DROP IN AN UNSTEADY UNIFORM PURE SHEAR FLOW AND RHEOLOGICAL PROPERTIES OF EMULSIONS

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Summary The velocity and pressure in an oscillatory uniform pure shear flow disturbed by a spherical drop placed in its center are found in the low Reynolds number approximation without using the quasi-steady approximation. With the use of the obtained formulas, the relation between the stress tensor and the strain rate tensor in a dilute emulsion of spherical drops is found. This relation is determined by some complex-valued viscosity.

INTRODUCTION

The problem about the distortion of a pure shear flow by a solid or fluid body is of a special interest since the result of its solution can be used in order to investigate theoretically the rheological properties of dilute dispersions. The viscosities of dilute suspensions of spherical particles [1] and emulsions of spherical drops [2] were calculated with the use of the solution of the above-mentioned problem for the steady case. The goal of this work is to solve this problem for essentially unsteady case in the hope that the obtained solution would allow revealing more complicated rheological behavior of diluted dispersions for sufficiently fast varying strain rates.

SPHERICAL DROP IN AN OSCILLATORY UNIFORM PURE SHEAR FLOW

Consider a spherical drop of radius a placed in the center of a uniform oscillatory pure shear flow with strain rate tensor $\hat{e}_\omega \cos(\omega t)$, where t and ω are the time and the angular frequency of the oscillations and $\hat{e}_\omega$ is some symmetric traceless tensor. The densities and viscosities of the liquid of the drop and of the ambient liquid are ρ_i and η_i and ρ_e and η_e, respectively. The Reynolds number is regarded as so small that the convective acceleration can be neglected. The surface tension of the interface between the liquids, σ_s, is so high that the deformation of the drop can be neglected.
The system of equations for the velocity and pressure, $\vec{v} = \vec{v}(\vec{r}, t)$ and $p = p(\vec{r}, t)$, consists of the continuity and Navier–Stokes equations for an incompressible fluid in the low Reynolds number approximation

$$\nabla \cdot \vec{v} = 0, \qquad \rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \eta \Delta \vec{v}. \tag{1}$$

Here, $\rho = \rho_\mathrm{i}$, $\eta = \eta_\mathrm{i}$ inside the drop, $\rho = \rho_\mathrm{e}$, $\eta = \eta_\mathrm{e}$ outside it, ∇ is the nabla operator, Δ is the Laplacian, $\vec{r}$ is the radius vector with the origin at the center of the drop, $\cdot$ denotes the contraction (or scalar product).
The boundary conditions on the interface between the liquids includes the impenetrability and no-slip conditions and the conditions for the jumps of the tangential and normal components of the stress vector, $\left(-p\hat{I} + \hat{\sigma}_\mathrm{v}\right) \cdot \vec{n}$,

$$\vec{v}\big|_\mathrm{e} \cdot \vec{n} = 0, \qquad \vec{v}\big|_\mathrm{i} \cdot \vec{n} = 0, \qquad [\vec{v}]_\mathrm{s} \times \vec{n} = 0, \qquad \vec{n} \times [\hat{\sigma}_\mathrm{v}]_\mathrm{s} \cdot \vec{n} = 0, \qquad [p]_\mathrm{s} = -\frac{2\sigma_\mathrm{s}}{a}, \tag{2}$$

where $\hat{\sigma}_\mathrm{v}$ is the viscous stress tensor, $\hat{\sigma}_\mathrm{v} = 2\eta \left(\nabla \vec{v}\right)^\mathrm{s}$. Here, $A\big|_\mathrm{i}$ and $A\big|_\mathrm{e}$ denote the values of the quantity A on the interface between the liquids approached from inside and outside the drop, respectively, $[A]_\mathrm{s} = A\big|_\mathrm{e} - A\big|_\mathrm{i}$ denotes the jump of the quantity A at the interface when moving from the inside to the outside, $\vec{n}$ is the external normal unit vector at a given point of the interface, $\nabla \vec{b}$ denotes the dyadic product of the nabla operator and the vector field $\vec{b} = \vec{b}(\vec{r})$, $\hat{I}$ is the identity tensor, $\hat{T}^\mathrm{s}$ denotes symmetric part of the tensor $\hat{T}$, $\times$ denotes the vector product.
The boundary conditions at infinity have the form

$$\vec{v} \sim \hat{e}_\omega \cdot \vec{r} \cos(\omega t), \quad p \sim \frac{1}{2}\rho_\mathrm{e} \vec{r} \cdot \hat{e}_\omega \cdot \vec{r} \sin(\omega t) \quad \text{as} \quad r \to \infty. \tag{3}$$

Besides, $\vec{v}(\vec{r}, t)$ and $p(\vec{r}, t)$ should be bounded for all the bounded values of $\vec{r}$ and t.
Seeking the steady-state oscillatory solution of the system of equations (1) with boundary conditions (2)–(3) in the form

$$\vec{v} = \nabla \times \left\{ \Re \left[f_\omega(r) e^{-i\omega t} \right] \vec{r} \cdot \hat{e}_\omega \times \vec{r} \right\}, \quad p = \Re \left[h_\omega(r) e^{-i\omega t} \right] \vec{r} \cdot \hat{e}_\omega \cdot \vec{r} + p_0, \tag{4}$$

where i is the imaginary unit, $\Re$ denotes the real part of a complex value, one obtains

$$f_\omega(r) = \begin{cases} f_{\mathrm{i}\omega} \left[\dfrac{\psi\left(z_{\mathrm{i}\omega} r/a\right)}{\psi(z_{\mathrm{i}\omega})} - 1 \right], & r < a, \\ f_{\mathrm{e}\omega} \left[\dfrac{\phi\left(z_{\mathrm{e}\omega} r/a\right)}{\phi(z_{\mathrm{e}\omega})} - \dfrac{a^5}{r^5} \right] + \dfrac{1}{3}\left(1 - \dfrac{a^5}{r^5}\right), & r \geq a, \end{cases} \tag{5}$$

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$$
h_\omega(r) = \begin{cases} -\dfrac{3}{2}\dfrac{\eta_\mathrm{i}}{a^2} z_{\mathrm{i}\omega}^2 f_{\mathrm{i}\omega}, & r < a, \\ \dfrac{\eta_\mathrm{e}}{a^2}\dfrac{a^5}{r^5} z_{\mathrm{e}\omega}^2 f_{\mathrm{e}\omega} + \dfrac{1}{2}\dfrac{\eta_\mathrm{e}}{a^2}\left(1 - \dfrac{2}{3}\dfrac{a^5}{r^5}\right) z_{\mathrm{e}\omega}^2, & r > a, \end{cases} \qquad p_0 = \begin{cases} \dfrac{2\sigma_\mathrm{s}}{a}, & r < a, \\ 0, & r > a, \end{cases} \tag{6}
$$

$$
f_{\mathrm{i}\omega} = \frac{5}{3}\frac{\eta_\mathrm{e}\left[2\phi_1(z_{\mathrm{e}\omega}) + \phi_2(z_{\mathrm{e}\omega})\right]\psi(z_{\mathrm{i}\omega})}{\eta_\mathrm{e}\psi_1(z_{\mathrm{i}\omega})\phi_2(z_{\mathrm{e}\omega}) - \eta_\mathrm{i}\psi_2(z_{\mathrm{i}\omega})\phi_1(z_{\mathrm{e}\omega})}, \qquad f_{\mathrm{e}\omega} = \frac{5}{3}\frac{\phi(z_{\mathrm{e}\omega})\left[2\eta_\mathrm{e}\psi_1(z_{\mathrm{i}\omega}) + \eta_\mathrm{i}\psi_2(z_{\mathrm{i}\omega})\right]}{\eta_\mathrm{e}\psi_1(z_{\mathrm{i}\omega})\phi_2(z_{\mathrm{e}\omega}) - \eta_\mathrm{i}\psi_2(z_{\mathrm{i}\omega})\phi_1(z_{\mathrm{e}\omega})}, \tag{7}
$$

$$
\psi(x) = \left(\frac{3}{x^5} - \frac{1}{x^3}\right)\sin x - \frac{3}{x^4}\cos x, \qquad \phi(x) = \left[\frac{1}{i}\left(\frac{3}{x^5} - \frac{1}{x^3}\right) - \frac{3}{x^4}\right] e^{ix}, \tag{8}
$$

$$
\psi_1(x) = \left(-\frac{15}{x^5} + \frac{6}{x^3}\right)\sin x + \left(\frac{15}{x^4} - \frac{1}{x^2}\right)\cos x, \qquad \phi_1(x) = \left(\frac{1}{i}\frac{1}{x^3} - \frac{1}{x^2}\right) e^{ix}, \tag{9}
$$

$$
\psi_2(x) = \left(\frac{30}{x^5} - \frac{15}{x^3} + \frac{1}{x}\right)\sin x + \left(-\frac{30}{x^4} + \frac{5}{x^2}\right)\cos x, \qquad \phi_2(x) = \left[\frac{1}{i}\left(-\frac{5}{x^3} + \frac{1}{x}\right) + \frac{5}{x^2}\right] e^{ix}, \tag{10}
$$

$$
z_{\mathrm{e}\omega} = (1+i)\sqrt{\frac{\omega\rho_\mathrm{e}}{2\eta_\mathrm{e}}}\, a, \qquad z_{\mathrm{i}\omega} = (1+i)\sqrt{\frac{\omega\rho_\mathrm{i}}{2\eta_\mathrm{i}}}\, a. \tag{11}
$$

Note that the limit case $\eta_\mathrm{i}/\eta_\mathrm{e} \to \infty$ corresponds to the oscillatory uniform pure shear flow distorted by a rigid sphere of radius a placed in its center. For this case,

$$
f_\omega(r) = \begin{cases} 0, & r < a, \\ -\dfrac{5}{3}\dfrac{\phi(z_{\mathrm{e}\omega})}{\phi_1(z_{\mathrm{e}\omega})}\left[\dfrac{\phi\left(z_{\mathrm{e}\omega} r/a\right)}{\phi(z_{\mathrm{e}\omega})} - \dfrac{a^5}{r^5}\right] + \dfrac{1}{3}\left(1 - \dfrac{a^5}{r^5}\right), & r \geq a, \end{cases} \tag{12}
$$

$$
h_\omega(r) = -\frac{5}{3}\frac{\eta_\mathrm{e}}{a^2}\frac{a^5}{r^5}\frac{z_{\mathrm{e}\omega}^2\phi(z_{\mathrm{e}\omega})}{\phi_1(z_{\mathrm{e}\omega})} + \frac{1}{2}\frac{\eta_\mathrm{e}}{a^2}\left(1 - \frac{2}{3}\frac{a^5}{r^5}\right) z_{\mathrm{e}\omega}^2, \quad r > a. \tag{13}
$$

RHEOLOGICAL PROPERTIES OF EMULSIONS OF SPHERICAL DROPS

Consider a dilute emulsion of drops identical to that specified above. Following the procedure described in [3] (§ 22) and using the obtained solution (4)–(11), one obtains the relation between the stress and strain rate tensors, $\hat{\sigma}_{\mathrm{d}\omega}(t)$ and $\hat{e}_{\mathrm{d}\omega}\cos(\omega t)$, in this emulsion for the angular frequency ω

$$
\hat{\sigma}_{\mathrm{d}\omega}(t) = -p_\mathrm{d}(t)\hat{I} + 2\eta_\mathrm{e}\Re\left\{\left[1 + \left(-f_{\mathrm{e}\omega}\frac{z_{\mathrm{e}\omega}^2}{2} - \frac{z_{\mathrm{e}\omega}^2}{30}\right)c_\mathrm{v}\right] e^{-i\omega t}\right\}\hat{e}_{\mathrm{d}\omega}, \tag{14}
$$

where $p_\mathrm{d}(t)$ is the pressure in the emulsion, c_v is the volume concentration of the emulsion. Thus, the rheological properties of the emulsion are determined by the complex-valued viscosity

$$
\tilde{\eta}_{\mathrm{d}\omega} = \eta'_{\mathrm{d}\omega} + i\eta''_{\mathrm{d}\omega} = \eta_\mathrm{e} + \left\{-\frac{5}{6}\frac{z_{\mathrm{e}\omega}^2\phi(z_{\mathrm{e}\omega})\left[2\eta_\mathrm{e}\psi_1(z_{\mathrm{i}\omega}) + \eta_\mathrm{i}\psi_2(z_{\mathrm{i}\omega})\right]}{\eta_\mathrm{e}\psi_1(z_{\mathrm{i}\omega})\phi_2(z_{\mathrm{e}\omega}) - \eta_\mathrm{i}\psi_2(z_{\mathrm{i}\omega})\phi_1(z_{\mathrm{e}\omega})} - \frac{z_{\mathrm{e}\omega}^2}{30}\right\}\eta_\mathrm{e} c_\mathrm{v}. \tag{15}
$$

In the limit $\omega \to 0$, (15) yields the expression for the viscosity obtained by Taylor [2]

$$
\eta_\mathrm{d} = \eta_\mathrm{e} + \frac{1}{2}\frac{2\eta_\mathrm{e} + 5\eta_\mathrm{i}}{\eta_\mathrm{e} + \eta_\mathrm{i}}\eta_\mathrm{e} c_\mathrm{v}. \tag{16}
$$

The limit case $\eta_\mathrm{i}/\eta_\mathrm{e} \to \infty$ corresponds to a dilute suspension of rigid spherical particles in a liquid with viscosity η_e. For this case,

$$
\tilde{\eta}_{\mathrm{d}\omega} = \eta_\mathrm{e} + \left[\frac{5}{6}\frac{z_{\mathrm{e}\omega}^2\phi(z_{\mathrm{e}\omega})}{\phi_1(z_{\mathrm{e}\omega})} - \frac{z_{\mathrm{e}\omega}^2}{30}\right]\eta_\mathrm{e} c_\mathrm{v} = \eta_\mathrm{e} + \left(-\frac{z_{\mathrm{e}\omega}^2}{30} - \frac{5iz_{\mathrm{e}\omega}}{6} + \frac{5}{3} + \frac{5i}{6}\frac{1}{z_{\mathrm{e}\omega} + i}\right)\eta_\mathrm{e} c_\mathrm{v}. \tag{17}
$$

In the limit $\omega \to 0$, (17) yields the expression for the viscosity, $\eta_\mathrm{d} = \eta_\mathrm{e} + \frac{5}{2}\eta_\mathrm{e} c_\mathrm{v}$, obtained by Einstein [1].

CONCLUSIONS

For sufficiently fast varying strain rates, the rheological properties of emulsions of spherical drops and of suspensions of spherical rigid particles are determined by some complex-valued viscosities, but not by real-valued ones like for ordinary viscous liquids. In spite of some resemblance to viscoelasticity, this rheological behavior of emulsions and suspensions is different from the latter and is the consequence of the essential unsteadiness of the flow.

References

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