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On automorphism groups of some modules

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Abstract. In this paper we investigated ZG-module A, where G is either nilpotent or soluble group of infinite special rank. It is described the construction of these modules in the case, if for any proper subgroup H of infinite special rank of group G the quotient module $A/C_A(H)$ is a noetherian Z-module.

Let F be a field and A be a vector space over the field F. Let $GL(F, A)$ be a group of all automorphisms of the vector space A. The group $GL(F, A)$ and all its subgroups are called the linear groups. The group $GL(F, A)$ is studied well for finite dimensional vector space A. The case of infinite dimensional vector space A was investigated not enough. There are no general approaches for study of $GL(F, A)$ in this case. The one of approaches is the application of finiteness conditions for study of these groups. Such approach is highly effective. The evidence of this fact is the example of the theory of finitary linear groups [1]. The other successful application of finiteness conditions was shown in the paper [2]. In the paper [3] it was investigated the subgroups G of group $GL(F, A)$ which possessed the propery that for any subgroup H of infinite special rank of group G the quotient space $A/C_A(H)$ had finite dimension.

If G is a subgroup of $GL(F, A)$ then A can be considered as FG-module. The generalization of this situation is the case of RG-module where R is a commutative ring which is similar to the field (the integral domain, the Dedekind domain, the principal ideal domain etc.). The finite dimensional vector space can be generalized to the R-module with the minimality and maximality conditions. The natural first step of this generalization is the investigation of Z-modules.

The principal results of this article are the theorems.

Theorem 1. Let A be a ZG-module, G be a nilpotent group of infinite special rank. Suppose that for any proper subgroup H of infinite special rank of group G the quotient module $A/C_A(H)$ is a noetherian Z-module. Then A is a noetherian Z-module.

Proof. It is sufficient to consider the case when G has infinite special rank. We consider firstly the case of abelian group G of infinite special rank. Then the group G can be presented as $G = G_1K$ where G_1 is a proper subgroup of infinite special rank and K has special rank 1. >From the theorem's conditions it follows that the quotient module $A/C_A(G_1)$ is a noetherian Z-module. We can choose the subgroup N of group G_1 such that either N is an elementary abelian p-group or free abelian group of infinite special rank. Therefore $N = N_1XN_2$, where the special ranks of subgroups N_1 and N_2 are infinite. Since the subgroup N_1K is a proper subgroup of A and the special rank of N_1K is infinite, then the quotient module $A/C_A(K)$ is a noetherian Z-module. From this fact and the inclusion $C_A(N_1K) \leq C_A(K)$ it follows that the quotient module $A/C_A(K)$ is a noetherian module. Since $G = G_1K$ and the submodule $C_A(G)$ is trivial then $C_A(G_1) \cap C_A(K) = 0$. Then the quotient module $A/(C_A(G_1) \cap C_A(K))$ is imbeded in the direct sum of quotient modules $A/C_A(G_1)$ and $A/C_A(K)$. Since these quotient modules are noetherian Z-modules then the quotient module $A/(C_A(G_1) \cap C_A(K))$ is a noetherian Z-module. Therefore the module A is a noetherian Z-module too.

Let G be a nonabelian nilpotent group of infinite special rank now. It is possible to consider that the special rank of the center $Z(G)$ of the group G is infinite. Otherwise it exists the number k for the upper central series

$$E = Z_0 < Z_1 < Z_2 < ... < Z_{n-1} < Z_n = G$$

of group G such that the special rank of the factor Z_{k+1}/Z_k is infinite. Therefore it is possible to consider the quotient group G/Z_k. The group G can be presented as the product $G = DK$, where D and K are the proper subgroups of G and the subgroups D and K contain the center $Z(G)$. Since the quotient modules $A/C_A(K)$ and $A/C_A(D)$ are noetherian Z-modules then the quotient module $A/(C_A(D) \cap C_A(K))$ is a noetherian Z-module too. Therefore the module A is a noetherian Z-module. The theorem is proved.

Theorem 2. Let A be a ZG-module. Suppose that G is a soluble group of infinite special rank and for any proper subgroup H of infinite special rank of group G the quotient module $A/C_A(H)$ is a noetherian Z-module. Then the one of the following statements is valid:

1) A is a noetherian Z-module;

2) the group G is presented as the product $G = BN$, where N is a quasicyclic subgroup, B is the minimal normal subgroup of group G and $B \cap N = E$.

Proof. It is sufficient to consider the case when G is a soluble group of infinite special rank, which has not the decomposition (2). Then one from the term G_k of derived series

$$E = G_0 < G_1 < G_2 < ... < G_{n-1} < G_n = G$$

of the group G has infinite special rank. Therefore it is sufficient to consider the case when the last non-identity term of derived series G_1 has the infinite spesial rank. The group G is presented as the product $G = DK$, where D and K are the proper subgroups of group G and the subgroups D and K contain the commutator subgroup G_1. If we shall reason like the case of nilpotent group, we shall prove that A is a noetherian Z-module. The theorem is proved.

References

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