Long-time asymptotics for the Buckley-Leverett models of development of oil and gas fields

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***Abstract* — Asymptotic solutions for two-phase flows in porous media are studied. Dimensionless numbers for characterization of these flows are introduced. These numbers we use to construct asymptotic presentations of solutions. We give explicit formulae for the initial terms of this asymptotic.**

***Index Terms* — Long-time asymptotic, multiphase flow, Buckley-Leverett model, porous medium.**

Introduction

In this paper we consider the Buckley-Leverett filtration model [1] for a two-phase system (for instance, “water – oil”) in a plane domain with injection and producing wells.

We assume that:

– the fluids are incompressible and immiscible;
– the porous media is non-deformable;
– capillary forces are not taken into account.

Corresponding differential equations we write in dimensionless form with a parameter which we call characteristic. We use this parameter to construct explicit asymptotic representations of solutions of the equations.

The model

The multiphase flow model for transport in porous medium is governing by two types of differential equations (see, for example, [1] [2] and [3]):

i) Mass conservation for phase *i*

$$m\frac{\partial(\rho_i s_i)}{\partial t}+\operatorname{div}(\rho_i U_i)=0,$$

where m is a porosity, ρ_i is a phase density, and s_i is a phase saturation.

ii) Momentum conservation, or Darcy law, for phase *i*:

$$U_i=-k\frac{f_i(s_i)}{\mu_i}\operatorname{grad} p_i$$

where U_i is a volumetric velocity, f_i is a relative permeability (see [4], for example), μ_i is a dynamic viscosity, p_i is a phase pressure. Putting these equations together, we get equations of the form

$$m\frac{\partial(\rho_i s_i)}{\partial t}=k\operatorname{div}\left(\frac{f_i(s_i)\rho_i}{\mu_i}\operatorname{grad} p_i\right). \tag{1}$$

In order to write the last equations in the dimensionless form we introduce characteristic values: T_i – for time interval, L – characteristic length, and M_i, F_i, P_i – for characteristic viscosity, relative permeability and pressure respectively.

Then in dimensionless variables $t/T, x/L, \mu_i/M_i$, p_i/P_i, f_i/F_i Eq.1 takes the form

$$\varepsilon_i\frac{\partial(\rho_i s_i)}{\partial t}=k\operatorname{div}\left(\frac{f_i(s_i)\rho_i}{\mu_i}\operatorname{grad} p_i\right), \tag{2}$$

where

$$\varepsilon_i = \frac{mM_i L^2}{F_i T P_i},$$

and we continue to use the same notations t, x, f_i, μ_i, p_i for dimensionless variables.

We call ε_i **characteristic number** of the phase i and say that i th phase is a **fast,** if $\varepsilon_i > 1$ and **slow,** if $\varepsilon_i < 1$.

In this paper, we'll consider only two-phase models, and will assume that one phase, say the first one, is slow.

In the Buckley-Leverett model there are two phases only and is assumed that densities ρ_i and viscosities μ_i are constant, and capillary forces are neglected, i.e. $p_1 = p_2 = p$.

For this case system Eq.2 takes the following form:

$$\begin{cases} \varepsilon_1 \dfrac{\partial \sigma}{\partial t} = k \operatorname{div}\left(\dfrac{f_1(\sigma)}{\mu_1} \operatorname{grad} p \right), \\ -\varepsilon_2 \dfrac{\partial \sigma}{\partial t} = k \operatorname{div}\left(\dfrac{f_2(1-\sigma)}{\mu_2} \operatorname{grad} p \right), \end{cases} \tag{3}$$

where $\sigma = s_1$.

Assuming that the first phase is slow, i.e. $\varepsilon_1 = \varepsilon < 1$, we'll rewrite system Eq.3 in the following form:

$$\begin{cases} \varepsilon \dfrac{\partial \sigma}{\partial t} = \operatorname{div}(h_1(\sigma) \operatorname{grad} p), \\ \operatorname{div}(h_2(\sigma) \operatorname{grad} p) = 0, \end{cases} \tag{4}$$

where

$$h_1(\sigma) = k \frac{f_1(\sigma)}{\mu_1},$$

and

$$h_2(\sigma) = \frac{k\varepsilon_2 f_1(\sigma)}{\mu_1} + \frac{k\varepsilon_1 f_2(1-\sigma)}{\mu_2}.$$

To study asymptotics in ε for solutions of system Eq.4 we represent functions p and σ in the following asymptotic form:

$$p(t,x) \sim \sum_{k\geq 0} p_k(t,x) \frac{\varepsilon^k}{k!},$$

$$\sigma(t,x) \sim \sum_{k\geq 0} \sigma_k(t,x) \frac{\varepsilon^k}{k!}.$$

Then, for the initial term of the asymptotic, we get the following system of equations:

$$\begin{cases} \operatorname{div}(h_1(\sigma_0) \operatorname{grad} p_0) = 0, \\ \operatorname{div}(h_2(1-\sigma_0) \operatorname{grad} p_0) = 0. \end{cases}$$

Straightforward computations show, that this system is equivalent to the following:

$$\begin{cases} \Delta(p_0) = 0, \\ (\operatorname{grad} p_0, \operatorname{grad} \sigma_0) = 0, \end{cases} \tag{5}$$

where Δ is the Laplace operator, and $(\cdot,\cdot)$ is the scalar product. Assume that we have "oil-field" $D \subset R^2$, with wells $a_j \in D, j = 1, \ldots, m$.

Remark that the pseudo-group of conformal transformations consists of symmetries of system Eq.5. Therefore, due to Riemann mapping theorem, we could consider only the case, when D is the unit disk.

Let

$$p_0^{(j)}(x) = \frac{1}{2\pi} \ln\left| x - a_j \right|$$

be the fundamental solution of the Laplace operator at point a_j, i.e.

$$\Delta p_0^{(j)} = \delta_{a_j}.$$

Here δ_{a_j} is the Dirac delta function located at the well a_j. Then the function

$$\bar{p}_0 = \sum_{j=1}^{m} c_j(t) p_0^{(j)}$$

satisfies the following relation:

$$\Delta \bar{p}_0 = \sum_{j=1}^{m} c_j(t) \delta_{a_j}.$$

Adding harmonic in D function $\tilde{p}_0$, such that

$$\left.\frac{\partial \tilde{p}_0}{\partial n}\right|_{\partial D} = -\left.\frac{\partial \bar{p}_0}{\partial n}\right|_{\partial D},$$

we get a function p_0, such that

$$\Delta p_0 = \sum_{j=1}^{m} c_j(t) \delta_{a_j},$$

and

$$\left.\frac{\partial p_0}{\partial n}\right|_{\partial D} = 0.$$

Straightforward computations and Dini formula show that the following result holds.

Theorem 1. *Function*

$$p_0(t,x) = \frac{1}{2\pi} \sum_{j=1}^{m} \ln|x - a_j|\, c_j(t) + \\ -\frac{1}{4\pi^2} \sum_{j=1}^{m} c_j(t) \int_0^{2\pi} \frac{(x - a_j, n) \ln(|x|^2 - 2(x,n) + 1)}{1 + |a_j|^2 - 2(x, a_j)} d\psi \tag{6}$$

where $n = (\cos\psi, \sin\psi)$ *is a unit normal vector to the circle, satisfies the following relations*

$$\Delta p_0 = \sum c_j(t)\delta_{a_j},$$

and

$$\left.\frac{\partial p_0}{\partial n}\right|_{\partial D} = 0.$$

In order to find the saturation σ_0, we denote by V_j function, complex adjoin to p_{0j}, i.e.

$$V_{j,x} = p_{0j,y},$$

$$V_{j,y} = -p_{0j,x}$$

or

$$V_j(x) = \int_\gamma (p_{0j,x}dy - p_{0j,y}dx),$$

here γ is a path in $D\setminus\{a_j\}$, connecting a point *x* with a fixed point $o \in D$.

This is a multivalued function, and its value changed on integer 1, when we path γ runs around the well a_j. Then the function

$$V(t,x) = \sum_j c_j(t)\, V_j(x),$$

is adjoint to $p_0(t,x)$, and its value will be changed on $c_j(t)$, when path γ runs along small circle around the well a_j.

Let's for a given moment of time *t*, denote by $G_t \subset R$ the abelian additive group, generated by numbers $c_1(t),\ldots,c_m(t)$.
We say that value t is **regular**, if the group G_t is closed in *R*, and *t* is **irregular** in the opposite case.

Theorem 2. *i) For regular moments of time, there is a function* $c(t)$*, such that* $c_j(t) = n_j c(t)$*, where.*n_j *are integer numbers.*
ii) For irregular moments of time t, the group G_t *is dense in R, and values of function* $V(t,x)$ *are dense in R.*

Remark now, that the saturation function σ_0, as well as, function *V* is the first integral for the vector field grad p_0.

Therefore,

$$\sigma_0(t,x) = F(V(t,x),t),$$

for some function F. Moreover, the function F should be periodic in the sense, that

$$F(V + c(t),t) = F(V,t).$$

Summarizing, we get the following description of the initial term of asymptotic.

Theorem 3. *Initial term* (p_0,σ_0) *of the asymptotic has form Eq.6 for pressure, and*

$$\sigma_0(t,x) = F(V(t,x),t),$$

where $F(V,t)$ *is periodic function in V with period c(t).*

Remark that productivities $c_j(t)$ related to mass rate in the following way.

The functions

$$\mathrm{div}\left(\frac{k\rho_i f_i(\sigma_i)}{\mu_i}\,\mathrm{grad}\ p\right)$$

represent mass rate for phase *i*.

Using relations

$$\mathrm{div}(\mathrm{grad}\ p_0) = \sum_j c_j(t)\delta_{a_j}$$

and

$$(\mathrm{grad}\ \sigma_0, \mathrm{grad}\ p_0) = 0,$$

we see that the mass rate for phase *i* at well a_j equals to

$$\frac{k\rho_i}{\mu_i} c_j(t) \int_0^{2\pi} f_i(\sigma_0)\,d\varphi \cdot$$

THE FIRST TERM OF ASYMPTOTIC

Let's $\varepsilon_2 = \lambda\varepsilon$ for some $\lambda \in R$ and compute the first term of asymptotic.

The routine computations give us the following system of linear differential equations for functions p_1 and σ_1 describing the first term of asymptotic:

$$\begin{cases} \Delta p_1 = \dfrac{A'}{BA' - AB'}\dfrac{\partial\sigma_0}{\partial t}, \\ (\mathrm{grad}\ p_0, \mathrm{grad}\ \sigma_1) = \dfrac{A}{A'B - B'A}\dfrac{\partial\sigma_0}{\partial t} - \\ \quad -(\mathrm{grad}\ p_1, \mathrm{grad}\ \sigma_0), \end{cases} \tag{7}$$

where

$$A = \frac{\lambda k f_1(\sigma_0)}{\mu_1} + \frac{k f_2(1-\sigma_0)}{\mu_2} \tag{8}$$

and $B = k f_1(\sigma_0)/\mu_1$ are known functions.

Remark, that Eq.7 is the Poisson equation and could be solved on the unit disk by means of the Neuman function.

To solve Eq.8 as equation with respect to function σ_1, assuming that functions σ_0, p_0, p_1 are known, we consider functions p_0 and σ_0 as new coordinates, and express all functions involved in this equation as functions in p_0 and σ_0.

Then Eq.8 will take the form

$$\frac{\partial\sigma_1}{\partial p_0} = C(p_0,\sigma_0),$$

where $C(p_0,\sigma_0)$ is the following function

$$C(p_0,\sigma_0)=\frac{A}{(A'B-B'A)\,|\,\mathrm{grad}\;p_0\,|^2}\times$$

$$\times\frac{\partial\sigma_0}{\partial t}-\frac{(\mathrm{grad}\;p_0,\mathrm{grad}\;p_1)}{|\,\mathrm{grad}\;p_0\,|^2}$$

written in coordinates (p_0,σ_0).

Thuse we have the following representation of the saturation:

$$\sigma_1=\int C(p_0,\sigma_0)dp_0.$$

CONCLUSION

Constructed asymptotics of the solutions of the Buckley-Leverett filtration equations (3) allows to calculate pressure and saturation analytically. The functions $c_j(t)$ can be considered as control parameters. They can be used to construct optimal regime of development of oil fields.

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