 may be generalized as follows.

PROPOSITION 5. *Suppose that on G there exists a family of periodic paths $\omega_1, \ldots, \omega_r$ with periods $p_1, \ldots, p_r$. Then there exists a flow m on G such that*

$$m_{ij} = \sum_{t=1}^{r} \alpha_t k^t_{ij}/p_t,$$

where $\alpha_t \geq 0$, $\sum_t \alpha_t = 1$, and k^t_{ij} is the number of passages of the path ω_t through the edge $i \to j$.

Proposition 4 can be obtained from Proposition 5 by taking $\alpha_t = p_t/N$, $N = \sum_t p_t$. It follows from Propositions 2 and 5 that any flow in $\mathcal{M}(G)$ can be obtained as described in Proposition 5.

3. *Invariant measures and flows on the symbolic image*

Consider a homeomorphism $f : M \to M$, a measured partition C and the symbolic image G generated by C. The maximal diameter of cells of the partition C will be denoted by d. Let us study the space of flows $\mathcal{M}(G)$ under successive subdivisions of C. Let the partition C be subdivided, i.e. divide each cell $M(i)$ into cells $M(i1), M(i2), \ldots$ such that

$M(i) = \bigcup_k M(ik)$. Thus we obtain a new partition NC and a new symbolic image NG, with $\{(ik)\}$ as the indices of vertices. The natural mapping $s : \mathrm{N}G \to G$ has a very simple form: $s(ik) = i$. This mapping is a mapping of directed graphs, that is: if on NG there is an edge $(ik) \to (jl)$, then on G there exists the edge $i \to j$. The mapping s allows us to transfer any flow on NG to the flow on G:

$$s^* : \mathcal{M}(NG) \to \mathcal{M}(G),$$

as was described in the previous section. It is clear that, in general, $s^*(\mathcal{M}(\mathrm{N}G)) \neq \mathcal{M}(G)$.

Definition 3. Two flows $m \in \mathcal{M}(\mathrm{N}G)$ and $m^* \in \mathcal{M}(G)$ are said to be consistent if $s^*(m) = m^*$.

Consider successive subdivisions $C_1, C_2, C_3, \ldots$ such that the maximal diameters of the partitions, $d_1, d_2, d_3, \ldots$, tend to zero. Such a sequence generates a sequence of symbolic images $G_1, G_2, G_3, \ldots$ and mappings $s : G_k \to G_{k-1}$ and $s^* : \mathcal{M}(G_k) \to \mathcal{M}(G_{k-1})$. Thus we obtain the sequences

$$G_1 \xleftarrow{s} G_2 \xleftarrow{s} G_3 \xleftarrow{s} \ldots.$$

and

$$\mathcal{M}(G_1) \xleftarrow{s^*} \mathcal{M}(G_2) \xleftarrow{s^*} \mathcal{M}(G_3) \xleftarrow{s^*} \ldots.$$

The mapping $f : M \to M$ can be treated as an infinite graph with vertices $x \in M$ and edges $x \to f(x)$. For any symbolic image G, there is a mapping $s : M \to G$ of the form $s(x) = \{i \mid x \in M(i)\}$, i.e. a point x is mapped to the index of the cell that contains x. This mapping is a mapping of directed graphs. We obtain a sequence of the form

$$G_1 \xleftarrow{s} G_2 \xleftarrow{s} G_3 \xleftarrow{s} \cdots \xleftarrow{s} \{f : M \to M\}. \tag{7}$$

For any symbolic image G, there exists a mapping $s^* : \mathcal{M}(f) \to \mathcal{M}(G)$ given by the formula

$$s^*(\mu) = m = \{m_{ij} = \mu(M(i) \cap f^{-1}(M(j)))\},$$

where $M(i)$ and $M(j)$ are cells of the symbolic image G. The sequence (7) generates the sequence

$$\mathcal{M}(G_1) \xleftarrow{s^*} \mathcal{M}(G_2) \xleftarrow{s^*} \mathcal{M}(G_3) \xleftarrow{s^*} \cdots \xleftarrow{s^*} \mathcal{M}(f). \tag{8}$$

Suppose that on each symbolic image G_k there is a flow $m^k \in \mathcal{M}(G_k)$ and that these flows are consistent, i.e.

$$s^*(m^{k+1}) = m^k.$$

By using Lebesgue measure we construct the measure μ_k on M for each k:

$$\mu_k(A) = \sum_i m_i^k v(A \cap M(i))/v(M(i)), \tag{9}$$

where A is a measured set, the $M(i)$ are the cells of C_k and v is Lebesgue measure normalized on M. Therefore, we get the sequence of measures $\{\mu_k\}$ on the manifold M.

THEOREM 1. *Consider successive subdivisions of the partitions C_k with maximal diameters $d_k \to 0$. If m^k is a consistent sequence of flows on the symbolic images G_k, then on M there exists a f-invariant measure μ such that*

$$\mu = \lim_{k\to\infty} \mu_k,$$

where the convergence is considered in the weak topology.

Proof. Let $m^k = \{m^k_{ij}\}$ be a consistent sequence of flows on the symbolic images G_k. Let ϕ be a continuous function on the compact set M and let $C_k = \{M(i)\}_k$ be a partition of M. To each cell $M(i)$ we ascribe the measure m^k_i of the vertex $i \in G_k$. Take a point $x_i \in M(i)$ and construct the integral sum

$$F_k(\phi) = \sum_i \phi(x_i) m^k_i.$$

We shall show that the limit

$$\lim_{k\to\infty} F_k(\phi) = F(\phi) \tag{10}$$

exists. It is enough to show that $F_k(\phi)$ is Cauchy sequence. Let C_l, $l > k$, be a subdivision of the partition C_k so that the cells $M(ir) \in C_l$, $r = 1, 2, \ldots$, form a partition of the cell $M(i) \in C_k$. Since the sequence of flows is consistent, we have

$$m^k = s^*(m^l)$$

and

$$m^k_i = \sum_r m^l_{ir}, \tag{11}$$

where m_{ir} is the measure of the cell $M(ir) \in C_l$ (or of the vertex $(ir) \in G_l$). Let us estimate the difference

$$|F_k(\phi) - F_l(\phi)| = \left|\sum_i \phi(x_i) m^k_i - \sum_{ir} \phi(x_{ir}) m^l_{ir}\right|.$$

Taking into account the equality (11) and the uniform continuity of ϕ on the compact set M, we get

$$\begin{aligned} |F_k(\phi) - F_l(\phi)| &= \left|\sum_{ir} (\phi(x_i) - \phi(x_{ir})) m^l_{ir}\right| \le \sum_{ir} |\phi(x_i) - \phi(x_{ir})| m^l_{ir} \\ &\le \sup_{|x-y|\le d_k} |\phi(x) - \phi(y)| \sum_{ir} m^l_{ir} = \alpha(d_k), \end{aligned}$$

where $\alpha(d)$ is the modulus of continuity of the function ϕ and d_k is the maximal diameter of C_k. Since $\alpha(d) \to 0$ as $d \to 0$, the sequence $F_k(\phi)$ is a Cauchy sequence and therefore the limit (10) exists.

In the same way, we can show that this limit does not depend on the choice of $x_i \in M(i)$. Thus, the linear functional $F(\phi)$ is well-defined. It is bounded, because $|F(\phi)| \le \sup_M |\phi|$ and $F(\phi) \ge 0$ as $\phi > 0$. According to the Riesz representation theorem [**9**], there exists a measure μ such that

$$F(\phi) = \int_M \phi \, d\mu.$$

Since the measure μ_k is defined according to the formula (9), the measure μ_k on a cell $M(i)$ differs from Lebesgue measure by a constant factor, and the measure of each cell $\mu_k(M(i)) = m_i^k$ coincides with the measure of the vertex $i \in G_k$. Next, we show that

$$\lim_{k\to\infty} \mu_k = \mu$$

in the weak topology. It suffices to show that for any continuous function ϕ,

$$\int_M \phi \, d\mu_k \to \int_M \phi \, d\mu$$

as $k \to \infty$. By the mean value theorem for each cell $M(i)$, there exists a point x_i^* in the closure $\overline{M(i)}$ such that

$$\int_{M(i)} \phi \, d\mu_k = \phi(x_i^*)\mu_k(M(i)) = \phi(x_i^*)m_i^k.$$

Hence

$$\int_M \phi \, d\mu_k = \sum_i \int_{M(i)} \phi \, d\mu_k = \sum_i \phi(x_i^*)m_i^k.$$

So, it is enough to show that

$$\lim_{k\to\infty} \sum_i \phi(x_i^*)m_i^k = \lim_{k\to\infty} \sum_i \phi(x_i)m_i^k,$$

where $|x_i^* - x_i| \le d_k$. This can be proved in the same way as above by using the modulus of continuity of the function ϕ.

It is known [**9**] that the invariance of a measure μ with respect to f follows from the equality

$$\int_M \phi \, d\mu = \int_M \phi(f) \, d\mu,$$

where ϕ is any continuous function on M. Consider the integral sum $F_k(\phi) = \sum_i \phi(x_i)m_i^k$, where

$$m_i^k = \sum_j m_{ij}^k = \sum_r m_{ri}^k.$$

We have

$$F_k(\phi) = \sum_i \phi(x_i) \sum_r m_{ri}^k = \sum_{ir} \phi(x_i)m_{ri}^k.$$

In each term $\phi(x_i)m_{ri}^k$, we replace the point x_i by a point $x_{ri} \in f(M(r)) \cap M(i)$ and get

$$F_k(\phi) = \sum_{ir} \phi(x_{ri})m_{ri}^k + \sum_{ir}(\phi(x_i) - \phi(x_{ri}))m_{ri}^k = \sum_{ir} \phi(x_{ri})m_{ri}^k + \varepsilon^*,$$

where ε^* is estimated through the modulus of continuity of ϕ and $\varepsilon^*(d_k) \to 0$ as $k \to \infty$. For each point $x_{ri} \in f(M(r)) \cap M(i)$ there exists a point $z_{ri} \in M(r) \cap f^{-1}(M(i))$ such that $f(z_{ri}) = x_{ri}$, i.e. $z_{ri} = f^{-1}(x_{ri})$. We therefore have

$$F_k(\phi) = \sum_{ir} \phi(f(z_{ri}))m_{ri}^k + \varepsilon^*,$$

where all points z_{ri}, $i = 1, 2, \ldots$, lie in $M(r)$. Let us replace these points by a single point $z_r \in M(r)$. We obtain the equalities

$$
\begin{aligned}
F_k(\phi) &= \sum_{ir} \phi(f(z_r))m^k_{ri} + \sum_{ir}(\phi(f(z_{ri})) - \phi(f(z_r)))m^k_{ri} + \varepsilon^* \\
&= \sum_{r} \phi(f(z_r)) \sum_{i} m^k_{ri} + \varepsilon^{**} + \varepsilon^* = \sum_{r} \phi(f(z_r))m^k_r = F_k(\phi(f)) + \varepsilon^{**} + \varepsilon^*,
\end{aligned}
$$

where ε^{**} is estimated through the modulus of continuity of ϕ and $\phi(f)$, and $\varepsilon^{**}(d_k) \to 0$ as $k \to \infty$. Passing to the limit as $k \to \infty$, we obtain the equality

$$F(\phi) = F(\phi(f)),$$

which says that the measure μ is f-invariant, as desired. The proof of the theorem is thus complete. □

As indicated earlier, according to (1) each invariant measure μ generates a sequence of flows m^k on the symbolic images for any sequence of subdivisions $C_1, C_2, C_3, \ldots$; it is easy to verify that the sequence is consistent. The above theorem guarantees the converse, namely that each consistent sequence of flows m^k on the symbolic images G_k with $d_k \to 0$ generates an invariant measure.

COROLLARY 1. *Every invariant measure μ can be obtained by the method described in Theorem 1.*

So, an invariant measure and a consistent sequence of flows are interconvertible. Now we shall study sequences of flows which are not consistent. Consider a sequence of symbolic images $G_1, G_2, \ldots\ G_t, \ldots$ of the homeomorphism f with respect to a sequence of subdivisions $C_1, C_2, \ldots, C_t, \ldots,$ with $d_t \to 0$ as $t \to \infty$. Fix a flow m^t on each symbolic image G_t. Using Lebesgue measure for m^t, we construct a sequence of measures μ_t on M by (9). On each symbolic image G_τ we define the sequence of flows $\{m^{k,\tau} : k = 0, 1, \ldots\}$ as the projection of the flows $m^{\tau+k}$ via the mapping $s^* : \mathcal{M}(G_{\tau+k}) \to \mathcal{M}(G_\tau)$. For the space of flows on G, we introduce the distance function $\rho(m^1, m^2) = \sum_i |m^1_i - m^2_i|$, where m^*_i is a measure of the vertex i (or the cell $M(i)$).

Definition 4. A sequence of flows $\{m^t\}$ is said to converge if the sequence of projections $\{m^{k,\tau}\}$ converges in the distance ρ on each G_τ as $k \to \infty$.

THEOREM 2. *If the sequence of flows $\{m^t\}$ converges, then the corresponding sequence of measures $\{\mu_t\}$ converges to an invariant measure in the weak topology.*

Proof. On each symbolic image G_τ we fix the flow $m^{*,\tau} = \lim_{k\to\infty} m^{k,\tau}$. By construction, the flows $m^{*,\tau}$ are consistent, i.e. $s^*(m^{*,\tau+1}) = m^{*,\tau}$. According to Theorem 1, the consistent sequence $\{m^{*,\tau}\}$ generates the sequence $\{\mu_{*,\tau}\}$ that converges to an invariant measure μ in the weak topology.

We now show that the sequence $\{\mu_t\}$ converges to μ in the weak topology as well. Let $t = \tau + k$, let G_τ be the symbolic image with respect to a partition C_τ, and let $M(i)$ be the cells of C_τ. We have

$$\int \phi \, d\mu_t - \int \phi \, d\mu = \sum_i \phi(x_i)m^t_i - \sum_i \phi(x^*_i)m^*_i,$$

where $m_i^t = \mu_t(M(i))$, $m_i^* = \mu(M(i))$, and the points x_i and x_i^* lie in $\overline{M(i)}$ and are determined by the mean value theorem applied to each $\overline{M(i)}$. By the assumption, we have that

$$\begin{aligned}\sum_i |m_i^t - m_i^*| &= \sum_i |\mu_t(M(i)) - \mu^*(M(i))| \\ &= \sum_i |m^{k,\tau}(M(i)) - m^{*,\tau}(M(i))| \\ &= \rho(m^{k,\tau}, m^{*,\tau}) \to 0\end{aligned}$$

as $k \to \infty$. Hence

$$\begin{aligned}\left|\int \phi\, d\mu_t - \int \phi\, d\mu\right| &= \left|\sum_i [\phi(x_i)(m_i^t - m_i^*) + (\phi(x_i) - \phi(x_i^*))m_i^*]\right| \\ &\le \|\phi\| \sum_i |m_i^t - m_i^*| + \alpha(d_t)\sum_i m_i^* \\ &= \|\phi\|\rho(m^{k,\tau}, m^{*,\tau}) + \alpha(d_\tau),\end{aligned}$$

where $\|\phi\| = \sup_M |\phi(x)|$, d_τ is the maximal diameter of C_τ and $\alpha(d)$ is the modulus of continuity of the function ϕ.

To prove that $\mu_t \to \mu$ in the weak topology, it is enough to show that for a given function ϕ and an $\epsilon > 0$, there is a number t_0 such that

$$\left|\int \phi\, d\mu_t - \int \phi\, d\mu\right| < \epsilon$$

for $t > t_0$. For any given number $\epsilon/2$ and function ϕ, we fix τ and $d_\tau > 0$ so that $\alpha(d_\tau) < \epsilon/2$. For fixed values of τ, $\epsilon/2$ and $\|\phi\|$, we find k such that $\rho(m^{k,\tau}, m^{*,\tau}) < \epsilon/(2\|\phi\|)$ if $1 \le \|\phi\|$ or $\rho(m^{k,\tau}, m^{*,\tau}) < \epsilon/2$ if $\|\phi\| < 1$. Set $t_0 = \tau + k$. Then for $t > t_0$ we have

$$\left|\int \phi\, d\mu_t - \int \phi\, d\mu\right| \le \|\phi\|\rho(m^{k,\tau}, m^{*,\tau}) + \alpha(d_\tau) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\int \phi\, d\mu_t \to \int \phi\, d\mu$ as $t \to \infty$, and therefore the sequence of measures $\{\mu_t\}$ converges to μ in the weak topology. □

If the sequence $\{m^t\}$ is consistent, i.e. if $s^*(m^{t+1}) = m^t$, then $m^{k,\tau} = m^\tau$ and the sequence of projections $\{m^{k,\tau}\}$ converges to m^τ. So, Theorem 2 is a generalization of Theorem 1. These theorems cannot, however, be applied to a sequence of general form. The next theorem describes the properties of an arbitrary sequence of flows.

THEOREM 3. *Suppose that a sequence of symbolic images $\{G_t\}$ of the homeomorphism f and a sequence of flows $\{m^t\}$ are fixed and that $d_t \to 0$ as $t \to \infty$. Then:*

(1) *there exists a subsequence indexed by $t_k \to \infty$ such that μ_{t_k} (constructed via (9)) converges in the weak topology to a measure μ that is invariant with respect to f;*

(2) *if some subsequence of measures μ_{t_l} converges in the weak topology to a measure μ^*, then μ^* is invariant with respect to f.*

Proof. Consider one of the symbolic images, G_1, say. The set of flows on G_1 forms the convex polyhedron $\mathcal{M}_1$. Each flow $m = \{m_{ij}\}$ is represented by a point in R^N where N is the number of vertices of $\mathcal{M}_1$. By means of the natural mapping $s^* : \mathcal{M}(G_t) \to \mathcal{M}(G_1) = \mathcal{M}_1$, we transform the flows m^t on the graph G_1 and denote the transformed flows by m_1^t, i.e. $m_1^t = s^*(m^t)$. So, on the compact set $\mathcal{M}_1$, we have the sequence $\{m_1^t\}$ from which we can take a convergent subsequence$\{m_1^{t_k}\}$. Let $m_1^* = \lim_{k\to\infty} m_1^{t_k}$. Next, consider the symbolic image G_2, for which we construct a flow m_2^* as the limit of some subsequence of the projections $s^*(m^{t_k})$ on $\mathcal{M}_2$. Following this procedure, we construct the flow m_t^* on each symbolic image G_t. If we now take the diagonal subsequence $\{m^{t_\tau}\}$, then on each symbolic image G_t the sequence $s^*(m^{t_\tau})$ converges to m_t^*. According to Theorem 2, the subsequence of measures μ_{t_τ} converges to an invariant measure μ in weak topology. Thus, assertion (1) of the theorem is proved.

To prove the second assertion, suppose that there is a subsequence of measures that is convergent to μ in the weak topology. Without loss of generality we can assume that the original sequence converges to μ. By assertion (1), we can take a subsequence $\{\mu_{t_k}\}$ that converges to an invariant measure μ^*. Hence $\mu^* = \mu$ by uniqueness of the limit, and μ is invariant. The theorem is thus proved. □

The results we have obtained are applicable to sequences of flows on symbolic images with $d_k \to 0$. In practice, it is desirable to have results concerning an individual flow on a symbolic image for a small positive diameter d.

THEOREM 4. *For any neighborhood U (in weak topology) of the set $\mathcal{M}(f)$, there exists a positive d_0 such that for any partition C with the maximal diameter of cells satisfying $d < d_0$ and any flow m on the symbolic image G constructed with respect to C, the measure μ (constructed via m and (9)) lies in U.*

Proof. Suppose, to the contrary, that there exists a neighborhood U of $\mathcal{M}(f)$ such that for any partition C_k with maximal diameter d_k on the symbolic image G_k there is a flow m_k for which the measure μ_k does not lie in U. We may assume that $d_k \to 0$. According to the previous theorem, there exists a subsequence $\{m_{k_p}\}$ such that the subsequence of measures $\{\mu_{k_p}\}$ constructed via (9) converges to an invariant measure $\mu \in \mathcal{M}(f)$. This means that for some number k_{p_0}, the subsequence μ_{k_p} with $p > p_0$ is in U, which is a contradiction. □

Theorem 4 guarantees that any measure constructed by means of a symbolic image is a good approximation of some invariant measure, provided that the diameter of the partition is small enough. In practice, we can construct no more than a finite number of symbolic images, hence the obtained result provides a basis for practical computation. Moreover, this theorem allows us to consider the set $\mathcal{M}(G)$ of all flows on a symbolic image G as an approximation of the set of invariant measures $\mathcal{M}(f)$, provided that the diameter of the partition is small enough.

4. *Stochastic Markov chains*

A stochastic Markov chain [**11**] is defined by a collection of states $\{i = 1, 2, \dots n\}$ together with probabilities P_{ij} of transition from state i to state j. The matrix of transition probabilities, $P = (P_{ij})$, is a stochastic matrix, i.e. $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ for each i. A probability distribution $p = (p_1, p_2, \dots, p_n)$, with $\sum_i p_i = 1$, is *stationary* if p is a left fixed vector of the matrix P. It should be noted (see, e.g., [**9**]) that a stochastic matrix is sometimes defined as the transpose of that described above, in which case a stationary distribution would be a right eigenvector.

Each flow $m = \{m_{ij}\}$ on a graph G generates a stochastic Markov chain such that the states $\{i\}$ are vertices $\{i \mid m_i \neq 0\}$ and the transition probability $i \to j$ is given by

$$P_{ij} = m_{ij}/m_i.$$

The resulting stochastic matrix $P = (m_{ij}/m_i)$ has a stationary distribution of the form $(m_1, m_2, \ldots, m_n)$. So any flow $m = \{m_{ij}\}$ on a graph G generates a stochastic Markov chain for which the distribution (m_i) of measures of vertices is stationary.

It turns out that the converse is also true: for any stochastic matrix $P = (P_{ij})$ and associated stationary distribution $p = (p_i)$, there exists a flow $m = \{m_{ij}\}$ such that the distribution of measures on vertices is $m_i = p_i$. In fact, let P be a stochastic matrix and suppose that $pP = p$. Consider a graph G with n vertices $\{i\}$ and edges $\{i \to j$ if $P_{ij} > 0\}$. Let us construct a distribution on the edges of the form $m_{ij} = P_{ij} p_i$ and prove that the constructed distribution is a flow on G. Since P is a stochastic matrix, we have the equality $\sum_j P_{ij} = 1$ for each i. It follows that

$$\sum_j m_{ij} = \sum_j P_{ij} p_i = p_i \sum_j P_{ij} = p_i.$$

Since $pP = p$, we have $\sum_k p_k P_{ki} = p_i$ for each i. Thus we obtain

$$\sum_k m_{ki} = \sum_k p_k P_{ki} = p_i = \sum_j m_{ij},$$

i.e. Kirchhoff's law is valid for the distribution m_{ij}. Moreover, $\sum_{ij} m_{ij} = \sum_i p_i = 1$.

>From the above it follows that the flow technology on the graph is equivalent to the stochastic-matrix method. The papers [**3, 4, 6, 7**] use a stochastic matrix of the form

$$P_{ij} = v(M(i) \cap f^{-1}(M(j)))/v(M(i)),$$

where v is Lebesgue measure and the $M(i)$ are cells of a partition. In these papers, the SBR measure is constructed via the stochastic-matrix method under some additional conditions. It is clear that the construction of a stochastic matrix through a non-Lebesgue measure leads, in general, to an invariant measure that differs from SBR measure. For example, a stochastic matrix can be obtained from the zero–one matrix $\Pi = (\pi_{ij})$ where $\pi_{ij} = 1$ if the edge $i \to j$ exists and $\pi_{ij} = 0$ otherwise. By setting $P_{ij} = \pi_{ij}/(\sum_k \pi_{ik})$, we get a stochastic matrix, a flow and an approximation of an invariant measure.

5. *The balance method*

We have considered the construction of a flow on a graph based on a collection of periodic paths. Now we look at another method, which allows us to obtain an invariant flow from any non-invariant distribution in an iterative manner. Let the matrix $P = (p_{ij})$ be an arbitrary distribution of non-negative values on the edges of a graph G. The set of edges $G(P^+) = \{i \to j \mid p_{ij} > 0\}$ taking positive values of P is called the *support* of P. We will construct a flow on the support $G(P^+)$. From what we have shown above, the support necessarily has to contain a periodic path in order for a flow on $G(P^+)$ to exist.

Let G be a graph with n vertices. A matrix $x = (x_{ij})$ is a flow on G if the following conditions are satisfied:

- $x_{ij} \geq 0$;
- $\sum_{ij} x_{ij} = 1$;
- $\sum_j x_{ij} = \sum_k x_{ki}$ for $i = 1, 2, \ldots, n$.

Moreover, we require that the support of x be in $G(P^+)$, i.e. that $x_{ij} = 0$ if $p_{ij} = 0$. A similar computational task arises in the theory of convex programming. Our discussion is based on the paper [**2**], in which the method we need is substantiated. The task of computing a flow on a graph may be considered as a special example of a transport problem. The Leningrad architect G. V. Sheleikhovsky solved such a problem in the 1940s [**15**] by using the balance method. Starting from an arbitrary distribution, he recalculated the distribution in a sequential way such that at each step only one equality is required to be satisfied while the others escape attention. By repeating such a process in a cyclic manner, he obtained a sequence of distributions that converged rapidly to the desired solution. We shall solve our task in the same way.

PROPOSITION 6. *Let $P = (p_{ij})$ be an arbitrary non-negative distribution on G such that $G(P^+)$ contains a periodic path. Then there exists an algorithm that constructs the flow $Q = (q_{ij})$ on G, with $G(Q^+) \subset G(P^+)$, which maximizes the function*

$$g(x) = \sum_{ij} x_{ij} \ln \frac{p_{ij}}{x_{ij}} = \sum_{ij} x_{ij} \ln p_{ij} - \sum_{ij} x_{ij} \ln x_{ij}$$

in the space of flows on $G(P^+)$.

Proof. Let d be the number of the edges of the graph $G(P^+)$, with $d \leq n^2$. If the elements with $p_{ij} > 0$ are put in sequential order $(ij) \to k = 1, 2, \ldots, d$ (for instance, row by row from top to bottom), then x can be considered as a point in the space $S = \{x \in \mathbb{R}^d \mid x_k = x_{ij} > 0\}$. Set

$$B_0 = \left\{x \in R^d \;\middle|\; x_{ij} \geq 0, \sum_{ij} x_{ij} = 1\right\},$$

$$B_i = \left\{x \in R^d \;\middle|\; x_{ij} \geq 0, \sum_{j} x_{ij} = \sum_k x_{ki}\right\}$$

for $i = 1, 2, \ldots, n$. The intersection $D = \bigcap_i B_i \cap B_0$ is a compact set that lies in the closure $\overline{S}$; hence the function g reaches its maximum on D. Our goal is to find the point at which the maximum is attained. In [**2**], the following problem of convex programming was solved.

Let $f(x)$ be a strictly convex function that is continuously differentiable on a convex set $S \subset \mathbb{R}^d$ and continuous on $\overline{S}$. It is required to minimize the function f under the linear restrictions

$$Ax = b, \quad x \in \overline{S},$$

where $b \in \mathbb{R}^m$ and A is a matrix with r columns and m rows.

We shall apply the results of [**2**] to the function $f = -g$, which is strictly convex. Our linear restrictions consist of $n + 1$ equations; the first of these gives the normalization

$$\sum_{ij} x_{ij} = 1, \tag{12}$$

and the next n equations describe the invariance (Kirchhoff's law):

$$\sum_j x_{ij} - \sum_k x_{ki} = 0, \quad i = 1, 2, \ldots, n. \tag{13}$$

The necessary condition for existence of a solution to the optimization task is the existence of a solution to the system (12–13). The existence of a periodic path in $G(P^+)$ guarantees that there is a flow on $G(P^+)$ which is the solution needed.

Consider the following iteration steps.

(1) Take an arbitrary point (matrix) $x^0 \in S$.

(2) If a point x^t is known, choose the p_tth equation and find a point $x^{t+1} \in B_{p_t}$ that satisfies this equation but which may not satisfy the others. The method of computing such a point will be described later.

In the iterative loop we have to take all $n + 1$ equations. The sequence $\{x^t\}$ thus obtained is called a relaxation sequence, and the index sequence $\{p_t\}$ is called a relaxation control. It is possible to take the relaxation control in ordinary cyclic order, i.e. first solve the normalization equation and then take consecutive invariance equations.

It was shown in [**2**] that for $\{x^t\}$ to solve the extremum problem the points x^{t+1} and x^t have to satisfy the following system of equations:

$$\operatorname{grad} f(x^{t+1}) = \operatorname{grad} f(x^t) + \lambda A_p, \tag{14}$$

$$(A_p, x^{t+1}) = b_p, \tag{15}$$

where $\operatorname{grad} f$ is the gradient of f, λ is an unknown parameter, (*,*) denotes the inner product, A_p is the row of the left-hand side of the pth equation, and b_p is the right-hand side of the same equation. In other words, x^{t+1} is a solution of the pth equation. For the function

$$f(x) = -g(x) = \sum_{ij} x_{ij} \ln x_{ij}/p_{ij},$$

we have

$$(\operatorname{grad} f(x))_{ij} = \ln \frac{x_{ij}}{p_{ij}} + 1.$$

For the normalizing equation (12), we get the system

$$\ln \frac{x_{ij}^{t+1}}{p_{ij}} = \ln \frac{x_{ij}^t}{p_{ij}} + \lambda,$$
$$\sum_{ij} x_{ij}^{t+1} = 1.$$

Hence we obtain the ordinary transformation of normalization,

$$x_{ij}^{t+1} = \frac{x_{ij}^t}{\sum_{kl} x_{kl}^t}. \tag{16}$$

For the ith equation of the invariance (13), we get the following system of equations:

$$\ln \frac{x_{ij}^{t+1}}{p_{ij}} = \ln \frac{x_{ij}^{t}}{p_{ij}} + \lambda,$$
$$\ln \frac{x_{ki}^{t+1}}{p_{ki}} = \ln \frac{x_{ki}^{t}}{p_{ki}} - \lambda,$$
$$\sum_{j \neq i} x_{ij}^{t+1} - \sum_{k \neq i} x_{ki}^{t+1} = 0.$$

Then

$$\exp \lambda = \left(\frac{\sum_{k \neq i} x_{ki}^{t}}{\sum_{j \neq i} x_{ij}^{t}} \right)^{\frac{1}{2}}$$

and we obtain the transformation in the form

$$x_{ij}^{t+1} = x_{ij}^{t} \left(\frac{\sum_{m \neq i} x_{mi}^{t}}{\sum_{l \neq i} x_{il}^{t}} \right)^{\frac{1}{2}} \tag{17}$$

for $j \neq i$,

$$x_{ki}^{t+1} = x_{ki}^{t} \left(\frac{\sum_{l \neq i} x_{il}^{t}}{\sum_{m \neq i} x_{mi}^{t}} \right)^{\frac{1}{2}} \tag{18}$$

for $k \neq i$, and

$$x_{ii}^{t+1} = x_{ii}^{t}. \tag{19}$$

The formulas (17), (18) and (19) describe the transformation of the ith row and ith column of the matrix x^t in step t. The other elements of x^t do not change. According to [**2**], the sequence thus obtained has a limit if the function

$$D(x, y) = f(x) - f(y) - (\operatorname{grad} f(y), x - y)$$

is such that $D(x, y^k) \to 0$ when $x \in \overline{S}$, $y^k \to x$ and $y^k \in S$. In our case,

$$D(x, y) = \sum_{ij} (y_{ij} - x_{ij}) + \sum_{ij} x_{ij} \ln x_{ij} - \sum_{ij} x_{ij} \ln y_{ij},$$

and the property is easily checked. Moreover, in this case (see [**2**]) the limit does not depend on the relaxation control but only on the initial value x^0. For the limit value

$$Q = \lim_{t \to \infty} x^t,$$

to solve the extremum problem it suffices to take the initial value as the point of the global minimum of f, i.e. to set $x_{ij}^0 = p_{ij} \exp(-1)$. Taking into account the normalization, we can start the iteration process with

$$x_{ij}^0 = p_{ij}, \tag{20}$$

and the proposition is proved. □

Consider a relaxation control that results in fast convergence to the desired solution. Let $x = (x_{ij})$ be an arbitrary non-negative distribution on the edges of G. We give an index $i = 0, 1, \ldots, n$ to each of the equations in (2) and (3) so that $i = 0$ corresponds to the normalization equation (3) and each $i > 0$ corresponds to the invariance property for the ith vertex. For the matrix $x = (x_{ij})$ and each i, we determine the residual

$$a_0 = \left|1 - \sum_{ij} x_{ij}\right|,$$

$$a_i = \left|\sum_j x_{ij} - \sum_k x_{ki}\right|, \quad i > 0.$$

PROPOSITION 7. *Let the hypotheses of the previous proposition be fulfilled. Select the initial distribution*

$$x_{ij}^0 = p_{ij}$$

and the relaxation control such that for each x^t, the index p_t realizes the maximum of the residual:

$$a_{p_t} = \max\{a_i : i = 0, 1, \ldots, n\}.$$

When $p_t = 0$, x^{t+1} is defined by (16); when $p_t > 0$, x^{t+1} is defined by (17)–(19) where $i = p_t$. Then the obtained sequence converges to the flow $Q = (q_{ij})$ which maximizes the function

$$g(x) = \sum_{ij} x_{ij} \ln \frac{p_{ij}}{x_{ij}} = \sum_{ij} x_{ij} \ln p_{ij} - \sum_{ij} x_{ij} \ln x_{ij}$$

in the space of flows on $G(P^+)$, with $G(Q^+) \subset G(P^+)$.

This proposition was proved in a more general form in [**2**]. Moreover, as mentioned above, in our case the limit value $Q = (q_{ij})$ does not depend on the relaxation control but only on the initial value. The proposition is useful in regard to computational practice, since it offers a possibility of increasing, essentially, the speed of convergence. Notice that since the transformations (16)–(19) contain the normalization, an initial value Lx^0, $L > 0$, that is proportional to the original one will give the same limit value Q. In computing applications of the balance method, one must remember that it is necessary for a support of any flow to be in the set of recurrent edges. It follows that the support of the initial value has to be in the same set and, moreover, that it is enough to construct a flow on each strongly connected component and then take the linear hull of these flows, if needed.

Example 1. The maximum flow on a graph.

Consider the zero–one adjacency matrix $\Pi = (\pi_{ij})$ of the graph G, i.e. $\pi_{ij} = 1$ if the edge $i \to j$ exists and $\pi_{ij} = 0$ otherwise. Let us apply Proposition 6 or Proposition 7 with $P = \Pi$ and initial value $x^0 = \Pi$. As a result of the normalization we get $x^1 = \Pi/N$, where N is the number of edges of the graph G. According to the propositions, the relaxation sequence converges to the distribution $Q = (q_{ij})$, which is a flow on G that maximizes the function

$$g(x) = -\sum_{ij} x_{ij} \ln x_{ij}$$

on the set of all flows $\mathcal{M}(G)$. Notice that if the normalization condition is satisfied but not the Kirchhoff law, the maximum of g is obtained from $x_{ij} = \text{const} = \pi_{ij}/N$, in which case $g_{\max} = \ln N$. If all the conditions are satisfied, then the flow Q maximizing the function g will be close to $x_{ij} = \text{const}$, i.e. Q is distributed to the maximal possible extent on G.

Example 2. Application of the balance method to the Ikeda mapping.

The Ikeda mapping arises in the modeling of optical recording media (crystals) [**8**] and is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} d & + & a(x\cos\tau(x, y) - y\sin\tau(x, y)) \\ & & b(x\sin\tau(x, y) + y\cos\tau(x, y)) \end{pmatrix}, \tag{21}$$

$$\tau(x, y) = 0.4 - \frac{6}{1 + x^2 + y^2}, \tag{22}$$

where $d > 0$, $0 < a < 1$ and $0 < b < 1$. The mapping contracts area and has a global attractor. In [**13, 14**] it was shown that if $d = 2$, $a = -0.9$ and $b = 0.9$, then the mapping switches the orientation and has a global attractor in the domain $M = [-10, 10] \times [-10, 10]$.

To locate the global attractor, we use ten subdivisions and construct the sequence of symbolic images $G_1, \ldots, G_{10}$. In the top picture of Figure 1, the covering of the attractor lies in the (x, y) plane. It is constructed on G_{10} and consists of 96 543 cells of size 0.019×0.019. The invariant flow is constructed on G_{10} by using the balance method with adjacency matrix Π as the initial value. The relaxation method leads to the matrix $Q = (q_{ij})$ of size $96\,543 \times 96\,543$ which maximizes the function $g(x) = -\sum_{ij} x_{ij} \ln x_{ij}$. The distribution of the invariant measure is shown in the bottom picture of Figure 1, where the measure of each cell is represented by the z-value. Since Q maximizes the function g, the invariant measure is distributed to the maximum allowable extent on the attractor.

This numerical experiment was performed by postgraduate student E. Petrenko at St. Petersburg University.

Example 3. Estimation of entropy.

Now we use the technique we have developed to estimate the entropy with respect to a measure. Suppose an invariant flow $m = \{m_{ij}\}$ is constructed on a symbolic image G of the mapping f. As remarked earlier, the flow m can be considered as an approximation of an invariant measure μ if the diameter d is small enough. The flow m on G generates a Markov chain [**11**, pp. 47 and 328] whose states are the vertices of G and whose transition probabilities are

$$p_{ij} = \frac{m_{ij}}{m_i}, \quad m_i = \sum_j m_{ij}.$$

Under these conditions, the probability matrix $P = (p_{ij})$ has a stationary distribution $(m_1, m_2, \ldots, m_n)$.

The entropy for the stationary distribution is computed by the formula

$$h_m = -\sum_i m_i \sum_j p_{ij} \ln p_{ij}$$

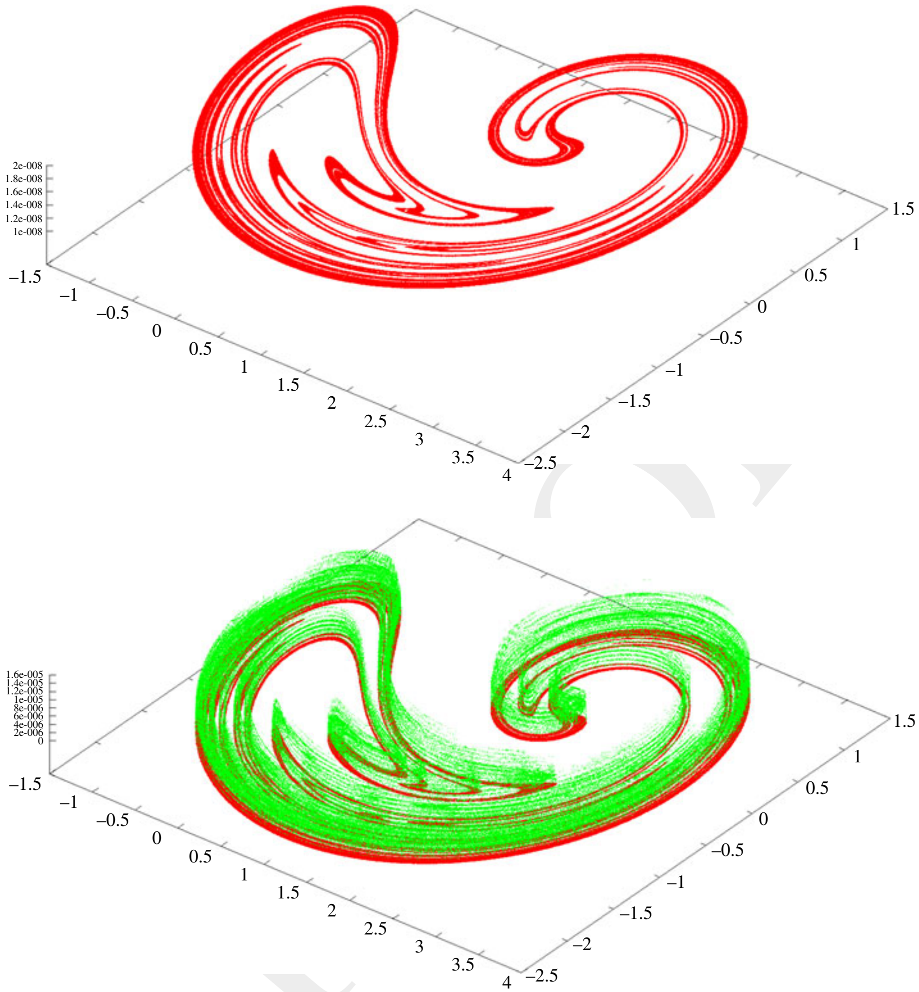

FIGURE 1. The balance method for the modified Ikeda mapping.

(see [**11**, p. 443]). Upon substituting $p_{ij} = m_{ij}/m_i$, we find that

$$h_m = -\sum_{ij} m_{ij} \ln m_{ij} + \sum_{i} m_i \ln m_i. \tag{23}$$

Thus, the entropy can be calculated directly via the flow $m = \{m_{ij}\}$ The entropy of f with respect to the invariant measure μ can be estimated by the formula (23), where the flow m approximates μ. However, justifying this method for arbitrary f is a problem (although, with hyperbolic structure for f, the estimation of entropy through SBR measures is well-known [**4, 7, 16**]).

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